CHAPTER 4: PRINCIPAL BUNDLES

4.1 Lie groups

A *Lie group* is a group G which is also a smooth manifold such that the multiplication map $G \times G \to G$, $(a,b) \mapsto ab$, and the inverse $G \to G, a \mapsto a^{-1}$, are smooth.

Actually, one can prove (but this is not easy) that it is sufficient to assume continuety, smoothness comes free. (This was one of the famous problems listed by David Hilbert in his address to the international congress of mathematicians in 1900. The result was proven by D. Montgomery and L. Zippin in 1952.)

Examples The vector space $\mathbb{R}^n$ is a Lie group. The group multiplication is just the addition of vectors. The set $GL(n,\mathbb{R})$ of invertible real $n \times n$ matrices is a Lie group with respect to the usual matrix multiplication.

Theorem. *Any closed subgroup of a Lie group is a Lie group.*

The proof is complicated. See for example S. Helgason: Differential Geometry, Lie Groups and Symmetric Spaces, section II.2.

The theorem gives an additional set of examples of Lie groups: The group of real orthogonal matrices, the group of complex unitary matrices, the group of invertible upper triangular matrices ....

For a fixed $a \in G$ in a Lie group we can define a pair of smooth maps, the *left translation* $l_a : G \to G, l_a(g) = ag$, and the right translation $r_a : G \to G, r_a(g) = ga$. We say that a vector field $X \in D^1(G)$ is *left (resp. right) invariant* if $(l_a)_* X = X$ (resp. $(r_a)_* X = X$) for all $a \in G$.

Since the left (right) translation is bijective, a left (right) invariant vector field is uniquely determined by giving its value at a single point, at the identity, say. Thus as a vector space, the space of left invariant vector fields can be identified as the tangent space $T_e G$ at the neutral element $e \in G$.

Theorem. *Let X, Y be a pair of left (right) invariant vector fields. Then $[X,Y]$ is left (right) invariant.*

Proof. Denote $f = l_a : G \to G$. Recall that

$$X'_i = (f_* X)_i = \frac{\partial y_i}{\partial x_j} X_j,$$

where we have written the map f in terms of local coordinates as $y = y(x)$. Then

$$\begin{aligned}
[X', Y']_i &= X'_j \partial'^j Y'_i - Y'_j \partial'^j X'_i = X_j \partial^j Y'_i - Y_j \partial^j X'_i \\
&= X_j \partial^j (\frac{\partial y_i}{\partial x_k} Y_k) - Y_j \partial^j (\frac{\partial y_i}{\partial x_k} X_k) \\
&= \frac{\partial y_i}{\partial x_k} (X_j \partial^j Y_k - Y_j \partial^j X_k) + \frac{\partial^2 y_i}{\partial x_j \partial x_k} (X_j Y_k - Y_j X_k).
\end{aligned}$$

The second term on the right vanishes since the second derivative is symmetric. Thus we have $[X', Y'] = [X, Y]'$, i.e., $[(l_a)_* X, (l_a)_* Y] = (l_a)_* [X, Y]$. Thus the commutator is left invariant.

It follows that the left invariant vector field form a *Lie algebra.* This Lie algebra is denoted by $Lie(G)$ and it is called the Lie algebra of the Lie group G. Observe that $\dim Lie(G) = \dim T_g G = \dim G$.

Example 1 Let $G = \mathbb{R}^n$. The property that a vector field $X = X_i \partial^i$ is left (right) invariant means simply that the coefficient functions $X_i(x)$ are constants. Thus left invariant vector fields can be immediately identified as vectors $(X_1, \ldots, X_n)$ in $\mathbb{R}^n$. Constant vector fields commute, thus $Lie(\mathbb{R}^n)$ is a commutative Lie algebra.

Example 2 Let $G = GL(n, \mathbb{R})$. Let X be a left invariant vector field and $z = X(1) = \frac{d}{dt} e^{tz}|_{t=0}$. Then

$$X(g) = \frac{d}{dt} g e^{tz}|_{t=0} = gz.$$

This implies that

$$(X \cdot f)(g) = \frac{d}{dt} f(g e^{tz})|_{t=0} = (\frac{d}{dt} g e^{tz})_{ij}|_{t=0} \frac{\partial}{\partial x_{ij}} f(x)|_{x=g} = g_{ik} z_{kj} \frac{\partial}{\partial g_{ij}} f.$$

When Y is another left invariant vector field with $w = Y(1)$, then

$$\begin{aligned}
[X, Y] &= [g_{ik} z_{kj} \frac{\partial}{\partial g_{ij}}, g_{lm} w_{mp} \frac{\partial}{\partial g_{lp}}] \\
&= g_{ik} z_{kj} w_{jp} \frac{\partial}{\partial g_{ip}} - g_{lm} w_{mp} z_{pj} \frac{\partial}{\partial g_{lp}} \\
&= g_{ik} [z, w]_{kp} \frac{\partial}{\partial g_{ip}}.
\end{aligned}$$

That is, the commutator of the vector fields X, Y is simply given by the commutator $[z, w]$ of the parameter matrices.

Example 3 The group $SO(n) \subset GL(n, \mathbb{R})$ of rotations in $\mathbb{R}^n$. Each antisymmetric matrix L defines a 1-parameter group of rotations by $R(t) = e^{tL}$. The tangent vector of this curve at $t = 0$ is L. We can define a left invariant vector field as above as $X(g) = gL$. The commutator of a pair of antisymmetric matrices is again antisymmetric. The condition that L is antisymmetric is necessary in order that it is tangential to the orthogonal group at the identity: Take a derivative of $R(t)^t R(t) = 1$ at $t = 0$! Thus the Lie algebra of $SO(n)$ consists precisely of all antisymmetric real $n \times n$ matrices. When $n = 2$ we recover the 1-dimensional group of rotations in the plane (the Lie algebra is commutative) and when $n = 3$ we get the 3-dimensional group of rotations in $\mathbb{R}^3$ and its Lie algebra is the angular momentum algebra.

The complex unitary group $U(n)$ has as its Lie algebra the algebra of antihermitean matrices. This is shown by differentiating $R(t)^* R(t) = 1$ at $t = 0$ for $R(t) = e^{tL}$. The Lie algebra of $SU(n)$ is given by antihermitean traceless matrices. Here $SU(n) \subset U(n)$ is the subgroup consisting of matrices of unit determinant.

In the case of a matrix Lie group we have an *exponential mapping* $\exp : Lie(G) \to G$ from the Lie algebra to the corresponding Lie group, which is given through the usual power series expansion $e^X = 1 + X + \frac{1}{2!}X^2 \ldots$. This is because the left invariant vector fields are parametrized by the value of the tangent vector at the identity which is equal to the derivative of a 1-parameter group of matrices at the identity. The exponential mapping, which has a central role in Lie group theory, can be generalized to arbitrary Lie groups. If X is a left invariant vector field on the group then, at least locally, there is a unique curve $g(t)$ with $g(0) = 1$ and $g'(t) = X(g(t))$ by the theory of first order differential equations. In fact, it is easy to see that this solution is actually globally defined by a use of group multiplication. Since X is left invariant we have $g'(t) = \ell_{g(t)} \cdot X(1)$, that is, $X(1) = \ell_{g(t)}^{-1} \cdot g'(t)$. The exponential mapping is then defined as

$$e^X = g(1).$$

See S. Helgason, Chapter II, for more details.

Exercise Prove with the help of the chain rule of differentiation that $e^{tX}e^{sX} = e^{(t+s)X}$ in every Lie group, for real t, s and for any left invariant vector field X.

Let $X_1, \ldots, X_n$ be a basis of $Lie(G)$. Then

$$[X_i, X_j] = c_{ij}^k X_k$$

for some numerical constants c_{ij}^k, the so-called *structure constants.* Since the Lie bracket is antisymmetric we have $c_{ij}^k = -c_{ji}^k$ and by the Jacobi identity we have

$$c_{ij}^k c_{kl}^m + c_{li}^k c_{kj}^m + c_{jl}^k c_{ki}^m = 0$$

for all i, j, l, m. In terms of the left invariant vector fields X_i, any tangent vector v at $g \in G$ can be written as $v = v^i X_i(g)$. Let us define $\theta^i \in \Omega^1(G)$ as $\theta^i(g)v = v^i$. We compute the exterior derivative $d\theta^i$:

$$\begin{aligned} d\theta^i(g)(X_j, X_k) &= X_j\theta^i(X_k) - X_k\theta^i(X_j) - \theta^i([X_j, X_k]) \\ &= X_j\delta_{ik} - X_k\delta_{ij} - \theta^i(c_{jk}^l X_l) = -c_{jk}^i. \end{aligned}$$

On the other hand,

$$(\theta^i \wedge \theta^j)(X_k, X_l) = \theta^i(X_k)\theta^j(X_l) - \theta^i(X_l)\theta^j(X_k) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Thus we obtain *Cartan's structural equations,*

$$d\theta^i = -\frac{1}{2}c_{kl}^i \theta^k \wedge \theta^l.$$

Denote $X_i\theta^i = g^{-1}dg$. This is a $Lie(G)$-valued 1-form on G. It is tautological at the identity: $(g^{-1}dg)(v) = v$ for $v \in T_1G$. For $\theta = g^{-1}dg$ the structural equations can be written as

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0,$$

where $[\theta \wedge \theta] = [X_i, X_j]\theta^i \wedge \theta^j$.

