# The $\overline{\partial}$-method for non-linear inverse problems

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# Outline

1. A primer on complex analysis
2. The Davey-Stewartson II equation
3. Scattering transform, inverse scattering transform and the $\overline{\partial}$-equation
4. Numerical solution of the $\overline{\partial}$-equation

# 1. A primer on complex analysis

# A primer on complex analysis

Concerns properties of functions $f\colon \mathbb{C} \mapsto \mathbb{C}$.
Special attention is given to the operators $\overline{\partial}$ and $\partial$ defined by

$$\overline{\partial} = \partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \quad \partial = \partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}),$$

where the variables are $z = x_1 + ix_2$.
We call a function analytic in the domain $\Omega$ if

$$\overline{\partial} u = 0 \quad z \in \Omega.$$

**Liouville's theorem:**
Suppose u is analytic on $\mathbb{C}$ and bounded. Then $u$ is constant.

# Pseudo-analytic functions

The $\overline{\partial}$-equation has the form

$$\overline{\partial} u(z) = a(z)\overline{u}(z),\ z \in \Omega.$$

We say that such a $u$ is pseudoanalytic in $\Omega$.
In the sequel we will consider the $\overline{\partial}$-equation in the whole plane

$$\overline{\partial} u(z) = a(z)\overline{u}(z),\ z \in \mathbb{C}$$

assuming

$$a \in L^{2-\epsilon} \cap L^{2+\epsilon}(\mathbb{R}^2)$$

We would like to solve this equation in $L^q(\mathbb{C})$.

# The solid Cauchy operator

To transform the $\overline{\partial}$-equation to an integral equation we need a $\overline{\partial}^{-1}$-operator. This is achived by convolution with Green's function for $\overline{\partial}$ :

$$\overline{\partial} g(z) = -\delta_0(z) \text{ with } \lim_{|z|\to\infty} g(z) = 0$$
$$\Rightarrow g(z) = \frac{1}{\pi z}.$$

Solid Cauchy transform:

$$C\phi(z) = \overline{\partial}^{-1}\phi(z) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\phi(z')}{z - z'} d\mu(z').$$

Satisfies for $\phi \in \mathcal{S}(\mathbb{R}^2)$

$$\overline{\partial}\overline{\partial}^{-1}\phi = \phi, \quad \overline{\partial}^{-1}\overline{\partial}\phi = \phi$$

# Properties of $\overline{\partial}^{-1}$

Mapping properties:

$$C\colon L^p(\mathbb{R}^2) \to L^{\tilde{p}}(\mathbb{R}^2),\ 1 < p < 2,\ \frac{1}{\tilde{p}} = \frac{2-p}{2p},$$
$$C\colon L^{p_1} \cap L^{p_2}(\mathbb{R}^2) \to C^{\alpha}(\mathbb{R}^2),\ \alpha = 1 - \frac{2}{p_2}$$

For $a \in L^2(\mathbb{R}^2)$ the operator

$$u \mapsto C(au)$$

is compact in $L^r(\mathbb{R}^2)$ for any $r > 2$.

# Integral equation of the 2nd kind

Let us apply $C$ to the equation

$$\overline{\partial} u = a\overline{u} + f.$$

Then

$$\begin{aligned} & u - C(a\overline{u}) = C(f) \\ \Leftrightarrow \qquad & u(z) - \frac{1}{\pi}\int \frac{a(z')\overline{u}(z')}{z-z'} d\mu(z') = F(z), \quad F = C(f). \end{aligned}$$

Integral equation in $L^{\tilde{p}}(\mathbb{R}^2)$.
For $a \in L^2(\mathbb{R}^2)$ the operator $u \mapsto C(a\overline{u})$ is real-linear and compact on $L^r(\mathbb{R}^2),\ r > 2$.: Fredholms alternative states that either

1. $(I - C(a\ \overline{\cdot}\ ))^{-1}$ exists or
2. the homogeneous problem

$$(u - C(a\,\overline{u})) = 0$$

   has a nontrivial solution.

