CHAPTER 1 : DIFFERENTIABLE MANIFOLDS

1.1 The definition of a differentiable manifold

Let M be a topological space. This means that we have a family Ω of *open sets* defined on M. These satisfy

(1) $\emptyset, M \in \Omega$

(2) the union of any family of open sets is open

(3) the intersection of a finite family of open sets is open

We normally assume also the *Hausdorff property:* For any pair x, y of distinct points there is a pair of nonoverlapping open sets U, V such that $x \in U$ and $y \in V$.

In any topological space one can define the notion of convergence. A sequence $x_1, x_2, x_3, \dots$ converges towards $x \in M$ if any open set U such that $x \in U$ contains all the points x_n except a finite set.

The basic example of a topological space is $\mathbb{R}^n$ equipped with the Euclidean norm $||x||^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ for $x = (x_1, \dots, x_n)$. A set $U \subset \mathbb{R}^n$ is open if for any $x \in U$ there is a positive number $\epsilon = \epsilon(x)$ such that $y \in U$ if $||x - y|| < \epsilon$. The convergence is then the usual one: $x^{(n)} \to x$ if for any $\epsilon > 0$ there is an integer n_ϵ such that $||x - x^{(n)}|| < \epsilon$ for $n > n_\epsilon$.

Actually, all the spaces we study in (finite dimensional) differential geometry are *locally homeomorphic* to $\mathbb{R}^n$.

Definition. *A topological space M is called a smooth manifold* of dimension n if 1) there is a family of open sets U_α (with $\alpha \in \Lambda$) such that the union of all U_α's is equal to M, 2) for each α there is a homeomorphism $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ such that 3) the *coordinate transformations* $\phi_\alpha \circ \phi_\beta^{-1}$ on their domains of definition are smooth functions in $\mathbb{R}^n$.

Example 1 $\mathbb{R}^n$ is a smooth manifold. We need only one *coordinate chart* $U = M$ with $\phi : U \to \mathbb{R}^n$ the identity mapping.

Example 2 The same as above, but take $M \subset \mathbb{R}^n$ any open set.

Example 3 Take $M = S^1$, the unit circle. Set U equal to the subset parametrized by the polar angle $-0.1 < \phi < \pi + 0.1$ and V equal to the set $\pi < \phi < 2\pi$. Then $U \cap V$ consists of two intervals $\pi < \phi < \pi + 0.1$ and $-0.1 < \phi < 0 \sim 2\pi - 0.1 < \phi < 2\pi$.

The coordinate transformation is the identity map $\phi \mapsto \phi$ on the former and the translation $\phi \mapsto \phi + 2\pi$ on the latter interval.

Exercise Define a manifold structure on the unit sphere S^2.

Example 4 The group $GL(n, \mathbb{R})$ of invertible real $n \times n$ matrices is a smooth manifold as an open subset of $\mathbb{R}^{n^2}$. It is an open subset since it is a complement of the closed surface determined by the polynomial equation $\det A = 0$.

1.2 Differentiable maps

Let M, N be a pair of smooth manifolds (of dimensions m, n) and $f : M \to N$ a continuous map. If (U, ϕ) is a local coordinate chart on M and (V, ψ) a coordinate chart on N then we have a map $\psi \circ f \circ \phi^{-1}$ from some open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$. If the composite map is smooth for any pair of coordinate charts we say that f is smooth. The reader should convince himself that the condition of smoothness for f does not depend on the choice of coordinate charts. From elementary results in differential calculus it follows that if $g : N \to P$ is another smooth map then also $g \circ f : M \to P$ is smooth.

Note that we can write the map $\psi \circ f \circ \phi^{-1}$ as $y = (y_1, \dots, y_n) = (y_1(x_1, \dots, x_m), \dots$ $\dots, y_n(x_1, \dots, x_m))$ in terms of the Cartesian coordinates. Smoothness of f simply means that the coordinate functions $y_i(x_1, x_2, \dots, x_m)$ are smooth functions.

