The $\overline{\partial}$-method for non-linear inverse problems

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Lecture 1-2
Summer school on computational solution of inverse problems
University of Helsinki 2010

Content of this minicourse

Monday	Nonlinear wave motion in 1D, the Korteweg - deVries equation, solitons and the inverse scattering method
Wednesday	Complex analysis, Davey-Stewartson II system, $\overline{\partial}$-equation and its solution (theoretical and numerical)
Thursday	The inverse conductivity problem, solution by the $\overline{\partial}$ method in 2D and 3D

Outline

1. Solitary waves and the KdV-equation
2. The Borg-Levinson problem
3. An inverse scattering problem
4. The Gelfand-Levitan inversion
5. Inverse scattering method for the KdV-equation

1. Solitary waves and the KdV-equation

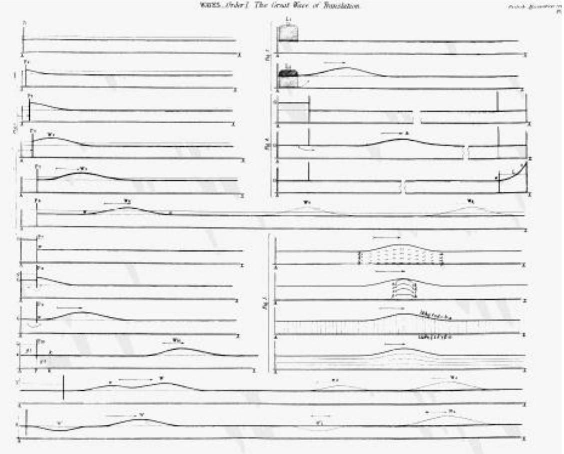
Waves of translation

First observed by **John Scott Russell** on the Union Canal at Hermiston, Scotland, in 1834, when he was conducting experiments for the design of canal boats. Described to the British Association for the Advancement of Science in 1844:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

Russell’s experimets

From Russell’s report



The linear wave equation

The linear wave equation in 1D has the form

$$u_{tt} - c^2 u_{xx} = 0, \qquad x \in \mathbb{R},\ t > 0.$$

Equivalent to

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0.$$

Equiped with asymptotic conditions and initial conditions

$$\lim_{|x|\to\infty} u = 0, \quad u(x,0) = \phi(x),\ u_t(x,0) = \psi(x).$$

The solution has the form of two waves traveling in opposite directions

$$u(x,t) = f(x-ct) + g(x+ct),$$

$$f(x) = \frac{\phi(x) - c^{-1}\psi(x)}{2}, \quad g(x) = \frac{\phi(x) + c^{-1}\psi(x)}{2}.$$

The Korteweg - de Vries equation

Solitary waves were studied further by Joseph Boussinesq (1871) and Lord Rayleigh (1876).

A reasonable mathematical model of waves on shallow water was formulated by Korteweg and de Vries (1895) - the KdV equation:

$$u_t + u_{xxx} + 6uu_x = 0 \qquad x \in \mathbb{R},\ t > 0.$$

- Assumption: wave height is small compared to the depth, which again is small compared to the wave length.
- This is a non-linear, dispersive wave-equation.

The equation is equipped with the conditions

$$\lim_{|x| \to \infty} u = 0, \quad u(x, 0) = \phi(x).$$

Traveling wave solution

Let's look for traveling wave solutions to the KdV equation:

$$u(x,t) = f(x - ct),$$

such that u and its derivatives tend to zero for large x.
Wave speed equals c.

Insert u into

$$u_t + u_{xxx} + 6uu_x = 0 \qquad x \in \mathbb{R},\ t > 0.$$

Gives the ODE

$$-cf' + f''' + 6ff' = 0.$$

Solving the ODE

Integrating the ODE gives

$$\begin{aligned} & -cf' + f''' + 6ff' = 0 \\ \Leftrightarrow \quad & -cf + f'' + 3f^2 = a \\ \Leftrightarrow \quad & -\frac{c}{2}f^2 + \frac{1}{2}(f')^2 + f^3 = af + b \end{aligned}$$

Decay condition implies $a = b = 0$. The solution to the ODE now turns out to be

$$f(y) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(y - y_0)\right), \quad \text{with } y_0 \in \mathbb{R}.$$

Recall

$$\operatorname{sech}(y) = \frac{1}{\cosh(y)} = \frac{2}{e^y + e^{-y}}.$$

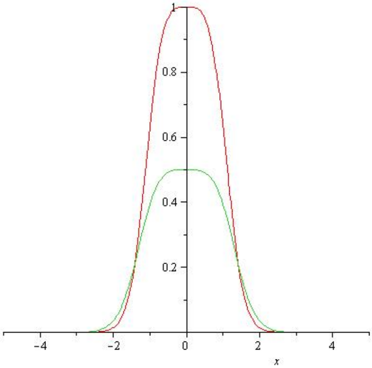
These traveling wave solutions were already found by Joseph Boussinesq and Lord Raleigh.