# Liouville's theorem for pseudoanalytic functions

Suppose $a \in L^2(\mathbb{R}^2)$. Then if $u \in L^{\tilde{p}}(\mathbb{R}^2)$ satisfies the equation

$$\overline{\partial} u = a\overline{u}$$

we have $u = 0$.

**Proof:** Define $w = u \exp(-C(a\frac{\overline{u}}{u}))$. Then (if $a \in L^{2-\epsilon} \cap L^{2+\epsilon}(\mathbb{R}^2)$) $w$ is bounded and satisfies

$$\overline{\partial} w = 0.$$

Hence $w = 0$ which implies $u = 0$.
The case $a \in L^2(\mathbb{R}^2)$ is more delicate...

# $\overline{\partial}$-equation with asymptotic condition

As a consequence of the previous result we have

**Lemma:**
Suppose $a \in L^p \cap L^2(\mathbb{R}^2)$. Then the equation

$$\overline{\partial} m = a\overline{m}$$

has a unique solution which satisfies $m - 1 \in L^{\tilde{p}}(\mathbb{R}^2)$.

**Proof:** The function $m - 1$ satisfies the integral equation

$$m - 1 = C(a\overline{(m-1)}) + C(a).$$

This is a Fredholm equation of the second kind in $L^{\tilde{p}}(\mathbb{R}^2)$. Uniqueness for the homogeneous problem is ensured by the previous result.

# 2. The Davey-Stewartson II equation

# Davey-Stewartson equation

Davey and Stewartson studied in 1974 three-dimensional packets of surface waves on water of finite depth. It is a partial differential equations for a complex (wave-amplitude) field $u$. In complex notation $(z = x_1 + ix_2,\ \overline{z} = x_1 - ix_2)$ the equation are in a particular case, the Davey-Stewartson II equation (DSII), given by

$$iu_t + u_{zz} + u_{\overline{z}\overline{z}} + 2u(\partial_{\overline{z}}^{-1}(|u|^2)_z + \partial_z^{-1}(|u|^2)_{\overline{z}}) = 0, \quad z \in \mathbb{C},\ t > 0.$$

The equation is equipped with an initial condition

$$u(z, 0) = u_0(z)$$

and asymptotic condition

$$\lim_{|z|\to\infty} u(z, t) = 0.$$

This system is an example of a soliton equation in two spatial + one time dimenions.

# The linearized DSII

The linearized DSII

$$2iu_t + u_{x_1x_1} - u_{x_2x_2} = 0,\ u(z,0) = u_0,\ \lim_{|z|\to\infty} u(z,t) = 0.$$

Solved via the Fourier transform

$$\hat{\phi}(k) = \int_{\mathbb{R}^2} \phi(x) e^{-ix\cdot k} dx, \quad x, k \in \mathbb{R}^2$$

$$\phi(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \phi(k) e^{ix\cdot k} dk.$$

Then

$$2i\hat{u}_t = (k_1^2 - k_2^2)\hat{u}.$$

Linear ODE for $\hat{u}(k,t)$, trivially solved by

$$\hat{u}(k,t) = \hat{u}_0(k) e^{-\frac{1}{2}i(k_1^2-k_2^2)t}$$

$$\Leftrightarrow \qquad u(x,t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{u}_0(k) e^{-\frac{1}{2}i(k_1^2-k_2^2)t + ix\cdot k} dk.$$

## Solving linearized DSII by $\overline{\partial}$-method

The linearized DSII is equivalent to the two compatibility conditions for the spectral function $m(z,t,k)$ depending on spectral parameter $k \in \mathbb{C}$ :

$$m_{\overline{z}} + ikm = u, \quad (1)$$

$$m_t + im_{zz} + ik^2 m = im_{\overline{z}} + ku. \quad (2)$$

The equations (1)- (2) constitute a Lax pair for the linearized DSII.

Strategy: Construct solution to (1), take the $\partial_{\overline{k}}$ derivative to get scattering / inverse scattering transform. Then look for evolution of the scattering data...

