$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{-x-4y}$. $D = D_x \times D_y$, missä $D_x = \{1, \frac{11}{10}, \frac{6}{5}\}$ ja $D_y = \{1, \frac{6}{5}, \frac{7}{5}\}$. Siis jakopisteiden x-koordinaatit ovat: $x_0 = 1$, $x_1 = \frac{11}{10}$ ja $x_2 = \frac{6}{5}$, sekä y-koordinaatit ovat: $y_0 = 1$, $y_1 = \frac{6}{5}$ ja $y_2 = \frac{7}{5}$.

Tutkitaan aluksi hieman funktion f käyttäytymistä.

(1) Kun $x' \le x''$, niin $-x' \ge -x''$ ja $-x' - 4y \ge -x'' - 4y$, joten $f(x', y) = e^{-x'-4y} \ge e^{-x''-4y} = f(x'', y)$.

(2) Vastaavasti, kun $y' \le y''$, niin $-4y' \ge -4y''$ ja $-x-4y' \ge -x-4y''$, joten $f(x, y') = e^{-x-4y'} \ge e^{-x-4y''} = f(x, y'')$.

Kohdista (1) ja (2) seuraa (mieti miksi), että funktiolla f on joukossa $T_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ suurin ja pienin arvo: $f(x_{i-1}, y_{j-1})$ ja $f(x_i, y_j)$. Siksi

$$M_{ij} = \sup\{f(x,y) \mid (x,y) \in T_{ij}\} = f(x_{i-1}, y_{j-1}) \text{ ja}$$
$$m_{ij} = \inf\{f(x,y) \mid (x,y) \in T_{ij}\} = f(x_i, y_j).$$

Nyt $x_i - x_{i-1} = \frac{1}{10}$ ja $y_j - y_{j-1} = \frac{1}{5}$ kaikilla $i, j \in \{1, 2\}$ ja yläsummaksi saadaan

$$S_D = \sum_{i=1}^{2}\sum_{j=1}^{2} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = \sum_{i=1}^{2}(M_{i1} + M_{i2}) \cdot \frac{1}{10} \cdot \frac{1}{5} = (M_{11} + M_{12} + M_{21} + M_{22})\frac{1}{50} = \underline{(e^{-5} + e^{-\frac{29}{5}} + e^{-\frac{51}{10}} + e^{-\frac{59}{10}})\frac{1}{50}} \approx 3{,}7 \cdot 10^{-4}.$$

Vastaavasti alasummaksi saadaan

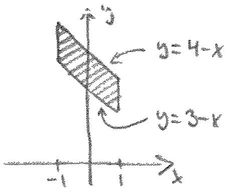
$$s_D = \sum_{i=1}^{2}\sum_{j=1}^{2} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = (m_{11} + m_{12} + m_{21} + m_{22})\frac{1}{50}$$
$$= \underline{(e^{-\frac{59}{10}} + e^{-\frac{67}{10}} + e^{-6} + e^{-\frac{34}{5}})\frac{1}{50}} \approx 1{,}5 \cdot 10^{-4}$$

Integraalin tarkka arvo:

$$\iint_A f(x,y)\,dx\,dy = \int_1^{6/5} dx \int_1^{7/5} dy\, e^{-x-4y} = \int_1^{6/5} dx \Big/_1^{7/5} \frac{e^{-x-4y}}{-4}$$
$$= -\frac{1}{4}\int_1^{6/5} (e^{-x-\frac{28}{5}} - e^{-x-4})\,dx = -\frac{1}{4} \Big/_1^{6/5} (-e^{-x-\frac{28}{5}} + e^{-x-4})$$
$$= -\frac{1}{4}(-e^{-\frac{34}{5}} + e^{-\frac{26}{5}} + e^{-\frac{33}{5}} - e^{-5}) = \underline{\frac{1}{4}(e^{-5} + e^{-\frac{34}{5}} - e^{-\frac{26}{5}} - e^{-\frac{33}{5}})} \approx 2{,}4 \cdot 10^{-4}.$$

(Tietenkin pätee: $s_D \le \iint_A f(x,y)\,dx\,dy \le S_D$.)

(2) $A = \{(x,y) \mid -1 \le x \le 1,\ 3-x \le y \le 4-x\}$. (Joukko A on x-projisoituva, katso monisteen sivu 71.)