A *left action* of a Lie group G on a manifold M is a smooth map $G \times M \to M$, $(g, x) \mapsto gx$, such that $g_1(g_2)x = (g_1g_2)x$ for all g_i, x and $1 \cdot x = x$ when 1 is the neutral element. Similarly, one defines the *right action* as a map $M \times G \to M$, $(x, g) \mapsto xg$, such that $x(g_1g_2) = (xg_1)g_2$ and $x \cdot 1 = x$.

The (left) action is *transitive* if for any $x, y \in M$ there is an element $g \in G$ such that $y = gx$ and the action is *free* if for any x, $gx = x$ only when $g = 1$. The action is *faithful* if $gx = x$ for all $x \in M$ only when $g = 1$. The *isotropy group* at $x \in M$ is the group $G_x \subset G$ of elements g such that $gx = x$.

Example 1 Let $H \subset G$ be a closed subgroup of a Lie group G. Then the left (right) multiplication on G defines a left (right) action of H on G.

Example 2 Let $H \subset G$ be a closed subgroup in a Lie group. Then the space $M = G/H$ of left cosets gH is a smooth manifold, see S. Helgason, section II.4. There is a natural left action given by $g' \cdot (gH) = g'gH$. In general, when G acts transitivly (from the left) on a manifold M, we can write $M = G/H$ with $H = G_x$ for any fixed element $x \in M$. The bijection $\phi : G/H \to M$ is given by $\phi(gH) = gx$.

For example, when $G = SO(3)$ and $H = SO(2)$ the quotient $M = SO(3)/SO(2)$ can be identified as the unit sphere S^2. Similarly, $SU(3)/SU(2)$ can be identified as the sphere S^5. The sphere S^5 is equal to the set of points $(z_1, z_2, z_3) \in \mathbb{C}^3$ such that $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. The point $(1, 0, 0)$ is left invariant exactly by the elements in the subgroup $SU(2) \subset SU(3)$ operating in the z_2z_3-plane. On the other hand, $SU(3)$ acts transitivly on S^5 and so $S^5 = SU(3)/SU(2)$.

Let a left action of a Lie group G on a manifold M be given. Then for each $X \in Lie(G)$ there is a canonical vector field $\hat{X}$ on M defined by $\hat{X}(x) = \frac{d}{dt} e^{tX} \cdot x|_{t=0}$. Similarly, a right action gives a canonical vector field by differentiating $x \cdot e^{tX}$. In the case when $M = G$ and the left action is given by the left multiplication on the group, we have simply $\hat{X} = X$.

A Lie group G acts on itself also through the formula $g \mapsto g_0 g g_0^{-1}$ for $g_0 \in G$. This is called the adjoint action and is denoted by $Ad_{g_0}(g) = g_0 g g_0^{-1}$. Note that the adjoint action is a left action. Because of $Ad_{g_0}(1) = 1$, the derivative of the adjoint action at $g = 1$ gives a linear map, denoted by ad_{g_0}, from T_1G to T_1G, that is, we may view ad_g as a linear map

$$ad_g : Lie(G) \to Lie(G).$$

In the case of a matrix Lie group we have simply $ad_g(X) = gXg^{-1}$, matrix multiplication. Thus we have also

$$ad_g([X, Y]) = [ad_g(X), ad_g(Y)]$$

for all $X, Y \in Lie(G)$. This holds also in the case of an arbitrary Lie group. This means that ad_g is *an automorphism* of the Lie algebra of G.

Exercise Prove the above statement for an arbitrary Lie group.

We also observe, by the the chain rule for differentiation, that $ad_{gg'} = ad_g \circ ad_{g'}$ for all $g, g \in G$. This means that the map $g \mapsto ad_g$ is a representation of the Lie group in the vector space $Lie(G)$.

4.2. Definition of a principal bundle and examples

Let G be a Lie group and M a smooth manifold. *A principal G bundle over M* is a manifold which locally looks like $M \times G$.

Definition 4.2.1. *A smooth manifold P is a principal G bundle over the manifold M, if a smooth right action of G on P is given, i. e., a map $P \times G \to P$, $(p, g) \mapsto pg$, such that $p(gg') = (pg)g' \forall p \in P$ and g, g' in G, and if there is given a smooth map $\pi : P \to M$ such that*

(1) *$\pi(pg) = \pi(p)$ for all g in G.*
(2) *$\forall x \in M$ there exists an open neighborhood U of x and a diffeomorphism (local trivialization) $f : \pi^{-1}(U) \to U \times G$ of the form $f(p) = (\pi(p), \phi(p))$ such that $\phi(pg) = \phi(p)g \, \forall p \in \pi^{-1}(U), g \in G$.*

The manifold P is the *total space* of the bundle, M is the *the base space,* and π is the bundle projection. The *trivial bundle* $P = M \times G$ is defined by the projection $\pi(x, g) = x$ and by the natural right action of G on itself.

Consider two bundles $P_i = (P_i, \pi_i, M_i; G)$ with the same structure group G. A smooth map $\phi : P_1 \to P_2$ is a *G bundle map* , if $\phi(pg) = \phi(p)g$ for all p and g. Two bundles P_1 and P_2 are *isomorphic* if there is a bijective bundle map $P_1 \to P_2$. An isomorphism of a bundle onto itself is an *automorphism* .

If $H \subset G$ is a closed subgroup then G is a principal H bundle over the homogeneous space G/H. The right action of H on G is just the right multiplication in G and the projection is the canonical projection on the quotient.

Example 4.2.2. Take $G = SU(2)$ and $H = U(1)$

$$H : \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \varphi \in \mathbb{R}.$$

A general element g of G is

$$g = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix},$$

with $|z_1|^2 + |z_2|^2 = 1$. Writing z_1 and z_2 in terms of their real and imaginary parts we see that the group G can be identified with the unit sphere S^3 in $\mathbb{R}^4$. We can define a map $\pi : G \to S^2$ by $\pi(g) = g\sigma_3 g^{-1}$, where σ_3 is the matrix $diag(1, -1)$; elements of $\mathbb{R}^3$ are represented by Hermitian traceless 2×2 matrices. The Euclidean metric is given by $\|x\|^2 = -\det x$. The kernel of the map π is precisely $U(1)$; thus we have a $U(1)$ fibration over $S^2 = SU(2)/U(1)$ in S^3.

Exercise 4.2.3. Let $S_+ = \{x \in S^2 | x_3 \neq -1\}$ and $S_- = \{x \in S^2 | x_3 \neq +1\}$. Construct local trivializations $f_\pm : \pi^{-1}(S_\pm) \to S_\pm \times U(1)$.

The bundle $S^3 \to S^2$ is nontrivial; it is not isomorphic to $S^2 \times S^1$ for topological reasons. Namely, S^3 is a simply connected manifold whereas the fundamental group of $S^2 \times S^1$ is equal to $\pi_1(S^1) = \mathbb{Z}$ [M. Greenberg: Lectures on Algebraic Topology].

Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of the base space M of a principal bundle P and let $p \mapsto (\pi(p), \phi_\alpha(p)) \in U_\alpha \times G$ be a set of local trivializations. If $p \in \pi^{-1}(U_\alpha \cap U_\beta)$, we can write

$$\phi_\alpha(p) = \xi_{\alpha\beta}(p)\phi_\beta(p),$$

where $\xi_{\alpha\beta}(p) \in G$. Now $\phi_\alpha(pg) = \phi_\alpha(p)g$ and $\phi_\beta(pg) = \phi_\beta(p)g$ from which follows that $\xi_{\alpha\beta}(pg) = \xi_{\alpha\beta}(p)$ and thus $\xi_{\alpha\beta}$ can be thought of as a function on the base space $U_\alpha \cap U_\beta$. If $p \in \pi^{-1}(U_\alpha \cap U_\beta \cap U_\gamma)$ and $x = \pi(p)$, then $\phi_\alpha(p) = \xi_{\alpha\beta}(x)\phi_\beta(p) = \xi_{\alpha\beta}(x)\xi_{\beta\gamma}(x)\phi_\gamma(p)$ so that

$$\xi_{\alpha\beta}(x)\xi_{\beta\gamma}(x) = \xi_{\alpha\gamma}(x).$$

In general, a collection of G-valued functions $\{\xi_{\alpha\beta}\}$ for the covering $\{U_\alpha\}$ is a *one-cocycle (with values in G)* if the above equation holds for all x in $U_\alpha \cap U_\beta \cap U_\gamma$ and for all triples of indices.

If we make the transformations $\phi'_\alpha = \eta_\alpha \phi_\alpha$ for some functions $\eta_\alpha : U_\alpha \to G$, then

$$\xi_{\alpha\beta} \mapsto \xi'_{\alpha\beta} = \eta_\alpha^{-1}\xi_{\alpha\beta}\eta_\beta.$$

If we can find the maps η_α such that $\xi'_{\alpha\beta} = 1 \forall \alpha, \beta$, then $\xi_{\alpha\beta} = \eta_\alpha \eta_\beta^{-1}$ and we say that the one-cocycle ξ is a *coboundary*.