The wave of translation

Height of f proportional to c; width of f proportional to $\sqrt{c}$.

f can be

- fast, tall and thin Movie fast
- slow, low and fat. Movie slow



Solitons, interaction and stability

Invention of computers enabled a numerical investigation of solutions of the KdV equation by Zabusky and Kruskal in 1965.
Let us look at a numerical solution to the KdV equation with initial condition

$$u_0(x,0) = \frac{1}{2}\,\mathrm{sech}^2\left(\frac{1}{2}x\right) + \mathrm{sech}^2\left(\frac{\sqrt{2}}{2}(x - x_0)\right). \quad \text{Movie}$$

Let us take other initial conditions

$$u(x,0) = \mathrm{sech}^2(x/8). \quad \text{Movie}$$

$$u(x,0) = \frac{2}{1+x^2}. \quad \text{Movie}$$

Realized that the solution to the KdV equation consist of a superposition of solitary waves, *solitons*, moving left and a wave train moving right.

Conclusion from early studies and numerical experiments

- The KdV-equation has traveling wave solutions, solitons
- The solitons are remarkably stable
- The interaction of the solitons is non-linear, but the solitons comes out unchanged
- All initial data gives rise to solitons moving left and a wave train moving right.

2. The Borg-Levinson problem

The Sturm-Liouville problem

Let $0 \leq x \leq 1$ suppose $q \in L^2((0,1))$ is real. Consider the Schrödinger equation

$$Lu = \left(-\frac{d^2}{dx^2} + q\right) u = 0.$$

Eigenvalue problem

$$L\phi = \lambda\phi,$$
$$\phi(0) = 0,\ \phi'(1) = 0.$$

Sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \to \infty$ as $n \to \infty$, and associated eigenfunction ϕ_n.

We normalize ϕ_n such that $\phi_n'(0) = 1$.

Ambartsumyan's result

Example: for the free problem $q = 0$ we have

$$\lambda_n = n^2\pi^2, \phi_n(x) = \frac{\sin(n\pi x)}{n\pi}, \quad n \in \mathbb{N}.$$

Ambartsumyan (1929): The only operator with spectrum $\lambda_n = n^2\pi^2$ is the free operator, i.e. ($q = 0$).

Borg-Levinson

Direct problem: Determine the spectrum $\{\lambda_n\}$ for a potential q.
"Determine the tones of a guiter string from knowledge of the physical properties of the string"

Inverse problem: Determine the potential q from the spectrum $\{\lambda_n\}$.
" Determine the physical properties of a guitar string from its tones"

One sequence of eigenvalues is insufficient (clearly from symmetry)

Borg (1946) - Levinson (1949) theorem:
Knowledge of the spectrum λ_n and normalising constants $c_n = \|\phi_n\|_{L^2((0,1))}$ determines the potential.

Alternatively, another spectrum from different boundary conditions together with $\{\lambda_n\}$ suffices for uniqueness.

3. An inverse scattering problem

Schrödinger operator on the line

Assume q is real and satisfies

$$\int_{\mathbb{R}} (1 + |x|)|q(x)|dx \leq \infty.$$

Consider the Schrödinger operator in $L^2(\mathbb{R})$

$$Lu = \left(-\frac{d^2}{dx^2} + q\right) u.$$

The spectrum is now more complicated:

1. Eigenvalues: $\lambda_n \in \sigma_{\text{disc}} \subset \mathbb{R}$ such that there is an eigenfunction $\phi_n \in L^2(\mathbb{R})$ with

$$L\phi_n = \lambda_n \phi_n.$$

2. Continuous spectrum: $\lambda \in \sigma_{\text{cont}} \subset \mathbb{R}$ such that $(L - \lambda)$ is not bijective, but λ is not an eigenvalue.

Examples

Example 1: For $q = 0$, $\sigma_{\text{cont}} = \emptyset$ and $\sigma_{\text{disc}} = [0, \infty)$.

Example 2: For $q = 1/r$ (the Hydrogen atom) $\sigma_{\text{disc}} = \{-\frac{1}{n^2} : n \in \mathbb{N}\}$ and $\sigma_{\text{cont}} = [0, \infty)$.

In general under our assumption

$$\sigma_{\text{disc}} \subset \mathbb{R}_-, \quad \sigma_{\text{cont}} = \mathbb{R}_+.$$

As before there is more spectral information than just the actual spectrum (e.g. the impact of eigenvalues). Best described by a spectral measure $\rho(\lambda)$.
In **Example 1**

$$\rho(\lambda) = \lambda^{3/2} d\lambda, \; \lambda > 0.$$

Inverse problems: Does ρ determine q?

A scattering problem

The spectrum has a counterpart in scattering theory:

1. For the eigenvalues $\lambda_n = -k_n^2 \in \sigma_{\text{disc}}$, the normalized eigenfunction ψ_n ($\|\phi_n\| = 1$) satisfies

$$\psi_n = c_n e^{-k_n x} \text{ as } x \to \infty.$$

   The c_n's are called normalizing constants.