# Constructing the spectral function

Equation (1) is equivalent to

$$\partial_{\overline{z}}(me^{i(k\overline{z}+z\overline{k})}) = u(z,t)e^{i(k\overline{z}+z\overline{k})}.$$

Let us suppress notation

$$\partial_{\overline{z}}(me(z,\overline{k})) = u(z,t)e(z,\overline{k}), \quad e(z,k) = e^{i(zk+\overline{z}\overline{k})}. \tag{3}$$

Solved by the Solid Cauchy transform

$$m(z,t,k) = \frac{1}{\pi}\int \frac{u(z',t)}{z-z'}e(z'-z,\overline{k})d\mu(z').$$

Let us apply the $\partial_{\overline{k}}$ operator:

$$\partial_{\overline{k}} m(z,t,k) = \frac{1}{\pi}\int_{\mathbb{R}^2} u(z',t)e(z'-z,\overline{k})d\mu(z') \\ = \tilde{u}(k,t)e(z,-\overline{k}),$$

with

$$\tilde{u}(k,t) = \int_{\mathbb{R}^2} u(z',t)e(z'-z,\overline{k})d\mu(z') = \hat{u}(-2k,t).$$

Same structure as (3)!! Moreover $\lim_{k\to\infty} m = 0$ and hence

$$m(z,t,k) = \frac{1}{\pi^2}\int_{\mathbb{R}^2} \frac{\tilde{u}(k',t)}{k-k'} e(z,-\overline{k'})d\mu(k'). \tag{4}$$

Inserting the rhs in (3) yields

$$u(z,t) = \frac{1}{\pi^2}\int_{\mathbb{R}^2} \tilde{u}(k',t)e(z,-\overline{k'})d\mu(k').$$

The evolutionary part (2) can now be used to show that necessarily the spectral function $\tilde{u}(k,t)$ evolves according to

$$\tilde{u}_t + i(k^2 + \overline{k}^2)\tilde{u} = 0.$$

Solution

$$\tilde{u}(k,t) = \tilde{u_0}(k)exp(-i(k^2 + \overline{k}^2)t).$$

By inversion of the scattering transform

$$u(z,t) = \frac{1}{\pi^2}\int_{\mathbb{R}^2} \tilde{u}_0(k')e(z,-\overline{k'})e^{-i(k^2+\overline{k}^2)t}d\mu(k').$$

Same formula as before - but different method.

Second method generalizes to non-linear DSII!

# $\overline{\partial}$-method for DS II

Lax pair for spectral vector $\Psi(z,k,t) = (\psi_1(z,k,t), \psi_2(z,k,t))$ :

$$\partial_{\overline{z}}\psi_1 = u\psi_2, \quad \partial_z\psi_2 = \overline{u}\psi_1 \tag{5}$$

$$i(\psi_1)_t + (\psi_1)_{zz} + u(\psi_2)_{\overline{z}} - u_{\overline{z}}\psi_2 + 2\partial_{\overline{z}}^{-1}((|u|^2)_z\psi_1) = 0 \tag{6}$$

$$-i(\psi_2)_t + (\psi_2)_{\overline{zz}} - \overline{u}(\psi_1)_z + \overline{u}_z\psi_1 + 2\partial_z^{-1}((|u|^2)_{\overline{z}}\psi_2) = 0. \tag{7}$$

As before the equations above has a solution $(\psi_1, \psi_2)$ if and only if $u$ satisfies DSII.

The elliptic system (5) has an associated scattering / inverse scattering transform, which enables the solution of DSII.

# 3. Scattering transform, inverse scattering transform and the $\overline{\partial}$-equation

# The direct scattering transform

We suppress the $t$-variable and write the equation in the form

$$(D-Q)\Psi = 0, \quad D = \begin{pmatrix} \partial_{\overline{z}} & 0 \\ 0 & \partial_z \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix}.$$

and look for Jost solutions

$$\Psi(z,k) = e^{izk} m(z,k) = e^{izk} \begin{pmatrix} m_1(z,k) \\ m_2(z,k) \end{pmatrix}$$

with asymptotic behaviour

$$\lim_{|z|\to\infty} m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The equations for $m_1, m_2$ :

$$\partial_{\overline{z}} m_1 = u m_2, \quad (\partial_z + ik) m_2 = \overline{u} m_1.$$