Let $(P, \pi, M), (P', \pi', M')$ be a pair of principal G bundles and let $f : P \to P'$ be a bundle map. We define the induced map $\hat{f} : M \to M'$ by $\hat{f}(x) = \pi'(f(p))$, where p is an arbitrary element in the fiber $\pi^{-1}(x)$.

Theorem 4.2.4. *Let P and P' be a pair of principal G bundles over M. Let $\{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$ (respectively, $\{U_\alpha, \phi'_\alpha\}_{\alpha\in\Lambda}$) be a system of local trivializations for P (respectively, for P'). Let $\xi_{\alpha\beta}$ and $\xi'_{\alpha\beta}$ be the corresponding transition functions. Then there exists an isomorphism $f : P \to P'$ such that $\hat{f} = id_M$ if and only if the transition functions differ by a coboundary, that is, $\xi'_{\alpha\beta}(x) = \eta_\alpha(x)^{-1}\xi_{\alpha\beta}(x)\eta_\beta(x)$ in $U_\alpha \cap U_\beta$ for some functions $\eta_\alpha : U_\alpha \to G$.*

Proof. 1) Suppose first that $\xi'_{\alpha\beta} = \eta_\alpha^{-1}\xi_{\alpha\beta}\eta_\beta$ for all $\alpha, \beta \in \Lambda$. Define $f : P \to P'$ as follows. Let $p \in P$ and $x = \pi(p)$. Choose $\alpha \in \Lambda$ such that $x \in U_\alpha$. Using a local trivialization (U_α, ϕ'_α) at x we set $f(p) = (x, f_\alpha(p))$, where $f_\alpha(p) = \eta_\alpha(x)^{-1}\phi_\alpha(p)$. We have to show that the map is well-defined: If $x \in U_\alpha \cap U_\beta$ then $\phi_\beta(p) = \xi_{\beta\alpha}(x)\phi_\alpha(p)$ and thus

$$\begin{aligned} f_\beta(p) &= \eta_\beta(x)^{-1}\phi_\beta(p) = \eta_\beta(x)^{-1}\xi_{\beta\alpha}(x)\phi_\alpha(p) \\ &= \xi'_{\beta\alpha}(x)[\eta_\alpha(x)^{-1}\phi_\alpha(p)] = \xi'_{\beta\alpha}(x)f_\alpha(p). \end{aligned}$$

We conclude that $(x, f_\alpha(p))$ and $(x, f_\beta(p))$ represent the same element in P'. The equation $f(pg) = f(p)g$ follows from $\phi_\alpha(pg) = \phi_\alpha(p)g$.

2) Let $f : P \to P'$ be an isomorphism. We can define

$$\eta_\alpha(x) = \phi_\alpha(p)\phi'_\alpha(f(p))^{-1},$$

where $p \in \pi^{-1}(x)$ is arbitrary. It follows at once from the definition of the transition functions that the collection $\{\eta_\alpha\}_{\alpha\in\Lambda}$ satisfies the requirements.

Let $\{\xi_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ be a one-cocycle with values in G, subordinate to an open cover $\{U_\alpha\}$ on a manifold M. We can construct a principal G bundle P from this data. Let $C = \amalg(\alpha, U_\alpha \times G)$ be the disjoint union of all the sets $U_\alpha \times G$. Define an equivalence relation in C by $(\alpha, x, g) \sim (\alpha', x', g')$ if and only if $x = x'$ and $g' = \xi_{\alpha'\alpha}(x)g$. Set $P = C/\sim$. The action of G in P is given by $(\alpha, x, g)g_0 = (\alpha, x, gg_0)$. The smooth structure on P is defined such that the sets $U_\alpha \times G$ are smooth coordinate charts for P.

Exercise 4.2.5. Complete the construction of P above.

Let (P, π, M) be a principal G bundle. A *(global) section* of P is a map $\psi : M \to P$ such that $\pi \circ \psi = id_M$.

Exercise 4.2.6. Show that a principal bundle is trivial if and only if it has a global section.

A local section consists of an open set $U \subset M$ and a map $\psi : U \to P$ such that $\pi \circ \psi = id_U$. If $f : \pi^{-1}(U) \to U \times G$ is a local trivialization we can define a local section by $\psi(x) = f^{-1}(x, h(x))$, where $h : U \to G$ is an arbitrary (smooth) function.

Let $H \subset G$ be a closed subgroup. We say that the bundle P has been *reduced* to a principal H *subbundle* Q, if $Q \subset P$ is a submanifold such that $qh \in Q$ for all $q \in Q, h \in H, \pi(Q) = M$ and H acts transitively in each fiber $Q_x = \pi^{-1}(x) \cap Q$.

Any manifold M of dimension n carries a natural principal $GL(n, \mathbb{R})$ bundle, namely, the bundle FM of linear frames. The fiber F_xM at a point $x \in M$ consists of all frames (ordered basis) of the tangent space T_xM. The group $GL(n, \mathbb{R})$ acts in F_xM by $(f_1, f_2, ..., f_n)A = (\sum_{i=1}^n A_{i1}f_i, \sum_{i=1}^n A_{i2}f_i, ..., \sum_{i=1}^n A_{in}f_i)$, where the f_i's are tangent

vectors at x and $A = (A_{ij}) \in GL(n, \mathbb{R})$. One can construct a local trivialization by choosing a local coordinate system $(x_1, x_2, ..., x_n)$ in M. In local coordinates the vectors of a frame f can be written as $f_i = \sum f_{ij}\partial_j$. This defines a mapping $f \mapsto (f_{ij}) \in GL(n, \mathbb{R})$. The collection $(\partial_1, ..., \partial_n)$ of vector fields defines a local section of FM.

If the manifold M has some additional structure the bundle FM can generally be reduced to a subbundle. For example, if M is a Riemannian manifold with metric g, then we can define the subbundle $OFM \subset FM$ consisting of *orthonormal* frames with respect to the metric g. If in addition M is oriented, then it makes sense to speak of the bundle *SOFM of oriented orthonormal* frames: A frame $(f_1, \ldots, f_n)$ at a point x is oriented if $\omega(x; f_1, \ldots, f_n)$ is positive, where ω is a n form defining the orientation. The structure group of OFM is the orthogonal group $O(n)$ and of $SOFM$ the special orthogonal group $SO(n)$ consisting of orthogonal matrices with determinant=1.

Let $\mathbf{g}$ be the Lie algebra of the Lie group G. To any $A \in \mathbf{g}$ there corresponds canonically a one-parameter subgroup $h_A(t) = \exp tA$. We define a vector field $\hat{A}$ on the G bundle P such that the tangent vector $\hat{A}(p)$ at $p \in P$ is equal to $\frac{d}{dt}[p \cdot h_A(t)]|_{t=0}$. Let $g \in G$ be any fixed element. The right translation $r_g(p) = pg$ on P determines canonically a transformation $X \mapsto r_g^* X$ on vector fields: The tangent vector of the transformed field at a point p is simply obtained by applying the derivative of the mapping r_g to the tangent vector $X(pg^{-1})$.

Proposition 4.2.7. *For any $A \in \mathbf{g}$ the vector field $\hat{A}$ is equivariant, that is,* $(r_g)^* \hat{A} = \widehat{ad_g^{-1} A} \forall g \in G$.

Proof. Using a local trivialization,

$$\hat{A}(p) = \frac{d}{dt}(\pi(p), \phi(pe^{tA}))\Big|_{t=0}$$

and therefore

$$\begin{aligned}
((r_g)^* \hat{A})(p) &= T_{pg^{-1}} r_g \cdot \frac{d}{dt}(\pi(pg^{-1}), \phi(pg^{-1}e^{tA}))\Big|_{t=0} \\
&= \frac{d}{dt}(\pi(pg^{-1}), \phi(pg^{-1}e^{tA}g))\Big|_{t=0} \\
&= \frac{d}{dt}(0, \phi(pe^{tad_g^{-1}A}))\Big|_{t=0} \\
&= \widehat{ad_g^{-1} A}(p).
\end{aligned}$$

4.3. Connection and curvature in a principal bundle

Let E and M be a pair of manifolds, V a vector space and $\pi : E \to M$ a smooth surjective map.

Definition 4.3.1. *The manifold E is a vector bundle over M with fiber V, if*

(1) $E_x = \pi^{-1}(x)$ *is isomorphic with the vector space V for each $x \in M$*
(2) $\pi : E \to M$ *is locally trivial: Any $x \in M$ has an open neighborhood U with a diffeomorphism $\phi : \pi^{-1}(U) \to U \times V$, $\phi(z) = (\pi(z), \xi(z))$, where the restriction of ξ to a fiber E_x is a linear isomorphism onto V.*

The product $M \times V$ is the *trivial vector bundle* over M, with fiber V. In this case the projection map $M \times V \to M$ is simply the projection onto the first factor.

A direct sum of two vector bundles E and F over the same manifold M is the bundle $E \oplus F$ with fiber $E_x \oplus F_x$ at a point $x \in M$. The *tensor product bundle* $E \otimes F$ is the vector bundle with fiber $E_x \otimes F_x$ at $x \in M$.

