FUNKTIONAALIANALYYSI II, 2009
EXERCISES, SET 2

1. Is it true that if $f \in C^\infty(\mathbb{R})$ and $f(0) = 0$, then

$$f\,\frac{d\delta_0}{dx} = 0 \qquad \text{or} \quad f\,\frac{d^2\delta_0}{dx^2} = 0?$$

What about the case $f(x) = x^k$, $k \in \mathbb{N}$?

2. Write the distribution $T \in \mathcal{D}'(\mathbb{R})$,

$$T := \delta_5 - \frac{d\delta_0}{dx}$$

as a derivative of a continuous function on $\mathbb{R}$.

3. Let $Y : \mathbb{R} \to \mathbb{C}$ be the step function. Show that $d\delta_0/dx * Y = \delta_0$ and that $1 * d\delta_0/dx = 0$. Calculate

$$1 * (\frac{d\delta_0}{dx} * Y) \qquad \text{and} \quad (1 * \frac{d\delta_0}{dx}) * Y. \tag{0.1}$$

This seems to violate the associative law. What's wrong?

4. Let x and y be variables in $\mathbb{R}$. Write the functions

$$f(x, y) = e^{-x^2 - y^2} \quad \text{and} \quad g(x, y) = \sin(x + y) \tag{0.2}$$

as tensor products (or their linear combinations) of functions of one variables. The fucntions here do not have compact supports, but the idea and the definition of the tensor product is the same as in that case.

5. Let $f \in C^\infty(\mathbb{R}^n)$, and let $\gamma_{k,m}$ be as in (4.1) of the lecture notes. Show that the condition "$\gamma_{k,m}(f) < \infty$ for all k and m" is equivalent to the condition

$$\text{“} \lim_{|x|\to\infty} |x|^k |D^\alpha f(x)| = 0 \tag{0.3}$$

for all k and α".

6. Using the definition of the seminorms $\gamma_{k,m}$, show that the linear operator T,

$$(T\varphi)(x) = 5 \sin x \varphi(x) + \varphi(2x) \tag{0.4}$$

is continuous $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

7. Show that the function e^{ax} is not a tempered distribution on $\mathbb{R}$, if $a \neq 0$ is a constant.

8. Show that the operator $G : T \mapsto \sin xT$ is sequentially continuous $\mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$, when this space is endowed with the weak topology. Sequential continuity means that if the sequence $(T_j)_{j=1}^\infty$ satisfies $T_j \to T$ weakly in $\mathcal{S}'(\mathbb{R})$ as $j \to \infty$, then $GT_j \to GT$ weakly in $\mathcal{S}'(\mathbb{R})$.

9. Calculate the Fourier–transforms of the tempered distributions x^k, $k \in \mathbb{N}$ on $\mathbb{R}$. Also calculate the Fourier–transform of the polynomial P of two variables, $P(x) := x_1^2 x_2$, where $x = (x_1, x_2) \in \mathbb{R}^2$.

10. Show that if $f : \mathbb{R}^n \to \mathbb{C}$ and if there exist constants $C > 0$ and $a > n + 1$ such that

$$|f(x)| \le \frac{C}{(1 + |x|)^a} \tag{0.5}$$

then $\mathcal{F}f$ is at least m times differentiable for $m < a - n - 1$. (Differentiate under the integral sign.) This is an indication of the important basic intuition that the more rapidly f vanishes at the infinity, the more smooth is its Fourier tranformation. Conversely, if $\hat{f}$ vanishes at a certain rate at infinity, the f must have a corresponding amount of smoothness.