Quasiregular Mappings
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Problem Set 2
Winter 2009 / Vuorinen

1. Let f be an inversion in $S^{n-1}(a,r)$ as defined in 1.2(2)[CGQM]. Show that $f^{-1}=f$ and that $|x-a||f(x)-a|=r^2$ for all $x\in\mathbf{R}^n\setminus\{a\}$. By considering similar triangles show that the following identity holds for $x,y\in\mathbf{R}^n\setminus\{a\}$:

$$|f(x)-f(y)|=\frac{r^2|x-y|}{|x-a||y-a|}\ .$$

2. (a) For $0<t<1$ let $w(t)=t/\sqrt{1-t^2}$. Show that $q(0,w(t)e_1)=t$ and that

$$\frac{t}{s}<\frac{w(t)}{w(s)}<\frac{2t}{s}$$

for $0<s<t<\frac{1}{2}\sqrt{3}$.
(b) Let $q(A)=\sup\{q(x,y)\colon x,y\in A\}$ for $A\subset\overline{\mathbf{R}^n}$. Show that

$$q(Q(z,r))=q(\partial Q(z,r))=2r\sqrt{1-r^2}$$

for $0<r\le 1/\sqrt{2}$.

3.(a) Let $x,y\in\mathbf{B}^n$ with $s=q(0,x)$, $t=q(0,y)$. Show that

$$q(x,y)\ \le\ s\sqrt{1-t^2}+t\sqrt{1-s^2}\ \le\ t+s\ .$$

(b) Let $x,y\in\mathbf{R}^n\setminus\{0\}$ with $q(0,x)>q(0,y)$. Show that the strict inequality $q(x,y)>q(0,x)-q(0,y)$ holds.

4. For $x, y \in \mathbf{R}^n$ prove the following:

$$q(x,y) = \frac{|x-y|}{\sqrt{(1+|x||y|)^2 + (|x|-|y|)^2}}.$$

$$\frac{|x-y|}{\sqrt{|x-y|^2 + (1+|x||y|)^2}} \le q(x,y) \le \frac{|x-y|}{\sqrt{|x-y|^2 + (1-|x||y|)^2}}.$$

$$q(x,y) \le \frac{|x-y|}{2}$$

for $|x||y| \ge 1$, with equality for $x, y \in S^{n-1}$ or $x = y$.

$$q(x,y) \le |x-y|/(|x|+|y|),$$

with equality iff $|x||y| = 1$ or $x = y$.

5. Show that $B^n(a,r)$ and $B^n(v)$, where $r^2 < 1 + |a|^2$,

$$v = \frac{2r}{\sqrt{(1+(|a|+r)^2)(1+(|a|-r)^2)} + 1 + |a|^2 - r^2}$$

have equal spherical diameters. Note that $v < 1$. Conclusion: The inversion f_1 in 1.52[CGQM] is in fact the inversion in a euclidean sphere with radius v and center 0.

6. The lines $[-e_1, 0]$ and $[ae_1, \infty]$, $a > 0$, can be mapped onto $[-e_1, e_1]$ and $[be_1, \infty] \cup [-be_1, \infty]$ by a Möbius transformation. Give a definition for b in terms of a. Notice that $[x, \infty] = \{xt : t \ge 1\}$, if $x \in \mathbf{R}^n \setminus \{0\}$.

7. Are the following mappings Hölder or Lipschitz?

(a) $f\colon \mathbf{B}^2 \to \mathbf{R}^2$, $f(x,y) = \begin{cases} (x, y^2), & y > 0, \\ (x, -y^2), & y \le 0, \end{cases}$

(b) $f\colon \mathbf{R}^2 \to \mathbf{R}^2$, $f(x,y) = \begin{cases} (x, 2y), & y > 0, \\ (x, y), & y \le 0. \end{cases}$

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