Example 4.3.2. The tangent bundle TM of a manifold M is a vector bundle over M with fiber $T_xM \simeq \mathbb{R}^n$, where $n = dimM$. The local trivializations are given by local coordinates: If $(x_1, x_2, ..., x_n)$ are local coordinates on $U \subset M$, then the value of ξ for a tangent vector $w \in T_xM$, $x \in M$, is obtained by expanding w in the basis defined by the vector fields $(\partial_1, ..., \partial_n)$.

A section of a vector bundle E is a map $\psi : M \to E$ such that $\pi \circ \psi = id_M$. The space $\Gamma(E)$ of sections of E is a linear vector space; the addition and multiplication by scalars is defined pointwise. A principal bundle may or may not have global sections but a vector bundle always has nonzero sections. A section can be multiplied by a smooth function $f \in C^\infty(M)$ pointwise, $(f\psi)(x) = f(x)\psi(x)$.

Let (P, π, M) be a principal G bundle. *The space V of vertical vectors* in the tangent bundle TP is the subbundle of TP with fiber $\{v \in T_pP | \pi(v) = 0\}$ at $p \in P$. If P is trivial, $P = M \times G$, then the vertical subspace at $p = (x, g)$ consists of vectors tangential to G at g. In general, the dimension of the fiber V_p is equal to $\dim G$.

Definition 4.3.3. *A connection in the principal bundle P is a smooth distribution $p \mapsto H_p$ of subspaces of T_p such that*

(1) *The tangent space T_p is a direct sum of V_p and H_p $\forall p \in P$*
(2) *The distribution is equivariant, i.e., $r_g H_p = H_{pg}$ $\forall p \in P, g \in G$.*

Smoothness means that the distribution can be locally spanned by smooth vector fields. We shall denote by pr_h (respectively, pr_v) the projection in T_p to the *horizontal* subspace H_p (respectively, vertical subspace V_p).

Let $A \in \mathbf{g}$ and let $\hat{A}$ be the corresponding equivariant vector field on P. The field $\hat{A}$ is vertical at each point. Since the group G acts freely and transitively on P, the mapping $A \mapsto \hat{A}(p)$ is a linear isomorphism onto V_p for all $p \in P$. Thus for each $X \in T_pP$ there is a uniquely defined element $\omega_p(X) \in \mathbf{g}$ such that

$$\widehat{\omega_p(X)} = pr_v X$$

at p. The mapping $\omega_p : T_pP \to \mathbf{g}$ is linear, thus defining a differential form of degree one on P, with values in the Lie algebra $\mathbf{g}$. The form ω is the *connection form* of the connection H.

Proposition 4.3.4. *The connection form satisfies*

(1) $\omega_p(\hat{A}(p)) = A \, \forall A \in \mathbf{g}$,
(2) $r_a^* \omega = ad_a \omega \, \forall a \in G$.

Furthermore, each **g***-valued differential form on P which satisfies the above conditions is a connection form of a uniquely defined connection in P.*

Proof. The first equation follows immediately from the definition of ω. To prove the second, we first note that

$$(\widehat{ad_a^{-1}A})(p) = \frac{d}{dt} p e^{t ad_a^{-1} A}|_{t=0} = \frac{d}{dt} p a^{-1} e^{tA} a|_{t=0} = r_a \hat{A}(pa^{-1}).$$

By the equivariantness property of the distribution H_p, the right translations r_a commute with the horizontal and vertical projection operators. Thus [we write $(A)\hat{}$ for $\hat{A}$ in case of long expressions]

$$\begin{aligned}(ad_a \omega_p(X))\hat{}(pa) &= r_a^{-1} \cdot \widehat{\omega_p(X)}(p) \\ &= r_a^{-1}(pr_v X) = pr_v(r_a^{-1} X) \\ &= (\omega_p(r_a^{-1}X))\hat{}(pa).\end{aligned}$$

Taking account that $(r_a^* \omega)_p(X) = \omega_{pa}(r_a^{-1}X)$ we get the second relation.

Let then ω be any form satisfying both equations. We define the horizontal subspaces $H_p = \{X \in T_p | \omega_p(X) = 0\}$. If $X \in H_p \cap V_p$, then $X = \hat{A}(p)$ for some $A \in \mathbf{g}$ and $\omega_p(\hat{A}(p)) = A = \omega_p(X) = 0$, from which follows $H_p \cap V_p = 0$. By (1) and a simple dimensional argument we get $T_p = H_p + V_p$. For $X \in H_p$ and $a \in G$ we obtain

$$\omega_{pa}(r_a X) = (r_a^* \omega)_p(X) = ad_a^{-1} \omega_p(X) = 0,$$

and therefore $r_a X \in H_{pa}$, which shows that the distribution H_p is equivariant and indeed defines a connection in P.

Let ω be a connection form in (P, π, M). Let $U \subset M$ be open and $\psi : U \to P$ a local section. The pull-back $A = \psi^* \omega$ is a one-form on U. Consider another local section $\phi : V \to P$ and set $A' = \phi^* \omega$. We can write $\psi(x) = \phi(x) g(x)$ for $g : U \cap V \to G$, where $g(x)$ is a smooth G valued function. We want to relate A to A'. Noting that

$$T_x \psi = r_{g(x)} T_x \phi + (g^{-1} T_x g)\hat{}(\psi(x))$$

by the Leibnitz rule, we get

$$\begin{aligned}A_x(u) &= \omega_{\psi(x)}(T_x \psi \cdot u) = \omega_{\psi(x)}(r_{g(x)} T_x \phi \cdot u + (g^{-1} T_x g \cdot u)\hat{}(\psi(x))) \\ &= ad_{g(x)}^{-1} \omega_{\phi(x)}(T_x \phi \cdot u) + g^{-1} T_x g \cdot u.\end{aligned}$$

For a matrix group G we can simply write

$$A = g^{-1}A'g + g^{-1}dg.$$

The transformation relating A to A' is called *a gauge transformation* . Next we define the two-form

$$F = dA + \frac{1}{2}[A, A] \tag{4.3.5}$$

on U. The commutator of Lie algebra valued one-forms is defined by

$$[A, B](u, v) = [A(u), B(v)] - [A(v), B(u)]$$

for a pair u, v of tangent vectors. We shall compute the effect of a gauge transformation $(U, \psi) \to (V, \phi)$ on F:

$$\begin{aligned} F &= dA + \tfrac{1}{2}[A, A] \\ &= g^{-1}dA'g - [g^{-1}dg, g^{-1}A'g] - \tfrac{1}{2}[g^{-1}dg, g^{-1}dg] \\ &\quad + \tfrac{1}{2}[g^{-1}A'g + g^{-1}dg, g^{-1}A'g + g^{-1}dg] \\ &= g^{-1}(dA' + \tfrac{1}{2}[A', A'])g = g^{-1}F'g. \end{aligned}$$

The *curvature form* F is a pull-back under ψ of a gobally defined two-form Ω on P. The latter is defined by

$$\Omega_p(u, v) = a^{-1}F_x(\pi u, \pi v)a,$$

where $p \in \pi^{-1}(x)$, u, v tangent vectors at p and $a \in G$ is an element such that $p = \psi(x)a$. The left-hand side does not depend on the local section. Writing $p = \phi(x)a' = \psi(x)g(x)a'$ we get

$$a'^{-1}F'_x(\pi u, \pi v)a' = a'^{-1}g(x)^{-1}F_x(\pi u, \pi v)g(x)a' = a^{-1}F_x(\pi u, \pi v)a.$$

Since A is the pull-back of ω and F is the pull-back of Ω we obtain from 4.3.5

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \tag{4.3.6}$$

Exercise 4.3.7. Prove the *Bianchi identity* $dF + [A, F] = 0$. (The 3-form $[A, F]$ is defined by an antisymmetrization of $[A(u), F(v, w)]$ with respect to the triplet (u, v, w) of tangent vectors.)

Let (P, π, M) be a principal G bundle and $\rho : G \to AutV$ a linear representation of G in a vector space V. We define the manifold $P \times_G V$ to be the set of equivalence classes $P \times V/\sim$, where the equivalence relation is defined by $(p, v) \sim (pg^{-1}, \rho(g)v)$, for $g \in G$. There is a natural projection $\theta : P \times_G V \to M$, $[(p, v)] \mapsto \pi(p)$. The inverse image $\theta^{-1}(x) \cong V$, since G acts freely and transitively in the fibers of P. The linear structure in a fiber $\theta^{-1}(x)$ is defined by $[(p, v)] + [(p, w)] = [(p, v + w)]$, $\lambda[(p, v)] = [(p, \lambda v)]$. Local trivializations of $P \times_G V$ are obtained from local trivializations $p \mapsto (\pi(p), \phi(p)) \in M \times G$ of P by $[(p, v)] \mapsto (\pi(p), \rho(\phi(p))v)$. Thus $P \times_G V$ is a vector bundle over M, *the vector bundle associated to* P via the representation ρ of G.