Remark In a given topological space M one can often construct different inequivalent smooth structures. That is, one might be able to construct atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\alpha, \psi_\alpha)\}$ such that both define a structure of smooth manifold, say M_U and M_V, but the manifolds M_U, M_V are not diffeomorphic (see the definition below). A famous example of this phenomen are the spheres S^7, S^{11} (John Milnor, 1956). On the sphere S^7 there are exactly 28 inequivalent differentiable structures! On the Euclidean space $\mathbb{R}^4$ there is an infinite number of differentiable structures (S.K. Donaldson, 1983).

A diffeomorphism is a one-to-one smooth map $f : M \to N$ such that its inverse $f^{-1} : N \to M$ is also smooth. The set of diffeomorphisms $M \to M$ forms a group Diff(M). A smooth map $f : M \to N$ is *an immersion* if at each point $p \in M$ the rank of the derivative $\frac{dh}{dx}$ is equal to the dimension of M. Here $h = \psi \circ f \circ \phi^{-1}$ with the notation as before. Finally $f : M \to N$ is *an embedding* if f is injective and it is an immersion; in that case $f(M) \subset N$ is *an embedded submanifold.*

A smooth curve on a manifold M is a smooth map γ from an open interval of the real axes to M. Let $p \in M$ and (U, ϕ) a coordinate chart with $p \in U$. Assume that curves γ_1, γ_2 go through p, let us say $p = \gamma_i(0)$. We say that the curves are equivalent at p, $\gamma_1 \sim \gamma_2$, if

$$\frac{d}{dt}\phi(\gamma_1(t))|_{t=0} = \frac{d}{dt}\phi(\gamma_2(t))|_{t=0}.$$

This relation does not depend on the choice of (U, ϕ) as is easily seen by the help of the chain rule:

$$\frac{d}{dt}\psi(\gamma_1(t)) - \frac{d}{dt}\psi(\gamma_2(t)) = (\psi \circ \phi^{-1})' \cdot \left(\frac{d}{dt}\phi(\gamma_1(t)) - \frac{d}{dt}\phi(\gamma_2(t))\right) = 0$$

at the point $t = 0$. Clearly if $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$ at the point p then also $\gamma_1 \sim \gamma_3$ and $\gamma_2 \sim \gamma_1$. Trivially $\gamma \sim \gamma$ for any curve γ through p so that " $\sim$ " is an equivalence relation.

A tangent vector v at a point p is an equivalence class of smooth curves $[\gamma]$ through p. For a given chart (U, ϕ) at p the equivalence classes are parametrized by the vector

$$\frac{d}{dt}\phi(\gamma(t))|_{t=0} \in \mathbb{R}^n.$$

Thus the space T_pM of tangent vectors $v = [\gamma]$ inherits the natural linear structure of $\mathbb{R}^n$. Again, it is a simple exercise using the chain rule that the linear structure does not depend on the choice of the coordinate chart.

We denote by TM the disjoint union of all the tangent spaces T_pM. This is called the *tangent bundle* of M. We shall define a smooth structure on TM. Let $p \in M$ and (U, ϕ) a coordinate chart at p. Let $\pi : TM \to M$ the natural projection, $(p, v) \mapsto p$. Define $\tilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n$ as

$$\tilde{\phi}(p, [\gamma]) = (\phi(p), \frac{d}{dt}\phi(\gamma(t))|_{t=0}).$$

If now (V, ψ) is another coordinate chart at p then

$$(\tilde{\phi} \circ \tilde{\psi}^{-1})(x, v) = (\phi(\psi^{-1}(x)), (\phi \circ \psi^{-1})'(x)v),$$

by the chain rule. It follows that $\tilde{\phi} \circ \tilde{\psi}^{-1}$ is smooth in its domain of definition and thus the pairs $(\pi^{-1}(U), \tilde{\phi})$ form an atlas on TM, giving TM a smooth structure.

Example 1 If M is an open set in $\mathbb{R}^n$ then $TM = M \times \mathbb{R}^n$.

Example 2 Let $M = S^1$. Writing $z \in S^1$ as a complex number of unit modulus, consider curves through z written as $\gamma(t) = ze^{ivt}$ with $v \in \mathbb{R}$. This gives in fact a parametrization for the equivalence classes $[\gamma]$ as vectors in $\mathbb{R}$. The tangent spaces at different points z_1, z_2 are related by the phase shift $z_1 z_2^{-1}$ and it follows that TM is simply the product $S^1 \times \mathbb{R}$.

