Exercises 3

1. Show that
$$f(y;\mu,\phi) = \frac{\phi}{\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{\phi^2}\right) \tag{1}$$
does NOT belong to the exponential family.

2. Let $y_1, \dots y_n$ be a random sample from $Poisson(\mu)$ where mean value $\mu > 0$ is unknown. Show that $\hat{\mu} = \bar{y}$ maximizes the likelihood function
$$L(\mu) = \prod_{i=1}^{n} \left[\frac{\mu^{y_i}}{y_i!} e^{-\mu}\right]$$

3. Let $y_1, \dots y_n$ be a random sample from $Gamma(\beta, \alpha)$ distribution with density function
$$f(y;\beta,\alpha) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} y^{\alpha-1} e^{-\beta y}.$$
Derive the formulas for score function, expected information matrix and observed information matrix.

4. Suppose that $Y_1, \dots, Y_n$ are independent and satisfy the linear model
$$\mu_i = \mathrm{E}(Y_i) = \sum_{j=1}^{p} x_{ij}\beta_j$$
for given covariates x_{ij} and unknown parameters β.

   a) Show that if Y_i has the Laplace distribution
   $$f_{Y_i}(y_i;\mu_i,\sigma) = \frac{1}{2\sigma} \exp\left(-|y_i - \mu_i|/\sigma\right)$$
   then the maximum likelihood estimate of β is obtained by minimizing the L_1
   $$S_1(y,\hat{y}) = \sum |y_i - \hat{y}_i|$$
   over values of $\hat{y}$ satisfying the linear model.

b) Show that if Y_i is uniformly distributed over the range $(\mu_i - \sigma, \mu_i + \sigma)$, maximum-likelihood estimates are obtained by minimizing the L_∞-norm

$$S_\infty(y, \hat{y}) = \max_i |y_i - \hat{y}_i|.$$

5. Show that when the canonical link is used in a generalized linear model, the expected information and observed information are equal.