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Exercises 3
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1. Let (X_n) be a MC (Markov Chain) with state space $S \subseteq \mathbb{N}$ and let $n, m \geq 1$. Let $M \subset S^n$ and $T \subset S^m$. Let $A := \{(X_0, \ldots, X_{n-1}) \in M\}$, $B := \{(X_{n+1}, \ldots, X_{n+m}) \in T\}$ and $C := \{(X_1, \ldots, X_m) \in T\}$ be events. Show that

$$\mathbf{P}\,(\,B \,|\, X_n = i \text{ and } A\,) = \mathbf{P}\,(\,C \,|\, X_0 = i\,)\,.$$

2. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ & & & & 1 & \\ & & 1 & & & \\ \frac{1}{4} & & & & & \frac{3}{4} \\ & \frac{2}{3} & & & \frac{1}{3} & \\ & & \frac{1}{2} & \frac{1}{2} & & \end{pmatrix}$$

a) Find non-trivial absorption sets.

b) Denote $i \leftrightarrow j$ if $i = j$ or there are paths from state i to state j and from state j to state i. Show that $\leftrightarrow$ is an equivalence relation on the state space S and determine its equivalence classes.

3. Continuation to the Exercise 5 in the previous Exercises sheet. Consider a symmetric random walk on $\mathbb{N}$ (that is $p_{00} = 1$, $p_{i,i+1} = p_{i,i-1} = \frac{1}{2}$, when $i \geq 1$ and $p_{ij} = 0$ otherwise).

a) Determine the generating function $G_i(t)$ of the absorption time T_0 from every starting state i.

b) Solve the distribution of T_0 when $X_0 = 1$.

4. Consider such a branching process model in which we assume the probability for the event "an individual has no children" is p and the probability that an individual has exactly three children is $1-p$. Determine the extinction probability (with the assumption that $X_0 = 1$).

5. Let $S = \{0, 1, \dots, d\}$ ja $P = (p_{ij})_{i,j \in S}$ be a matrix with row sums all equal to 1. Suppose $(Y(i,n)\,;\, i = 0, 1, \dots, d, n \in \mathbb{N})$ are independent random variables and assume that

$$\mathbf{P}\,(\,Y(i,n) = j\,) = p_{ij}.$$

Let $X_0 \in S$ be a random variable that is independent of all r.v's $(Y(i,n))$. Define

$$X_{n+1} := Y(X_n, n), \quad n \geq 0.$$

a) Show that the events

$$\begin{aligned} A &:= \{X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} \quad \text{and} \\ B &:= \{X_0 = i_0, Y(i_0, 0) = i_1, \dots, Y(i_{n-1}, n-1) = i_n\} \end{aligned}$$

are same.

b) Show that the chain (X_n) is a Markov chain [Hint. Apply *a*)].