FUNKTIONAALIANALYYSI II, 2009
EXERCISES, SET 4
TO BE RETURNED ON WEDNESDAY DEC. 16TH AT LATEST, PERSONALLY OR TO THE MAILBOX OF J.T.

1. Let a, $b \in \mathbb{R}$ and $a < b$, and let $C(a,b)$ be the usual space of continuous mappings $f : [a,b] \to \mathbb{R}$ endowed with the sup-norm. Construct a bounded linear extension operator $E : C(0,1) \to C(-10,10)$ (having the properties $\|Ef\| \le \|f\|$ and $Ef(t) = f(t)$ for all $f \in C(0,1)$ and $t \in [0,1]$). Remark. There are many ways to do this.

2.–3. Let $1 \le p < \infty$, $m \in \mathbb{N}$ and $p < q < np/(n-mp)$. By constructing an explicit sequence of functions, show that the embedding $W^{m,p}(\mathbb{R}) \to L^q(\mathbb{R})$, existing by Theorem 6.4. of the lectures, cannot be compact, i.e., the identity operator is not a compact operator, i.e., the unit ball of the Banach space $W^{m,p}(\mathbb{R})$ is not a precompact subset of $L^q(\mathbb{R})$.

4. Is the operator

$$L := (1 - |x|^2) \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$$

elliptic on the domain $\Omega = \{|x| < 1\} \subset \mathbb{R}^3$?

5. Is the operator

$$a)\ L := \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\ , \qquad b)\ T := \Big(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\Big)^2$$

elliptic on $\Omega := \mathbb{R}^2$?

6.–8. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a regular enough (at least C^2) boundary. Consider the following Neumann problem:

$$-\Delta u + u = f \quad \text{in the domain } \Omega$$
$$\partial_\nu u = 0 \quad \text{in } \partial\Omega,$$

where $f \in C(\bar{\Omega})$ is given and ∂_ν denotes the partial derivative in the direction of the outer normal vector of the boundary. By a *classical solution* we mean a function $u \in C^2(\bar{\Omega})$ satisfying the above equalities.
a) Let us define the *weak solution* as a function $u \in W^{1,2}(\Omega)$, which satisfies

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega u\varphi = \int_\Omega f\varphi$$

for all $\varphi \in W^{1,2}(\Omega)$. By using a relevant *Green formula* show that a classical solution to the above Neumann problem is always a weak solution.
b) Prove the existence and uniqueness of the weak solution using the Lax–Milgram theorem.

9. Let $\Omega :=]-1,1[\subset \mathbb{R}$ and let the coefficient field $\mathbb{K}$ be equal to $\mathbb{R}$. The point evaluation mapping $T : f \mapsto f(0)$ is linear, but it is not a well defined and bounded

linear operator e.g. from $L^2(\Omega) \to \mathbb{R}$, $\Omega :=]-1,1[$ for example since the elements of $L^2(\Omega)$ are defined only almost everywhere. Explain how the Sobolev embedding theorem can be used to make sense of the definition of T as a mapping $W^{m,p}(\Omega) \to \mathbb{R}$, if, say, $m = 1$, $p = 2$.