2. For $\lambda = k^2 \in \sigma_{\text{cont}}$ there is a generalized eigenfunction $\psi(x,k)$ with $L\psi = \lambda\psi$ which satisfies the asymptotic condition

$$\begin{aligned} \psi &\sim e^{-ikx} + R(k)e^{ikx} && \text{as } x \to \infty, \\ \psi &\sim T(k)e^{-ikx} && \text{as } x \to -\infty. \end{aligned}$$

   $R(k)$ and $T(k)$ are called the reflection and transmission coefficients, respectively.
   The spectral measure $\rho(\lambda)$ is equivalent to the scattering data λ_n, c_n, R, T.

Direct/inverse scattering problem

Direct scattering problem:
Given the potential q, find the eigenvalues λ_n and normalizing constants c_n, and the reflection and transmission coefficients $R(k)$, $T(k.)$

Inverse scattering problem:
Given the eigenvalues λ_n and normalizing constants c_n, as well as the reflection and transmission coefficients $R(k)$, $T(k)$, determine and compute the potential q.

Inverse problem considered by Gelfand-Levitan (1951), Marchenko (1952), Kreĭn (1953)... and more recently by B. Simon (1999).

4. The Gelfand-Levitan inversion scheme

The inverse problem: Jost solution

Introduce the Jost solution $\phi_k(x)$

$$\phi_k(x) = e^{ikx} + \int_x^\infty K(x,s)e^{iks}ds.$$

Satisfies

$$\phi_k(x) \to e^{ikx} \text{ as } x \to \infty.$$

The kernel $K(x,y)$ can be computed from q (by solving a so-called Goursat problem).

Gelfand-Levitan integral equation

Remarkable property: K satisfies the Gelfand-Levitan integral equation

$$K(x, y) + B(x + y) + \int_{-\infty}^{\infty} K(x, s)B(s + y)ds = 0, \qquad (1)$$

where B is given by the spectral information

$$B(z) = \sum_{n=1}^{\infty} c_n^2 e^{-k_n z} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k)e^{ikz}dk.$$

The equations (1) is a solvable Fredholm equation of the second kind.

Inversion scheme

1. Given the scattering data $\lambda_n, c_n, R(k)$; compute $B(z)$.
2. Solve the Gelfand-Levitan integral equation for K.
3. Compute $q(x) = -2\frac{d}{dx}K(x,x)$.

Scattering transform:

$$q \mapsto (k_n, c_n, R(k)).$$

Inverse scattering transform

$$(k_n, c_n, R(k)) \mapsto q.$$

5. Inverse scattering method for the KdV-equation

Inverse scattering transform and KdV

Gardner, Greene, Kruskal and Miura, 1967 showed that the solutions of the KdV equation and the solitons can be explained by the inverse scattering transform!!

Denote by $u(x, t)$ the solution of the KdV equation. Consider t as a parameter, and let us regard the function $u(x, t)$ as a potential in the Schrödinger equation:

$$-\frac{d^2}{dx^2}\phi(x) + u(x, t)\phi(x) = \lambda\phi(x).$$

Let us denote the scattering data by $(\lambda_n(t), c_n(t), R(t))$.
The evolution of the scattering data turns out to be linear:

$$\frac{d}{dt}\lambda_n = 0,\ \frac{d}{dt}c_n(t) = 4k_n^3 c_n(t),\ \frac{d}{dt}R(t) = 8ik^3 R(t).$$

Thus

$$\lambda_n(t) = \lambda_n(0),\ c_n(t) = c_n(0)e^{4k_n^3 t},\ R(t) = R(0)e^{8ik^3 t}.$$

The solution $u(x,t)$ to the KdV equation can then be determined by the inverse scattering transform:

$$(\lambda_n, c_n(t), R(k,t)) \mapsto u(x,t).$$

Schematic:

$$\begin{array}{ccc} u_0(x) & \xrightarrow{\text{scattering transform}} & (\lambda_n, c_n(0), R(k,0)) \\ \downarrow & & \downarrow \text{Linear evolution} \\ u(x,t) & \xleftarrow[\text{inverse scattering transform}]{} & (\lambda_n, c_n(t), R(k,t)) \end{array}$$

Impact of the spectrum on the solution:

- Eigenvalues: solitons moving right
- Continuous spectrum: Wave train moving left

Lax pairs

Peter Lax discovered in 1968 a way to explain the inverse scattering method. Introduce two linear operators, L, B, and consider the two equations

$$L\psi = \lambda\psi$$
$$\psi_t = B\psi.$$

In the concrete problem we could take

$$L = -\frac{d^2}{dx^2} + u, \quad B = -4\frac{d^3}{dx^3} + 6u\frac{d}{dx} + 3\frac{\partial u}{\partial x}.$$

The two two linear equations are compatible if and only if u satisfies the KdV equation.
Moreover, one can show that the evolution of the spectrum λ is constant if and only if

$$L_t = [B, L] = BL - LB. \tag{2}$$

The operators L, B is called the Lax pair and the equation (2) is called the Lax equation.

Applications of solitons

Modern applications of solitons include

1. Plasma physics
2. Optical fiber communication

The ideas behind have led to computational methods and enormous insight in the field of Inverse Problems.
More on Wednesday...