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Exercises 5
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1. Consider a reflecting random walk on the interval $\{0, 1, \ldots, d\}$. In other words, suppose $p + q = 1$ and the transition probabilities satisfy

$$p_{ij} = \begin{cases} p & \text{when } j = i+1 \text{ and } 0 < i < d \\ q & \text{when } j = i-1 \text{ ja } 0 < i < d \\ 1 & \text{when } i = 0, j = 1 \text{ tai } i = d, j = d-1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Is the chain reversible ?

b) Determine the stationary distribution.

2. Suppose $T \sim \text{Exp}(\lambda)$ and $S \sim \text{Exp}(\mu)$ are independent. Show by calculating that

$$\mathbf{P}\,(\,T + S \le t + s \,|\, T + S > t, T < t\,) = \mathbf{P}\,(\,S \le s\,)\,.$$

3. Let $N(t)$ be Poisson process. Suppose $0 < t < u < v < s$ are given. What is the distribution of the random variable $N(v) - N(u)$ given we know that $N(s) - N(t) = n$?

4. The uniqueness of the stationary distribution. Suppose that (X_n) is finite and irreducible MC and $\alpha \in S$. Suppose $\widetilde{\pi}$ is some stationary distribution of (X_n). Denote

$$A_j^{(n)}(k) := \mathbf{P}_j\,(\,X_m \ne \alpha \text{ for } 1 \le m < n \text{ and } X_n = k\,)\,.$$

a) Represent the probability $A_j^{(n)}(k)$ with the help of hitting time τ_α and random variable X_n and deduce the estimate

$$A_j^{(n)}(k) \le \mathbf{P}_j\,(\,\tau_\alpha > n\,)$$

for $k \ne \alpha$.

$b)$ Show that

$$\sum_{k \neq \alpha} A_j^{(n)}(k) p_{ki} = A_j^{(n+1)}(i)$$

when $i \neq \alpha$ and $n \geq 1$.

$c)$ Show that

$$\widetilde{\pi}_j = \widetilde{\pi}_\alpha \sum_{m=1}^{n} A_\alpha^{(m)}(j) + \sum_{l \neq \alpha} \widetilde{\pi}_l A_l^{(n)}(j),$$

when $j \neq \alpha$ and $n \geq 1$ [Hint: induction, definition of the stationary distribution and part $b)$]

$d)$ Suppose $p_{\alpha j}^{(m)} > 0$ and suppose m is the smallest number with this property. Show that

$$\mathbf{P}_j \left(\tau_\alpha > n \right) \leq \frac{\mathbf{P}_\alpha \left(\tau_\alpha > n + m \right)}{p_{\alpha j}^{(m)}} \to 0,$$

when $n \to \infty$.