Kombinatoriikka (Combinatorics)
Excercise 2

In excercises 1-3 use combinatorial reasoning.

1. Show that for all $0 \leq b \leq a$ one has

$$\binom{a}{b} = \sum_{k=b}^{a} \binom{k-1}{b-1}$$

[Hint: each subset of $[a]$ has a greatest element.]

2. Show that for all $a, b \geq 0$, $n \leq a+b$ one has

$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$$

3. Show that for all $n \geq 1$ one has

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

4. A meeting is attended by n elegible voters. In how many ways can one form a simple majority (i.e. a set strictly larger than its complement) out of the voters?

5. An (unordered) partition of a set S is a set $\mathcal{A}$ of disjoint subsets of S that cover S i.e. each $s \in S$ is in some $A \in \mathcal{A}$. For example $\{\{1\}, \{2,4\}, \{3,5,6\}\}$ is a partition of the set $[6]$. The number B_n of partitions of the set $[n]$ is called the n:th Bell number B_n. Use combinatorial reasoning to show that the Bell numbers satisfy the following recursion:

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k}.$$