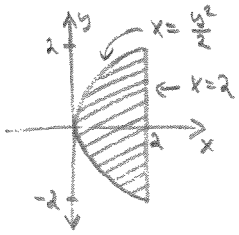


$$\iint_A y^2\,dxdy = \int_{-1}^{1} dx \int_{3-x}^{4-x} dy\, y^2 = \int_{-1}^{1} dx \Big/_{3-x}^{4-x} \frac{y^3}{3} =$$

$$\frac{1}{3}\int_{-1}^{1}\left((4-x)^3-(3-x)^3\right)dx = \frac{1}{3}\Big/_{-1}^{1}\left(-\frac{(4-x)^4}{4}+\frac{(3-x)^4}{4}\right) =$$

$$\frac{1}{12}\left(-3^4+2^4+5^4-4^4\right) = \frac{304}{12} = \frac{76}{3} = \underline{\underline{25\tfrac{1}{3}}}.$$

(3) $A = \{(x,y) \mid -2 \le y \le 2,\ y^2/2 \le x \le 2\}$. (Joukko A on y-projisoituva, katso moniste s. 71.)



$$\underline{a(A)} = \iint_A dxdy = \int_{-2}^{2} dy \int_{y^2/2}^{2} dx = \int_{-2}^{2} dy \Big/_{y^2/2}^{2} x = \int_{-2}^{2}\left(2-\frac{y^2}{2}\right)dy =$$

$$\Big/_{-2}^{2}\left(2y-\frac{y^3}{6}\right) = 4-\frac{8}{6}+4-\frac{8}{6} = \frac{32}{6} = \underline{\frac{16}{3}}.$$

$$\underline{\iint_A x\,dxdy} = \int_{-2}^{2} dy \int_{y^2/2}^{2} dx\, x = \int_{-2}^{2} dy \Big/_{y^2/2}^{2} \frac{x^2}{2} = \int_{-2}^{2}\left(2-\frac{y^4}{8}\right)dy = \Big/_{-2}^{2}\left(2y-\frac{y^5}{40}\right) =$$

$$4-\frac{32}{40}+4-\frac{32}{40} = 8-\frac{16}{10} = \frac{80-16}{10} = \frac{64}{10} = \underline{\frac{32}{5}}.$$

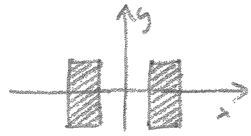
$$\underline{\iint_A y\,dxdy} = \int_{-2}^{2} dy \int_{y^2/2}^{2} dx\, y = \int_{-2}^{2} dy \Big/_{y^2/2}^{2} xy = \int_{-2}^{2}\left(2y-\frac{y^3}{2}\right)dy = \Big/_{-2}^{2}\left(y^2-\frac{y^4}{8}\right) =$$

$$4-2-4+2 = \underline{0}.$$

Siis joukon A keskiö on

$$\frac{1}{a(A)}\left(\iint_A x\,dxdy,\ \iint_A y\,dxdy\right) = \frac{3}{16}\left(\frac{32}{5},0\right) = \underline{\underline{\left(\frac{6}{5},0\right)}}.$$

(4) Esimerkiksi $A = A_1 \cup A_2$, missä $A_1 = [-2,-1]\times[-1,1]$ ja $A_2 = [1,2]\times[-1,1]$.



Koska $A_1 \cap A_2 = \emptyset$, niin (monisteen sivun 68 kohta (iv), huom. $a(\emptyset) = \iint_\emptyset dxdy = 0$)

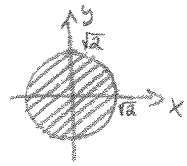
$$\underline{\iint_A x\,dxdy} = \iint_{A_1} x\,dxdy + \iint_{A_2} x\,dxdy = \int_{-1}^{1} dy \int_{-2}^{-1} dx\, x + \int_{-1}^{1} dy \int_{1}^{2} dx\, x =$$

$$\int_{-1}^{1} dy \Big/_{-2}^{-1} \frac{x^2}{2} + \int_{-1}^{1} dy \Big/_{1}^{2} \frac{x^2}{2} = \int_{-1}^{1}\left(\tfrac{1}{2}-2\right)dy + \int_{-1}^{1}\left(2-\tfrac{1}{2}\right)dy = \int_{-1}^{1}\left(\tfrac{1}{2}-2+2-\tfrac{1}{2}\right)dy = \int_{-1}^{1} 0\,dy = \underline{0} \text{ ja}$$

$$\underline{\iint_A y\,dxdy} = \iint_{A_1} y\,dxdy + \iint_{A_2} y\,dxdy = \int_{-2}^{-1} dx \int_{-1}^{1} dy\, y + \int_{1}^{2} dx \int_{-1}^{1} dy\, y =$$

$$\int_{-2}^{-1} dx \Big/_{-1}^{1} \frac{y^2}{2} + \int_{1}^{2} dx \Big/_{-1}^{1} \frac{y^2}{2} = \int_{-2}^{-1} 0\,dx + \int_{1}^{2} 0\,dx = \underline{0}.$$

Nyt nähdään, että joukon A keskiö on $(0,0) \notin A$.