Example 4.3.8. Let $P = SU(2), M = S^2 = SU(2)/U(1), G = U(1), V = \mathbb{C}$ and $\rho(\lambda) = \lambda^2$ for $\lambda \in U(1)$. The associated vector bundle $E = SU(2) \underset{U(1)}{\times} \mathbb{C}$ is in fact the tangent bundle of the sphere S^2. The isomorphism is obtained as follows. Fix a linear isomorphism of $\mathbb{C} \cong \mathbb{R}^2$ with the tangent space of S^2 at the point x, which has as its isotropy group the given $U(1)$. The map $E \to TS^2$ is defined by $(g, v) \mapsto D(g)v$, where $D(g)$ is the 2-1 representation of $SU(2)$ in $\mathbb{R}^3$. The tangent vectors of S^2 are represented by vectors in $\mathbb{R}^3$ by the natural embedding $S^2 \subset \mathbb{R}^3$.

4.4. Parallel transport

Let H be a connection in a principal G bundle (P, π, M). *A horizontal lift* of a smooth curve $\gamma(t)$ on the base manifold M is a smooth curve $\gamma^*(t)$ on P such that the tangent vector $\dot\gamma^*(t)$ is *horizontal* at each point on the curve and $\pi(\gamma^*(t)) = \gamma(t)$.

Lemma 4.4.1. *Let $X(t)$ be a smooth curve on the Lie algebra* $\mathbf{g}$ *of G, defined on an interval $[t_0, t_1]$. Then there exists a unique smooth curve $a(t)$ on G such that $\dot a(t)a(t)^{-1} = X(t) \forall t \in [t_0, t_1]$ and such that $a(t_0) = e$.*

Proof. See Kobayashi and Nomizu, vol. I, p. 69.

Proposition 4.4.2. *Let $\gamma(t)$ be a smooth curve on M and p an element in the fiber over $\gamma(t_0)$. Then there exists a unique horizontal lift $\gamma^*(t)$ of $\gamma(t)$ such that $\gamma^*(t_0) = p$.*

Proof. Choose first any (smooth) curve $\phi(t)$ on P such that $\pi(\phi) = \gamma$ and $\phi(t_0) = p$.

We are looking for the solution in the form $\gamma^*(t) = \phi(t)g(t)$, where $g(t)$ is a curve on G such that $g(t_0) = e$. Now $\gamma^*(t)$ is a solution if

$$\dot{\gamma}^*(t) = r_{g(t)} \cdot \dot{\phi}(t) + (g(t)^{-1}\dot{g}(t))\widehat{\ }[\phi(t)g(t)]$$

is horizontal. Let ω be the connection form of the connection H. A tangent vector on P is horizontal if and only if it is in the kernel of ω. We get the differential equation

$$\begin{aligned} 0 = \omega(\dot{\gamma}^*(t)) &= \omega(r_{g(t)}\dot{\phi}(t)) + \omega([g(t)^{-1}\dot{g}(t)]\widehat{\ }[\phi(t)g(t)]) \\ &= ad_{g(t)}^{-1}\omega(\dot{\phi}(t)) + g(t)^{-1}\dot{g}(t). \end{aligned}$$

Applying ad_g to this equation we get

$$\dot{g}(t)g(t)^{-1} = -\omega(\dot{\phi}(t)).$$

The solution $g(t)$ exists and is unique by the previous lemma.

Example 4.4.3. Let $P = M \times U(1)$. A connection form ω can be written as $\omega_{(x,g)}(u, a) = A_x(u) + g^{-1} \cdot a$, where u is a tangent vector at $x \in M$ and a is a tangent vector at $g \in U(1)$; the Lie algebra of $U(1)$ is identified by the set of purely imaginary complex numbers. Let $\gamma(t)$ be a curve on M. The horizontal lift of $\gamma(t)$ which goes through $(\gamma(t), g)$ at time $t = 0$ is $\gamma^*(t) = (\gamma(t), g(t))$ with

$$g(t) = g \cdot \exp\left(\int_0^t -A_{\gamma(s)}(\dot{\gamma}(s))ds\right).$$

In particular, for a *closed contractible curve*, $\gamma(0) = \gamma(1)$, we get by Stokes's theorem

$$g(1) = g \cdot \exp(-\int_S F),$$

where $F = dA$ is the curvature two-form and the integration is taken over any surface on M bounded by the closed curve γ.

We define *the parallel transport* along a curve $\gamma(t)$ on M as a mapping $\tau : \pi^{-1}(x_0) \to \pi^{-1}(x_1)$ ($x_0 = \gamma(t_0), x_1 = \gamma(t_1)$ points on the curve). The value $\tau(p_0)$ for $p_0 \in \pi^{-1}(x_0)$ is given as follows: Let $\gamma^*(t)$ be a horizontal lift of $\gamma(t)$ such that $\gamma^*(t_0) = p_0$. Then $\tau(p_0) = \gamma^*(t_1)$.

Exercise 4.4.4. Prove the following properties of the parallel transport.

(1) $\tau \circ r_g = r_g \circ \tau \forall g \in G$

(2) If γ_1 is a path from x_0 to x_1 and γ_2 is a path from x_1 to x_2 then the parallel transport along the composed path $\gamma_2 * \gamma_1$ is equal to the product of parallel transport along γ_1 followed by a parallel transport along γ_2.

(3) The parallel transport is a one-to-one mapping between the fibers $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$.

4.5. Covariant differentiation in vector bundles

Let E be a vector bundle over a manifold M with fiber V, $\dim V = n$. The vector space V is defined over the field $\mathbb{K} = \mathbb{R}$ or $K = \mathbb{C}$. A vector bundle can always be thought of as an associated bundle to a principal bundle. Namely, let P_x denote the space of all linear frames in the fiber E_x for $x \in M$. Using the local trivializations of E it is not difficult to see that the spaces P_x fit together and form naturally a smooth manifold P. Fix a basis $w = \{w_1, \dots, w_n\}$ in E_x. Then any other basis of E_x can be obtained from w by a linear tranformation $w'_i = \sum A_{ji} w_j$ and therefore P_x can be identified with the group $GL(n, \mathbb{K})$ of all linear transformations in $\mathbb{K}^n$; it should be stressed that this identification depends on the choice of w. However, we have a well-defined mapping $P \times GL(n, \mathbb{K}) \to P$ given by the basis transformations and this shows that P can be thought of as a principal $GL(n, \mathbb{K})$ bundle over M.

The vector bundle E is now isomorphic with the associated bundle $P \times_\rho \mathbb{K}^n$, where ρ is the natural representation of $GL(n, \mathbb{K})$ in $\mathbb{K}^n$. The isomorphism is defined as follows. Let $w \in P_x$ and $a \in \mathbb{K}^n$. We set $\phi(w, a) = \sum a_i w_i$. This gives a mapping from $P \times \mathbb{K}^n$ to E which is obviously linear in a. For a fixed w the mapping $a \mapsto \phi(w, a)$ gives an isomorphism between $\mathbb{K}^n$ and E_x. Let $w' = w \cdot g$ and $a' = \rho(g^{-1})a$ for some $g \in GL(n, \mathbb{K})$. We have to show that $\phi(w', a') = \phi(w, a)$; but this follows immediately from the definitions.

Often the bundle E can be thought of as an associated bundle to a principal bundle with a smaller structure group than the group $GL(n, \mathbb{K})$. This happens when there is some extra structure in E. For example, assume there is a *fiber metric* in E: This means that there is an inner product $< \cdot, \cdot >_x$ in each fiber E_x such that $x \mapsto < \psi(x), \psi(x) >_x$ is a smooth function for any (local) section ψ. We can then define the bundle of orthonormal frames in E

with structure group $U(n)$ in the complex case and $O(n)$ in the real case. The vector bundle E is now an associated bundle to the bundle of orthonormal frames.

We shall now assume that E is given as an associated vector bundle $P \times_\rho V$ to some principal bundle P, with a connection H, over M. Let G be the structure group of P. For each vector field X on M we can define a linear map ∇_X of the space $\Gamma(E)$ of sections into itself such that

(1) $\nabla_{X+Y} = \nabla_X + \nabla_Y$
(2) $\nabla_{fX} = f\nabla_X$
(3) $\nabla_X(f\psi) = (Xf)\psi + f\nabla_X\psi$

for all vector fields X, Y, smooth functions f and sections ψ. We shall give the definition in terms of a local trivialization $\xi : U \to P$, where $U \subset M$ is open. Locally, a section $\psi : M \to E$ can be written as

$$\psi(x) = (\xi(x), \phi(x)),$$

where $\phi : U \to V$ is some smooth function. Let A denote the pull-back $\xi^*\omega$ of the connection form ω in P. The representation ρ of G in V defines also an action of the Lie algebra $\mathbf{g}$ in V. We set

$$\nabla_X\psi = (\xi, X\phi + A(X)\phi),$$

where $A(X)$ is the Lie algebra valued function giving the value of the one-form A in the direction of the vector field X.

We have to check that our definition does not depend on the choice of the local trivialization. So let $\xi'(x) = \xi(x) \cdot g(x)$ be another local trivialization, where $g : U \to G$ is a smooth function. The vector potential with respect to the trivialization ξ' is $A' = g^{-1}Ag + g^{-1}dg$. Now $(\xi, \phi) \sim (\xi', \phi')$, where $\phi' = g^{-1}\phi$ (we simplify the notation by dropping ρ) and therefore $(\xi', X\phi' + A'(X)\phi')$ is equal to

$$\begin{aligned}(\xi', &-g^{-1}(Xg)g^{-1}\phi + g^{-1}X\phi + (g^{-1}Ag + g^{-1}Xg)g^{-1}\phi) \\ &= (\xi', g^{-1}(X\phi + A(X)\phi)) \sim (\xi, X\phi + A(X)\phi)\end{aligned}$$

which shows that ∇_X is well-defined.

