Quasiregular Mappings
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Problem Set 3
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1. Show that for all $a, x, y \in B^n$

$$\frac{|T_a x - T_a y|^2}{(1-|T_a x|^2)(1-|T_a y|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} \,.$$

Hint. Lectures, Ahlfors bracket.

2. Let $r \in (0,1)$. Find a point $a \in (0, re_1)$ such that $T_a(0) = -T_a(re_1)$.

3. For $\varphi \in (0, \frac{1}{2}\pi)$, let $x_\varphi = (\cos\varphi, \sin\varphi)$ and $y_\varphi = (\cos\varphi, -\sin\varphi)$. Then there exists a Möbius transformation $T_a \colon \mathbf{B}^2 \to \mathbf{B}^2$ with $T_a e_1 = e_1$, $T_a(-e_1) = -e_1$, $T_a(x_\varphi) = e_2 = -T_a(y_\varphi)$. Find $|a|$.

4. Let $x, y \in \mathbf{R}^n$ and let t_x be a spherical isometry with $t_x(x) = 0$. Show that

$$|t_x y| = \frac{|x-y|}{\sqrt{(1+|x|^2)(1+|y|^2) - |x-y|^2}} \,.$$

Let $\alpha \in [0, \frac{1}{2}\pi]$ be such that $\sin\alpha = q(x,y)$. Then α is the angle between the segments $[e_{n+1}, t_x x] = [e_{n+1}, 0]$ and $[e_{n+1}, t_x y]$ at e_{n+1} . Show that the above formula can be rewritten as $|t_x y| = \tan\alpha$.

5. For $x, y \in \mathbf{B}^n$ and $T_x \in \mathcal{M}(\mathbf{B}^n)$ show that

$$|T_x y| = \frac{|x-y|}{\sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)}} = \frac{s}{\sqrt{1+s^2}} \,,$$

where $s^2 = |x-y|^2/((1-|x|^2)(1-|y|^2))$.

6. Let $h : [0,\infty) \to [0,\infty)$ be strictly increasing with $h(0) = 0$ such that $h(t)/t$ is decreasing. Show that $h(x+y) \le h(x) + h(y)$ for all $x > 0$. With $h(t) = t^\alpha$, $\alpha \in (0,1)$, apply this result to show that if $d(x,y)$ is a metric then $d^\alpha(x,y) = d(x,y)^\alpha$ also is a metric.

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