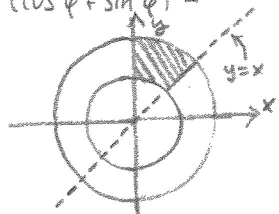


(5) (a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2+y^2$, ja $A = \{(x,y) \mid x^2+y^2 \le 2\}$.

Siirrytään napakoordinaatteihin muunnoksella $g: B \to A$, missä $g(r,\varphi) = (x(r,\varphi), y(r,\varphi)) = (r\cos\varphi, r\sin\varphi)$ ja $B = \{(r,\varphi) \mid 0 \le r \le \sqrt{2},\ 0 \le \varphi \le 2\pi\}$. Kuvaus g on kyllin siisti (jatkuvasti derivoituva melkein bijektio) ja sen Jacobin determinantin itseisarvo $|J_g(r,\varphi)|$ on r (Katso monisteen sivu 78). Niinpä

$$\iint_A (x^2+y^2)\,dx\,dy = \iint_B ((r\cos\varphi)^2 + (r\sin\varphi)^2)|J_g(r,\varphi)|\,dr\,d\varphi = \int_0^{\sqrt{2}} dr \int_0^{2\pi} d\varphi\, r^3 \overbrace{(\cos^2\varphi + \sin^2\varphi)}^{=1} =$$

$$\int_0^{\sqrt{2}} dr \Big/_0^{2\pi} r^3\varphi = 2\pi \int_0^{\sqrt{2}} r^3\,dr = 2\pi \Big/_0^{\sqrt{2}} \frac{r^4}{4} = \underline{\underline{2\pi.}}$$



(b) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = y$, ja $A = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ 0 \le x \le y\}$.

Kuten (a)-kohdassa siirrytään napakoordinaatteihin. Nyt $B = \{(r,\varphi) \mid 1 \le r \le 2,\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{2}\}$.

$$\iint_A y\,dx\,dy = \iint_B r\sin(\varphi)|J_g(r,\varphi)|\,dr\,d\varphi = \int_{\pi/4}^{\pi/2} d\varphi \int_1^2 dr\, r^2\sin\varphi = \int_{\pi/4}^{\pi/2} d\varphi \Big/_1^2 \frac{r^3}{3}\sin\varphi =$$

$$\int_{\pi/4}^{\pi/2} \frac{7}{3}\sin\varphi\,d\varphi = \Big/_{\pi/4}^{\pi/2} -\frac{7}{3}\cos\varphi = 0 + \frac{7}{3}\cdot\frac{\sqrt{2}}{2} = \underline{\underline{\frac{7\sqrt{2}}{6}.}}$$

(6) Huomaa, että kun $x = ar\cos t$ ja $y = br\sin t$, niin

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \iff \frac{a^2r^2\cos^2 t}{a^2} + \frac{b^2r^2\sin^2 t}{b^2} \le 1 \iff r^2\overbrace{(\cos^2 t + \sin^2 t)}^{=1} \le 1 \iff r^2 \le 1.$$

Tehdään viljeen muunnos: $g: B \to A$, missä $g(r,t) = (x(r,t), y(r,t)) = (ar\cos t, br\sin t)$, $B = \{(r,t) \mid 0 \le r \le 1,\ 0 \le t \le 2\pi\}$ ja $A = \{(x,y) \mid x^2/a^2 + y^2/b^2 \le 1\}$. Tämän muunnoksen Jacobin determinatti on

$$J_g(r,t) = \begin{vmatrix} D_r x(r,t) & D_t x(r,t) \\ D_r y(r,t) & D_t y(r,t) \end{vmatrix} = \begin{vmatrix} a\cos t & -ar\sin t \\ b\sin t & br\cos t \end{vmatrix} = abr\cos^2 t + abr\sin^2 t$$

$$= abr(\cos^2 t + \sin^2 t) = abr.$$

Koska muunnos $g: B \to A$ on kyllin siisti (jatkuvasti derivoituva melkein bijektio), niin

$$\iint_A dx\,dy = \iint_B |J_g(r,t)|\,dr\,dt = \int_0^1 dr \int_0^{2\pi} dt\, abr = \int_0^1 dr \Big/_0^{2\pi} abrt$$

$$= \int_0^1 ab2\pi r\,dr = \Big/_0^1 ab\pi r^2 = \underline{\underline{ab\pi.}}$$