Example 3 In general, $TM \neq M \times \mathbb{R}^n$. The simplest example for this is the unit sphere $M = S^2$. Using the spherical coordinates, for example, one can identify the tangent space at a given point (θ, ϕ) as the plane $\mathbb{R}^2$. However, there is no natural way to identify the tangent spaces at different points on the sphere; the sphere is not parallelizable. This is the content of the famous hairy ball theorem. Any smooth vector field on the sphere has zeros. (If there were a globally nonzero vector field on S^2 we would obtain a basis in all the tangent spaces by taking a (oriented) unit normal vector field to the given vector field. Together they would form a basis in the tangent spaces and could be used for identifying the tangent spaces as a standard $\mathbb{R}^2$.)

Exercise The unit 3-sphere S^3 can be thought of complex unitary 2×2 matrices with determinant =1. Use this fact to show that the tangent bundle is trivial, $TS^3 = S^3 \times \mathbb{R}^3$.

Let $f : M \to N$ be a smooth map. We define a linear map

$$T_p f : T_p M \to T_{f(p)} N, \text{ as } T_p f \cdot [\gamma] = [f \circ \gamma],$$

where γ is a curve through the point p. This map is expressed in terms of local coordinates as follows. Let (U, ϕ) be a coordinate chart at p and (V, ψ) a chart at $f(p) \in N$. Then the coordinates for $[\gamma] \in T_p M$ are $v = \frac{d}{dt}\phi(\gamma(t))|_{t=0}$ and the coordinates for $[f \circ \gamma] \in T_{f(p)} N$ are $w = \frac{d}{dt}\psi(f(\gamma(t)))|_{t=0}$. But by the chain rule,

$$w = (\psi \circ f \circ \phi^{-1})'(x) \cdot \frac{d}{dt}\phi(\gamma(t))|_{t=0} = (\psi \circ f \circ \phi^{-1})'(x) \cdot v$$

with $x = \phi(p)$. Thus in local coordinates the linear map $T_p f$ is the derivative of $\psi \circ f \circ \phi^{-1}$ at the point x. Putting together all the maps $T_p f$ we obtain a map

$$Tf : TM \to TN.$$

Proposition. *The map $Tf : TM \to TN$ is smooth.*

Proof. Recall that the coordinate charts $(U, \phi), (V, \psi)$ on M, N, respectivly, lead to coordinate charts $(\pi^{-1}(U), \tilde{\phi})$ and $(\pi^{-1}(V), \tilde{\psi})$ on TM, TN. Now

$$(\tilde{\psi} \circ Tf \circ \tilde{\phi}^{-1})(x, v) = ((\psi \circ f \circ \phi^{-1})(x), (\psi \circ f \circ \phi^{-1})'(x)v)$$

for $(x, v) \in \tilde{\phi}(\pi^{-1}(U)) \in \mathbb{R}^m \times \mathbb{R}^m$. Both component functions are smooth and thus Tf is smooth by definition.

If $f : M \to N$ and $g : N \to P$ are smooth maps then $g \circ f : M \to P$ is smooth and

$$T(g \circ f) = Tg \circ Tf.$$

To see this, the curve γ through $p \in M$ is first mapped to $f \circ \gamma$ through $f(p) \in N$ and further, by Tg, to the curve $g \circ f \circ \gamma$ through $g(f(p)) \in P$.