Exercise 4.5.1. Prove that ∇_X defined above satisfies (1)-(3).

The commutator of the *covariant derivatives* ∇_X is related to the curvature of the connection in the following way:

$$\begin{aligned}[\nabla_X, \nabla_Y]\psi &= (\xi, [X + A(X), Y + A(Y)]\phi) \\ &= (\xi, ([X, Y] + X \cdot A(Y) - Y \cdot A(X) + [A(X), A(Y)])\phi) \\ &= (\xi, (F(X, Y) + [X, Y] + A([X, Y]))\phi)\end{aligned}$$

where $F = dA + \frac{1}{2}[A, A]$. Thus we can write

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = F(X, Y)$$

when acting on the functions ϕ.

A section ψ is *covariantly constant* if $\nabla_X \psi = 0$ for all vector fields. From the above commutator formula we conclude that one can find at each point in the base space a local basis of covariantly constant sections of the vector bundle if and only if the curvature vanishes.

4.6. An example: The monopole line bundle

Construction of the basic monopole bundle

Let G be a Lie group and $\mathbf{g}$ its Lie algebra. Let us denote by ℓ_g the left translation $\ell_g(a) = ga$ in G. The left invariant Maurer-Cartan form $\theta_L = g^{-1}dg$ is the $\mathbf{g}$-valued one form on G which sends a tangent vector X at $g \in G$ to the element $\ell_g^{-1} X \in T_e G$ in the Lie algebra. Similarly, we can define the right invariant Maurer-Cartan form $\theta_R = dgg^{-1}$, $\theta_R(g; X) = r_g^{-1} X$. By taking commutators, we can define higher order forms. For example, the form $[g^{-1}dg, g^{-1}dg]$ sends the pair (X, Y) of tangent vectors at g to $2[\ell_g^{-1} X, \ell_g^{-1} Y] \in \mathbf{g}$.

Taking projections to one dimensional subspaces of $\mathbf{g}$ we get real valued one-forms on G.

Let $< \cdot, \cdot >$ be a bilinear form on $\mathbf{g}$ and $\sigma \in \mathbf{g}$. Then $\alpha =< \sigma, g^{-1}dg >$ is a well-defined one form. Let us compute the exterior derivative of α. Let X, Y be a pair of left invariant vector fields on G. Now

$$\begin{aligned}d\alpha(g; X, Y) &= X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X, Y]) \\ &= -\alpha([X, Y])\end{aligned}$$

since $\alpha(Y)(g) =< \sigma, \ell_g^{-1}Y >$ is a constant function on G and similarly for $\alpha(X)$. Since the left invariant vector fields on a Lie group span the tangent space at each point, we conclude

$$d\alpha = - < \sigma, \tfrac{1}{2}[g^{-1}dg, g^{-1}dg] > .$$

We have not yet defined the exterior derivative of a Lie algebra valued differential form, but motivated by the computation above we set

$$d(g^{-1}dg) = -\frac{1}{2}[g^{-1}dg, g^{-1}dg].$$

A bilinear form $< \cdot, \cdot >$ on $\mathbf{g}$ is *invariant* if

$$< [X, Y], Z >= - < Y, [X, Z] >$$

for all X, Y, and Z. Given an invariant bilinear form, the group G has a natural **closed** three-form c_3 which is defined by

$$c_3(g; X, Y, Z) =< \ell_g^{-1}X, [\ell_g^{-1}Y, \ell_g^{-1}Z] > .$$

Thus

$$c_3 =< g^{-1}dg, \tfrac{1}{2}[g^{-1}dg, g^{-1}dg] > .$$

Proposition 4.6.1. $dc_3 = 0$.

Proof.. Recall the definition of the exterior differentiation d: If ω is a n-form and $V_1, ..., V_{n+1}$ are vector fields, then

$$\begin{aligned} d\omega(V_1, ..., V_{n+1}) = &\sum_{i=1}^{n+1}(-1)^{i+1}V_i \cdot \omega(V_1, ..., \hat{V}_i, ..., V_{n+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([V_i, V_j], V_1, ..., \hat{V}_i, ..., \hat{V}_j, ..., V_{n+1}), \end{aligned}$$

where the caret means that the corresponding variable has been dropped. Let us compute dc_3 for left invariant vector fields $X_1, ..., X_4$. Taking account that $c_3(X_i, X_j, X_k)$ is a constant function we get

$$\begin{aligned} dc_3(X_1, ...X_4) = &- 2 < [X_1, X_2], [X_3, X_4] > +2 < [X_1, X_3], [X_2, X_4] > \\ &- 2 < [X_1, X_4], [X_2, X_3] > \\ =&2 < X_1, [[X_3, X_4], X_2] - [[X_2, X_4], X_3] + [[X_2, X_3], X_4] > \\ =&0 \end{aligned}$$

by Jacobi's identity.

If G is a group of matrices we can define an invariant form on $\mathbf{g}$ by $< X, Y >=$ $\mathrm{tr}\, XY$. Then the form c_3 can be written as

$$c_3 = \mathrm{tr}\,(g^{-1}dg)^3.$$

As an example we shall consider in detail the case $G = SU(2)$. Let $\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and define the one-form $\alpha = -\frac{1}{2}\mathrm{tr}\,\sigma_3 g^{-1}dg$. Remember that $SU(2) \to SU(2)/U(1) = S^2$ is a principal $U(1)$ bundle. The form α is invariant with respect to right translations $g \mapsto gh$ by $h \in U(1)$. Thus α is a connection form in the bundle $SU(2)$ [the Lie algebra of the structure group $U(1)$ can be identified with $i\mathbb{R}$]. Let us compute the curvature. The exterior derivative of α is $\frac{1}{4}\mathrm{tr}\,\sigma_3[g^{-1}dg, g^{-1}dg]$. A tangent vector at $x \in S^2$ can be represented by a tangent vector $\ell_g X$ at $g \in \pi^{-1}(x)\,, X \in \mathbf{g}$, such that X is orthogonal to the $U(1)$ direction, $\mathrm{tr}\,\sigma_3 X = 0$. The curvature in the base space S^2 is then $\Omega(X,Y) = \frac{1}{2}\mathrm{tr}\,\sigma_3[X,Y]$. The form Ω is $\frac{1}{2}\times$ the volume form on S^2: If {X,Y} is an ortonormal system at $x \in S^2$, then $[X,Y] = \pm\frac{i}{2}\sigma_3$ (exercise), the sign depending on the orientation. We obtain $\Omega(X,Y) = \pm\frac{i}{4}\mathrm{tr}\,\sigma_3^2 = \pm\frac{i}{2}$.

The basic monopole line bundle is defined as the associated bundle to the bundle $SU(2) \to S^2$, constructed using the natural one dimensional representation of $U(1)$ in $\mathbb{C}$.

Embedding $S^2 \subset \mathbb{R}^3$ and using Cartesian coordinates $\{x_1, x_2, x_3\}$ we can write the curvature form as

$$\Omega = \frac{1}{4r^3}\varepsilon^{ijk}x_i dx_j \wedge dx_k,$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$ is equal to 1 on S^2. However, we can extend Ω to the space $\mathbb{R}^3 \setminus \{0\}$ using the above formula. The coefficients of the linearly independent forms $dx_2 \wedge dx_3$, $dx_3 \wedge dx_1$ and $dx_1 \wedge dx_2$ form a vector $\vec{B} = \frac{1}{2r^3}(x_1, x_2, x_3) = \frac{\vec{x}}{r^3}$. The field $\vec{B}$ satisfies

(1) $$\vec{\nabla} \cdot \vec{B} = 0$$

(2) $$\vec{\nabla} \times \vec{B} = 0,$$

i.e., it satisfies Maxwell's equations in vacuum. On the other hand,

$$\int_{S^2} \vec{B} \cdot d\vec{S} = 2\pi \tag{3}$$

for *any* sphere containing the origin. Because of these properties, the field $\vec{B}$ can be interpreted as the magnetic field of a magnetic monopole located at the origin. The integral (3) multiplied by the dimensional constant $1/e$ (e is the unit electric charge) is called the *monopole strength.*

The first Chern class

The magnetic field of the monopole is the curvature of a circle bundle over the unit sphere S^2. The circle bundle we have constructed is a "generator" for the set of all circle bundles over S^2. In general, a principal $U(1)$ bundle over S^2 can be constructed from the transition function $\xi : S_- \cap S_+ \to U(1)$ (cf. 3.2.3). The intersection of the coordinate neighborhoods $S_\pm$ is homeomorphic with the product of an open interval with the circle S^1. It follows that the set of maps ξ decomposes to connected components labelled by the winding number of a map $S^1 \to U(1)$. Let ξ_1 be the transition function of the bundle $SU(2) \to S^2$ with respect to some fixed local trivializations on $S_\pm$. The winding number of ξ_1 is equal to one. The winding number of $\xi_n = (\xi_1)^n$ is equal to n. Let $P(n)$ be the bundle constructed from ξ_n. Let $A_\pm$ be the vector potentials on $S_\pm$ corresponding to the chosen local trivializations and the connection in $SU(2)$ described above.