# The Jost solutions

**Lemma:** Suppose $Q \in L^2(\mathbb{R}^2)$. Then there is a unique Jost solution.
*Proof:* The functions

$$m_\pm = m_1 \pm \overline{m_2} e(z, -k)$$

satisfy the $\overline{\partial}$-equation

$$\partial_{\overline{z}} m_\pm = \pm u e(z, -k) \overline{m_\pm}$$

together with the asymptotic condition $\lim_{|z|} m_\pm = 1$. Such an equation has a unique solution whenever $u \in L^2(\mathbb{R}^2)$.

In addition it can be shown that with respect to the spectral parameter $k$ the solution satisfies

$$\lim_{|k|\to\infty} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

# The scattering transform

The scattering data is now defined by the formula

$$S(k) = \frac{-i}{\pi} \int_{\mathbb{R}^2} e(z, -k)\overline{u}(z) m_1(z, k) d\mu(z).$$

The mapping $u \mapsto S$ will be called the scattering transform.

Notice the familiarity with the Fourier transform; we can think of it as a non-linear Fourier transform. For $u \in \mathcal{S}(\mathbb{R}^2)$ one can show that

- $S \in \mathcal{S}(\mathbb{R}^2)$
- $\|S\|_{L^2(\mathbb{R}^2)} = \|u\|_{L^2(\mathbb{R}^2)}$

Asymptotically

$$\lim_{|k| \to \infty} S(k) = \hat{u}(2k_1, -2k_2).$$

# The $\overline{\partial}$-equation

Applying the operator $\partial_{\overline{k}}$ to the Jost solution gives the following

$$\partial_{\overline{k}}(m_1 \pm m_2) = \pm S(k)e(z,-k)\overline{(m_1 \pm m_2)}.$$

Now this is again a $\overline{\partial}$-equation, now in the $k$ variable. These equations have unique solutions, and hence we can from the scattering data $S(k)$ compute the Jost solution $(m_1, m_2)$. Moreover we retrieve the potential

$$u(z) = \frac{-i}{2\pi}\int_{R^2} m_1(z,k)e(z,-k)\overline{S(k)}d\mu(k).$$

Facilitated by the Jost solution we then have the inverse scattering transform

$$S \mapsto u.$$

# Evolution of scattering transform

When $u(z,t)$ solves DSII, then we can consider the assocated scattering transform $S(k,t)$. From the evolutionary part of the Lax equation it follows that

$$\frac{d}{dt}S(k,t) = (k^2 + \overline{k}^2)S(k,t) \Leftrightarrow S(k,t) = S_0(k)e^{(k^2+\overline{k}^2)t}.$$

Applying the inverse scattering transform yields

$$u(z,t) = \frac{-i}{2\pi}\int_{\mathbb{R}^2} m_1(z,k,t)e(z,-k)\overline{S(k,t)}d\mu(k)$$

with $m_1(z,k,t)$ found from

$$\partial_{\overline{k}}(m_1(z,k,t) \pm m_2(z,k,t)) = \pm S(k,t)e(z,-k)\overline{(m_1(z,k,t) \pm m_2(z,k,t))}.$$

# Picture

The solution strategy for the DSII

$$
\begin{array}{ccc}
u_0(z) & \xrightarrow{\text{scattering transform}} & S_0(k) \\
\text{DSII} \downarrow & & \downarrow \text{Linear evolution} \\
u(z,t) & \xleftarrow[\text{inverse scattering transform}]{} & S(k,t) = S_0(k)e^{(k^2+\bar{k}^2)t}
\end{array}
$$

- The solution strategy opens up for theoretical analysis of DSII
- Could be used as basis for numerical computations

# 4. Numerical solution of the $\overline{\partial}$-equation

# Numerical solution of the $\overline{\partial}$-equation

With J. Mueller and S. Siltanen.