In terms of local coordinates x_i at p, y_i at $f(p)$ and z_i at $g(f(p))$ the chain rule becomes the standard formula,

$$\frac{\partial z_i}{\partial x_j} = \sum_k \frac{\partial z_i}{\partial y_k} \frac{\partial y_k}{\partial x_j}.$$

1.3 Vector fields

We denote by $C^\infty(M)$ the algebra of smooth real valued functions on M. *A derivation* of the algebra $C^\infty(M)$ is a linear map $d : C^\infty(M) \to C^\infty(M)$ such that

$$d(fg) = d(f)g + fd(g)$$

for all f, g. Let $v \in T_pM$ and $f \in C^\infty(M)$. Choose a curve γ through p representing v. Set

$$v \cdot f = \frac{d}{dt} f(\gamma(t))|_{t=0}.$$

Clearly $v : C^\infty(M) \to \mathbb{R}$ is linear. Furthermore,

$$v \cdot (fg) = \frac{d}{dt} f(\gamma(t))|_{t=0} g(\gamma(0)) + f(\gamma(0)) \frac{d}{dt} g(\gamma(t))|_{t=0} = (v \cdot f)g(p) + f(p)(v \cdot g).$$

A vector field on a manifold M is a smooth distribution of tangent vectors on M, that is, a smooth map $X : M \to TM$ such that $X(p) \in T_pM$. From the previous formula follows that a vector field defines a derivation of $C^\infty(M)$; take above $v = X(p)$ at each point $p \in M$ and the right- hand-side defines a smooth function on M and the operation satisfies the Leibnitz' rule.

We denote by $D^1(M)$ the space of vector fields on M. As we have seen, a vector field gives a linear map $X : C^\infty(M) \to C^\infty(M)$ obeying the Leibnitz' rule. Conversely, one can prove that any derivation of the algebra $C^\infty(M)$ is represented by a vector field.

One can develope an algebraic approach to manifold theory. In that the commutative algebra $\mathcal{A} = C^\infty(M)$ plays a central role. Points in M correspond to *maximal ideals* in the algebra $\mathcal{A}$. Namely, any point p defines the ideal $I_p \subset \mathcal{A}$ consisting of all functions which vanish at the point p.

The action of a vector field on functions is given in terms of local coordinates $x_1, \dots, x_n$ as follows. If $v = X(p)$ is represented by a curve γ then

$$(X \cdot f)(p) = \frac{d}{dt} f(\gamma(t))_{t=0} = \sum_k \frac{\partial f}{\partial x_k} \frac{dx_k}{dt}(t=0) \equiv \sum_k X_k(x) \frac{\partial f}{\partial x_k}.$$

Thus a vector field is locally represented by the vector valued function $(X_1(x), \dots, X_n(x))$.

In addition of being a real vector space, $D^1(M)$ is a *left module* for the algebra $C^\infty(M)$. This means that we have a linear left multiplication $(f, X) \mapsto fX$. The value of fX at a point p is simply the vector $f(p)X(p) \in T_pM$.

As we have seen, in a coordinate system x_i a vector field defines a derivation with local representation $X = \sum_k X_k \frac{\partial}{\partial x_k}$. In a second coordinate system x'_k we have a representation $X = \sum X'_k \frac{\partial}{\partial x'_k}$. Using the chain rule for differentiation we obtain the coordinate transformation rule

$$X'_k(x') = \sum_j \frac{\partial x'_k}{\partial x_j} X_j(x),$$

for $x'_k = x'_k(x_1, \dots, x_n)$.

We shall denote $\partial^k = \frac{\partial}{\partial x_k}$ and we use Einstein's summation convention over repeated indices,

Let $X, Y \in D^1(M)$. We define a new derivation of $C^\infty(M)$, the commutator $[X, Y] \in D^1(M)$, by

$$[X, Y]f = X(Yf) - Y(Xf).$$

We prove that this is indeed a derivation of $C^\infty(M)$.

$$\begin{aligned}[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) = X(fYg + gYf) - Y(fXg + gXf) \\ &= (Xf)(Yg) + fX(Yg) + (Xg)(Yf) + gX(Yf) - (Yf)(Xg) - fY(Xg) \\ &- (Yg)(Xf) - gY(Xf) = f[X,Y]g + g[X,Y]f.\end{aligned}$$

Writing $X = X_k\partial^k$ and $Y = Y_k\partial^k$ we obtain the coordinate expression

$$[X,Y]_k = X_j\partial^j Y_k - Y_j\partial^j X_k.$$

Thus we may view $D^1(M)$ simply as the space of first order linear partial differential operators on M with the ordinary commutator of differential operators. The commutator $[X,Y]$ is also called the *Lie bracket* on $D^1(M)$. It has the basic properties

(1) $[X,Y]$ is linear in both arguments
(2) $[X,Y] = -[Y,X]$
(3) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$

The last property is called the Jacobi identity. A vector space equipped with a Lie bracket satisfying the properties above is called a *Lie algebra.*

Other examples of Lie algebras:

Example 1 Any vector space with the bracket $[X,Y] = 0$.