We have $A_+ = A_- + \xi^{-1}d\xi$ on $S_- \cap S_+$ and therefore $nA_+ = nA_- + \xi_n^{-1}d\xi_n$. Thus nA is a connection in the bundle $P(n)$ and the curvature of $P(n)$ is n times the curvature form Ω of the (basic) monopole bundle. The monopole strength of the bundle $P(n)$ is $2\pi n/e$.

The cohomology class $[\Omega] \in H^2(S^2, \mathbb{R})$ is the *first Chern class* of the bundle. It depends only on the equivalence class of the bundle and not on the chosen connection; we shall return to the proof of the topological invariance of the Chern classes in a more general context later, but as an illustration of the general ideas we give a simple proof for the case at hand. Let $B_\pm$ be the vector potentials on $S_\pm$ of some connection in the bundle $P(n)$. We have $B_+ = B_- + n\xi^{-1}d\xi$ and therefore $A_+ - B_+ = A_- - B_-$ on $S_+ \cap S_-$. It follows that $A - B$ is a globally defined one-form on S^2; the difference of the curvatures corresponding to the connections A and B is equal to $d(A-B)$.

The first Chern class of a circle bundle (or an associated complex line bundle) over a manifold M can be evaluated from the knowledge of the $U(1)$ valued transition functions [R. Bott and L.W. Tu: Differential forms in algebraic topology]. In the example above we needed only one transition function ξ. A representative Ω for the Chern class can be constructed from a vector potential (A_+, A_-) such that $A_- = 0$ for $x_3 < \frac{1}{2}$, A_+ is equal to $\xi^{-1} d\xi$ on the strip $-\frac{1}{2} < x_3 < \frac{1}{2}$, and A_+ is contracted smoothly to zero when approaching the north pole $x_3 = 1$. The first Chern class is always quantized in the sense that the integral of the two-form Ω over any two-dimensional compact surface is 2π times an integer.

4.7. Chern classes

We shall consider polynomials $P(A)$ of a complex $N \times N$ matrix variable A which are invariant in the sense that $P(gAg^{-1}) = P(A)$ for all $g \in GL(N, \mathbb{C})$. For example, if we expand

$$\det\left(1 + \frac{\lambda}{2\pi i} A\right) = \sum_{n=0}^{N} \lambda^n P_n(A) \tag{4.7.1}$$

then the coefficients $P_n(A)$ are homogeneous invariant polynomials of degree n in A. These polynomials will play a special role in the following discussion.

To each homogeneous polynomial $P(A)$ one can associate a unique *symmetric* multilinear form $P(A_1, \dots A_n)$ such that $P(A, \dots, A) = P(A)$. The general formula for the n linear form in terms of $P(A)$ is

$$\begin{aligned} P(A_1, \dots, A_n) = & \frac{1}{n!}\{P(A_1 + \dots + A_n) \\ & - \sum_j P(A_1 + \dots + \hat{A}_j + \dots + A_n) \\ & - \sum_{j,j'} P(A_1 + \dots \hat{A}_j + \dots \hat{A}_{j'} \dots + A_n) - \dots\}, \end{aligned}$$

with $\hat{A}_j$ deleted. When $P(A)$ is invariant we clearly have $P(gA_1g^{-1}, \dots, gA_ng^{-1}) =$

$P(A_1, \dots, A_n)$. Writing $g = g(t) = \exp(tX)$ we get the useful formula

$$
\begin{aligned}
0 &= \frac{d}{dt} P(g(t)A_1 g(t)^{-1}, \dots, g(t)A_n g(t)^{-1})|_{t=0} \\
&= \sum_j P(A_1, \dots, [X, A_j], \dots, A_n). \qquad (4.7.2)
\end{aligned}
$$

If F_i is a $N \times N$ matrix valued differential form of degree k_i on a manifold M, $1 \le i \le n$, and P a symmetric n linear form then we can define a complex valued differential form $P(F_1, \dots, F_n)$ of degree $k_1 + \dots + k_n = p$ by

$$
P(F_1, \dots, F_n)(t_1, \dots, t_p) = \\
\left(\prod \frac{1}{k_i!}\right) \sum_\sigma \epsilon(\sigma) P(F_1(t_{\sigma(1)}, \dots, t_{\sigma(k_1)}), \dots, F_n(t_{\sigma(p-k_n+1)}, \dots, t_{\sigma(p)}))
$$

where the sum is taken over all permutations of the indices $1, 2, \dots, p$.

Let F be the curvature form of a vector bundle E over M with fiber $\mathbb{C}^N$. The curvature transforms in a change of a local trivialization as $F \mapsto gFg^{-1}$ and therefore $P(F, \dots, F)$ is well-defined, independent of the local trivialization, for any invariant symmetric polynomial P.

Proposition 4.7.3. *The symmetric homogeneous polynomial $P(F, \dots, F)$ of degree n in the curvature F is a closed form of degree $2n$.*

Proof. Locally we can write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Using the property $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{deg\alpha} \alpha \wedge d\beta$ of differential forms we have

$$
\begin{aligned}
dP(F, \dots, F) &= \sum_j P(F, \dots, dF, \dots, F) \\
&= \sum_j \{P(F, \dots, DF, \dots, F) - P(F, \dots, [A, F], \dots, F)\}. \qquad (4.7.4)
\end{aligned}
$$

The covariant derivative $DF = 0$ by the Bianchi identity and the sum of the terms involving $[A, F]$ is zero by (4.7.2).

In particular, the class in $H^{2n}(M, \mathbb{R})$ defined by the closed $2n$ form Re $P_n(F)$ is called the nth *Chern class* of the bundle E and is denoted by $c_n(E)$.

Theorem 4.7.5. *The Chern classes are topological invariants: They do not depend on the choice of connection in the vector bundle E.*

Proof. Let A_0 and A_1 be two connections in E and F_0, F_1 the corresponding curvatures. Define a one-parameter family $A_t = A_0 + t\eta$ of connections with $\eta = A_1 - A_0$;

note that the difference η transforms homogeneously in a change of local trivialization, $\eta \mapsto g\eta g^{-1}$. Let us introduce the notation $Q(A,B) = nP(A,B,\dots,B)$ when B is repeated $n-1$ times. Using

$$F_t = dA_t + \frac{1}{2}[A_t, A_t] = F_0 + tD\eta + \frac{1}{2}t^2[\eta,\eta],$$

where D is the covariant derivative determined by A_0, we get

$$\frac{d}{dt}P(F_t) = Q\left(\frac{d}{dt}F_t, F_t\right) = Q\left(D\eta, F_t\right) + tQ([\eta,\eta], F_t). \tag{4.7.6}$$

On the other hand,

(3.7.7)

$$\begin{aligned} dQ(\eta, F_t) =& Q(d\eta, F_t) - n(n-1)P(\eta, dF_t, F_t, \dots, F_t) \\ =& Q(d\eta, F_t) - n(n-1)P(\eta, dF_t, F_t, \dots, F_t) \\ &+ nP([A_0,\eta], F_t, \dots, F_t) - n(n-1)P(\eta, [A_0, F_t], \dots, F_t) \\ =& Q(D\eta, F_t) - n(n-1)P(\eta, DF_t, F_t, \dots, F_t) \\ =& Q(D\eta, F_t) + tn(n-1)P(\eta, [\eta, F_t], F_t, \dots, F_t) \end{aligned}$$

where we have used $DF_t = DF_0 + tD^2\eta + t^2[D\eta,\eta] = t[F_0,\eta] + t^2[D\eta,\eta] = t[F_t,\eta]$, since $[[\eta,\eta],\eta] = 0$ by Jacobi identity. By (3.7.2) we have

$$P([\eta,\eta], F_t, \dots, F_t) - (n-1)P(\eta, [\eta, F_t], F_t, \dots, F_t) = 0$$

or in other words,

$$Q([\eta,\eta], F_t) - n(n-1)P(\eta, [\eta, F_t], F_t, \dots, F_t) = 0.$$

Using (4.7.7) we get

$$dQ(\eta, F_t) = Q(D\eta, F_t) + tQ([\eta,\eta], F_t)$$

and with (4.7.6) we obtain

$$\frac{d}{dt}P(F_t) = dQ(\eta, F_t). \tag{4.7.8}$$

Integrating this result over the interval $0 \le t \le 1$ we get

$$P(F_1) - P(F_0) = d\int_0^1 Q(\eta, F_t)dt$$

which shows explicitly that the difference of the differential forms $P(F_1)$ and $P(F_0)$ is an exact form.