For the scattering / inverse cattering problem solving a $\overline{\partial}$-equation is important. We will now consider the numerical solution of such an equation: We would like to solve the integral equation

$$v(k) = 1 - \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{T(k')}{k - k'} \overline{v(k')} dk_1' dk_2', \qquad k = k_1' + ik_2' \in \mathbb{C}, \qquad (8)$$

or

$$v(k) = 1 - g * (T(k)\overline{v(k)}). \qquad (9)$$

Assume that $T$ is compactly supported in a bounded domain $\Omega$.

Inspired by G. Vainikko:
Let $S = [-s, s]^2$ be a square such that $\overline{\Omega} \subset S$. Choose $m \in \mathbb{Z}_+,\ M = 2^m,\ h = 2s/M$. Define a grid $\mathcal{G}_m \subset S$ by

$$\mathcal{G}_m = \{jh \,|\, j \in \mathbb{Z}_m^2\},$$
$$\mathbb{Z}_m^2 = \{j = (j_1, j_2) \in \mathbb{Z}^2 \mid -2^{m-1} \leq j_l < 2^{m-1}\}.$$

Grid approximation $\phi_h : \mathbb{Z}_m^2 \to \mathbb{C}$ of a function $\phi$ by

$$\phi_h(j) = \phi(jh), \qquad \text{for } j \in \mathbb{Z}_m^2.$$

Grid approximation of Green's function:

$$g_h(j) = \begin{cases} g(jh) & j \in \mathbb{Z}_m^2, j \neq (0,0) \\ 0 & j = (0,0) \end{cases}.$$

# Discrete approximation

The discrete convolution operator $A_h$

$$(A_h\phi_h)(j) = h^2 \sum_{l\in\mathbb{Z}_m^2} g_h(j-l)\phi_h(l), \qquad \text{for } j \in \mathbb{Z}_m^2.$$

Important fact:

$$A_h\phi_h = h^2\, \text{IFFT}\big(\text{FFT}(g_h)\cdot \text{FFT}(\phi_h)\big),$$

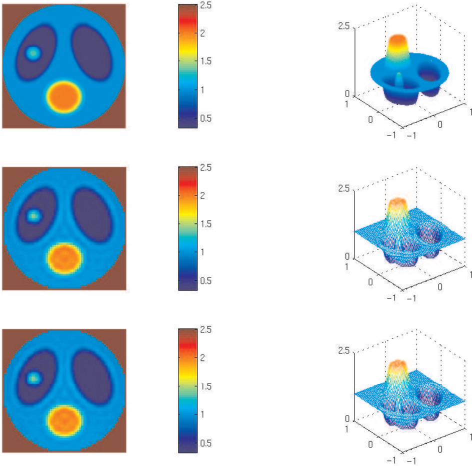
i.e. the implementation is fast.
We approximate the integral equation by the discrete eqation

$$[I + A_h(T_h \cdot^{-})]w_h = 1. \qquad (10)$$

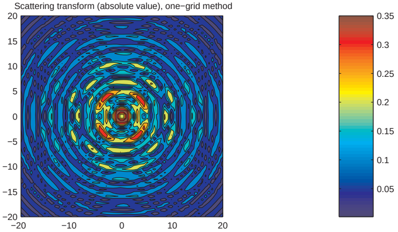
It has a solution for sufficiently large $m$. The equation is rela linear, so by keeping real and imaginary parts seperately we can solve the linear system using GMRES.

- Linear convergence of algorithm
- Complexity of algorithm is $\mathcal{O}(M^2 \log(M))$
- Complexity of scattering / inverse scattering transform is $\mathcal{O}(Mt \log(M))$
- Multigrid extension of algorithm is possible.
- Speed up possible (Huhtanen and Perämäki, 2010)

# Pictures of potential and reconstruction



# Picture of scattering data



# Conclusions

- Scattering /inverse scattering transform is a non-linear, generalized Fourier transform
- Facilitates the solution of DSII
- Several inverse problems can be solved using Scattering /inverse scattering - inverse conductivity problem tomorrow
- Radon transform and inversion formula can be found using similar ideas
- Novikov inversion formula for attenuated Radon transform can also be found using similar ideas