Department of Mathematics and Statistics
Stokastiset prosessit
Exercise 1
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Determine in the following two exercises the transition probabilities p_{ij} of the described Markov chains.

1. There is a total of N balls in two urns. At the time n there are X_n balls in the first one and $N - X_n$ in the second. At each step we select randomly one of the N balls and move it to the other urn.

2. There are N black and N white balls that are placed randomly in two urns so that there are N balls in each urn. At each step, we select randomly one ball from each urn and then we swap the balls. The chain describes the number of white balls in the first urn.

3. Let $Y_0, Y_1, \ldots$ be independent and identically distributed (i.i.d.) random variables such that $Y_0 \in \{H, T\}$ with $\mathbf{P}(Y_0 = H) = \frac{1}{2}$. Show that $X_n := (Y_n, Y_{n+1})$ is a Markov chain and compute its transition probabilities p_{ij}.

4. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables such that the state space is $S = \{1, 2, \ldots, N\}$ and $\mathbf{P}(Y_1 = i) = 1/N$ for every $i \in S$. Show that $X_n := \#\{Y_1, \ldots, Y_n\}$ is a Markov chain and compute its transition probabilities p_{ij}. [Note. In this exercise (and also later on) we denote by $\#A$ the number of elements in a set A.]

5. Suppose that the weather of day $n+1$ depends only on the weather of day n. Let us denote

$$p = \mathbf{P}(\text{tomorrow is dry} \mid \text{today is dry})$$
$$q = \mathbf{P}(\text{it rains tomorrow} \mid \text{it rains today})$$

Determine by using a Markov chain model the probability

$$\mathbf{P}(\text{it rains tomorrow and day after tomorrow} \mid \text{it rains today})$$

and determine also the distribution of the rainy period and its expectation.