Given a Hermitian inner product in the fibers of the vector bundle E it is always possible to choose a Hermitian connection, that is, a connection such that in an orthonormal basis the vector potential takes values in the Lie algebra of the unitary group $U(N)$. In that case the determinant $\det(1 + \frac{\lambda}{2\pi i} F)$ is real for any real parameter λ and the Chern classes are given by the expansion in powers of λ; the first two positive powers lead to

$$
\begin{aligned}
c_1(F) &= \frac{1}{2\pi i} \operatorname{tr} F \\
c_2(F) &= \frac{1}{2(2\pi i)^2} [-\operatorname{tr} F^2 + (\operatorname{tr} F)^2].
\end{aligned}
$$

The coefficients in the expansion can be best computed by diagonalizing the matrix F. Writing $F = diag(\alpha_1, \ldots, \alpha_N)$ one obtains

$$
\det\left(1 + \frac{\lambda}{2\pi i} F\right) = \prod_k \left(1 + \frac{\lambda \alpha_k}{2\pi i}\right) = \sum_n \left(\frac{\lambda}{2\pi i}\right)^n S_n(\alpha)
$$

with

$$
\begin{aligned}
S_0 = 1,\ S_1 = \operatorname{tr}\alpha,\ S_2 &= \frac{1}{2}(\operatorname{tr}\alpha)^2 - \frac{1}{2}\operatorname{tr}\alpha^2 \\
S_3 &= \frac{1}{6}(\operatorname{tr}\alpha)^3 - \frac{1}{2}\operatorname{tr}\alpha^2 \operatorname{tr}\alpha + \frac{1}{3}\operatorname{tr}\alpha^3
\end{aligned}
$$

etc. Note that c_n vanishes identically if $n > \frac{1}{2}\dim M$ or $n > N$. If $n = \frac{1}{2}\dim M$ then we can integrate the form $c_n(E)$ over M and the value of the integral is called *the Chern number* associated to the vector bundle E.

Example 4.7.9. Consider a vector bundle E over $M = S^4$ such that the transition functions take values in the group $SU(N)$, $N \geq 2$. Dividing S^4 to the upper and lower hemispheres $S^4_{\pm}$ the bundle is given by the transition function ϕ along the equator S^3. The vector potentials $A_{\pm}$ are then related by $A_- = \phi A_+ \phi^{-1} - d\phi\phi^{-1}$ on the equator. Using the formula $\operatorname{tr} F^2 = d\operatorname{tr}(F \wedge A - \frac{1}{3} A^3)$ we can compute the Chern number corresponding to the second Chern class,

$$\begin{aligned}
&\frac{1}{8\pi^2}\int_{S^4_+}\mathrm{tr}\,F_+^2+\frac{1}{8\pi^2}\int_{S^4_-}\mathrm{tr}\,F_-^2\\
&\qquad=\frac{1}{8\pi^2}\int_{S^3}\left[\mathrm{tr}\,(F_+\wedge A_+-\tfrac{1}{3}A_+^3)-\mathrm{tr}\,(F_-\wedge A_--\tfrac{1}{3}A_-^3)\right]\\
&\qquad=\frac{1}{8\pi^2}\int_{S^3}\left[\mathrm{tr}\,\tfrac{1}{3}(d\phi\phi^{-1})^3-d\mathrm{tr}\,(A_+\wedge d\phi\phi^{-1})\right]\\
&\qquad=\frac{1}{24\pi^2}\int_{S^3}\mathrm{tr}\,(d\phi\phi^{-1})^3.
\end{aligned}$$

Remark 4.7.10. The value of the integral above is an integer which depends only on the homotopy class of the map $\phi: S^3 \to SU(N)$. This follows from the fact that the form $\mathrm{tr}\,(dgg^{-1})^3$ on any Lie group is closed (section 3.6) and from Stokes' theorem applied to the integral $\int_0^1 dt \frac{d}{dt}\int_{S^3}\mathrm{tr}\,(d\phi_t {\phi_t}^{-1})^3$ for a 1-parameter family of maps $\phi_t; S^3 \to SU(N)$.

Since the equivalence class of the bundle E depends only on the homotopy class of the transition function ϕ, the Chern number $\int c_2(E)$ gives a complete topological characterization of E.

The *Chern character* $ch(E)$ of a vector bundle is defined as follows. It is a formal sum of differential forms of different degrees,

$$ch(E)=\mathrm{tr}\ \exp\left(\frac{1}{2\pi i}F\right),$$

where again F is the curvature form of E. When the exponential is evaluated as a power series we obtain

$$ch(E)=\sum_{k=0}^{\infty}\frac{1}{(2\pi i)^k k!}\mathrm{tr}\,F^k.$$

Clearly all the terms can be expressed using the Chern classes; the three first terms are

$$ch(E)=N+c_1(E)+\frac{1}{2}c_1(E)\wedge c_1(E)-c_2(E)+\ldots.$$

The Chern character is a convenient tool because one has

$$ch(E\oplus E')=ch(E)+ch(E')\quad ch(E\otimes E')=ch(E)\cdot ch(E').$$

This follows immediately from the definition and the elementary properties of the exponential function.

Above we have studied characteristic classes of complex vector bundles. The most important characteristic classes for real vector bundles are the *Pontrjagin classes* and they are constructed as follows.

Let $\pi : E \to M$ be a real vector bundle over the manifold M. We can always think a real vector bundle as an associated vector bundle to a principal bundle P with structure group $GL(n, \mathbb{R})$. We fix a metric in the fibers of E, so it makes sense to speak about orthonormal frames in the fibers. This means that we can consider E as an associated bundle to a principal $O(n)$ bundle; the principal bundle is simply the bundle of orthonormal frames.

Thus we are led to studying connections in principal $O(n)$ bundles. A connection form takes values in the Lie algebra of $O(n)$, that is, in the Lie algebra of real antisymmetric $n \times n$ matrices.

If we choose a local section of the principal $O(n)$ bundle then the curvature form F is a local matrix form on the base space M and in gauge transformations $F' = g^{-1}Fg$.

A real antisymmetric matrix can be brought to the canonical form

$$T^{-1}FT = \begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 \\ -\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_2 \dots & 0 \\ 0 & 0 & -\lambda_2 & 0 \dots & 0 \\ \dots\dots\dots & & & & \\ \dots\dots\dots & & & & \end{pmatrix}.$$

When $n = 2k$ is even the matrix consist of k antisymmetric 2×2 matrices on the diagonal; when $n = 2k+1$ then the last column and the last row consists of only zeros. The eigenvalues of the matrix are $\pm i\lambda_j$.

We define

$$p(F) = \det\left(1 + \frac{F}{2\pi}\right) = \prod_{i=1}^{k}\left(1 + \frac{\lambda_i^2}{4\pi^2}\right).$$

Clearly $p(F) = p(-F)$ so that p is a polynomial of even degree in the curvature tensor F. We write

$$p(F) = 1 + p_1(F) + p_2(F) + \dots$$

as a sum of homogeneous terms $p_j(F)$ of degree $2j$ in the curvature. Since F is a 2-form, $p_j(F)$ is a differential form on M of degree $4j$.

Note that $p_j(F)$ depends only on the eigenvalues of $F_{\mu\nu}$ and therefore it is invariant with respect to gauge transformations and thus gives a globally well-defined form on M.

Exactly as in the case of Chern classes we expand each p_j in powers of the curvature tensor F. The lowest Pontrjagin classes are

$$\begin{aligned}
p_1(F) &= -\frac{1}{2}(\frac{1}{2\pi})^2 \mathrm{tr}\, F^2 \\
p_2(F) &= \sum_{i<j}(\frac{\lambda_i}{2\pi})^2(\frac{\lambda_j}{2\pi})^2 = \frac{1}{2}\left[(\sum_i \frac{\lambda_i^2}{(2\pi)^2})^2 - \sum_i(\frac{\lambda_i}{2\pi})^4\right] \\
&= \frac{1}{8}(\frac{1}{2\pi})^4\left[(\mathrm{tr}\, F^2)^2 - 2\mathrm{tr}\, F^4\right] \\
p_3(F) &= \sum_{i<j<k}(\frac{\lambda_i}{2\pi})^2(\frac{\lambda_j}{2\pi})^2(\frac{\lambda_k}{2\pi})^2 = \dots \\
&= \frac{1}{48}(\frac{1}{2\pi})^6\left[-(\mathrm{tr}\, F^2)^3 + 6\mathrm{tr}\, F^2 \cdot \mathrm{tr}\, F^4 - 8\mathrm{tr}\, F^6\right].
\end{aligned}$$

We shall meet later another set of characteristic classes, called the *A-roof genus,* which are actually formed from the Pontrjagin classes. The definition is best set up in terms of eigenvalues of the matrix form F,

$$\hat{A}(F) = \prod_j \frac{x_j/2}{\sinh(x_j/2)} = \prod_j \left(1 + \sum_\ell (-1)^\ell \frac{2^{2\ell}-2}{(2\ell)!} B_\ell\, x_j^{2\ell}\right),$$

where B_ℓ are the Bernoulli numbers and $x_j = \lambda_j/2\pi$. In terms of Ponrjagin classes,

$$\hat{A}(F) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \frac{1}{967680}(-31p_1^3 + 44p_1p_2 - 16p_3) + \dots.$$

Further reading: Nakahara, Chapters 9-11. The proof above of the topological invariance of the Chern classes follows S.S. Chern: *Complex Manifolds without Potential Theory.* Princeton University Press (1979). On characteristic classes see also: J.W. Milnor and J.D. Stasheff: *Characteristic Classes.* Princeton University Press (1974).