Example 2 The angular momentum Lie algebra with basis L_1, L_2, L_3 and nonzero commutators $[L_1, L_2] = L_3$ + cyclically permuted relations.

Example 3 The space of $n\times n$ matrices with the usual commutator of matrices, $[X,Y] = XY - YX$.

Exercise Check the relations

$$[X, fY] = f[X,Y] + (Xf)Y, \text{ and } [fX, Y] = f[X,Y] - (Yf)X$$

for $X, Y \in D^1(M)$ and $f \in C^\infty(M)$.

Let $f : M \to N$ be a diffeomorphism and $X \in D^1(M)$. We can define a vector field $Y = f_*X$ on N by setting $Y(q) = T_pf \cdot X(p)$ for $q = f(p)$. In terms of local coordinates,

$$Y = Y_k\frac{\partial}{\partial y_k} = X_j\frac{\partial y_k}{\partial x_j}\frac{\partial}{\partial y_k}.$$

In the case $M = N$ this gives back the coordinate transformation rule for vector fields.

Let $X \in D^1(M)$. Consider the differential equation

$$X(\gamma(t)) = \frac{d}{dt}\gamma(t)$$

for a smooth curve γ. In terms of local coordinates this equation is written as

$$X_k(x(t)) = \frac{d}{dt}x_k(t), k = 1, 2, \ldots, n.$$

By the theory of ordinary differential equations this system has locally, at a neighborhood of an initial point $p = \gamma(0)$, a unique solution. However, in general the solution does not need to extend to $-\infty < t < +\infty$ except in the case when M is a compact manifold. The (local) solution γ is called *an integral curve* of X through the point p.

The integral curves for a vector field X define a (local) *flow* on the manifold M. This is a (local) map

$$f : \mathbb{R} \times M \to M$$

given by $f(t, p) = \gamma(t)$ where γ is the integral curve through p. We have the identity

$$f(t + s, p) = f(t, f(s, p)), \tag{1}$$

which follows from the uniqueness of the local solution to the first order ordinary differential equation. In coordinates,

$$\frac{d}{dt}f_k(t, f(s, x)) = X_k(f(t, f(s, x)))$$

and

$$\frac{d}{dt}f_k(t + s, x) = X_k(f(t + s, x)).$$

Thus both sides of (1) obey the same differential equation. Since the initial conditions are the same, at $t = 0$ both sides are equal to $f(0, f(s, x)) = f(s, x)$, the solutions must agree.

Denoting $f_t(p) = f(t, p)$, observe that the map $\mathbb{R} \to \mathrm{Diff}_{loc}(M)$, $t \mapsto f_t$, is a homomorphism,

$$f_t \circ f_s = f_{t+s}.$$

Thus we have a *one parameter group of (local) transformations* f_t on M. In the case when M is compact we actually have globally globally defined transformations on M.

Example Let $X(r, \phi) = (-r \sin \phi, r \cos \phi)$ be a vector field on $M = \mathbb{R}^2$. The integral curves are solutions of the equations

$$x'(t) = -r(t) \sin \phi(t)$$
$$y'(t) = r(t) \cos \phi(t)$$

and the solutions are easily seen to be given by $(x(t), y(t)) = (r_0 \cos(\phi + \phi_0), r_0 \sin(\phi + \phi_0))$, where the initial condition is specified by the constants ϕ_0, r_0. The one parameter group of tranformations generated by the vector field X is then the group of rotations in the plane.

Further reading: M. Nakahara: *Geometry, Topology and Physics,* Institute of Physics Publ. (1990), sections 5.1 - 5.3 . S. S. Chern, W.H. Chen, K.S. Lam: *Lectures on Differential Geometry,* World Scientific Publ. (1999), Chapter 1.