Quasiregular Mappings
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Problem Set 1 (2009-01-19)
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1. Perhaps the simplest non-injective map is $f\colon z \mapsto z^2$. Find a domain $D \subset \mathbb{C}$ such that $f|_D\colon D \to fD$ is not closed.

2. Find a Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}, \quad ad-bc \neq 0,$$

which maps $H^2 = \{(x,y) \in \mathbf{R}^2 \colon y > 0\}$ onto $B^2 = \{(x,y) \in \mathbf{R}^2 \colon x^2 + y^2 < 1\}$ such that $(-1,0,1) \mapsto (1,i,-1)$ $[i = (0,1)]$.

3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be Hölder continuous with exponent $\beta > 1$. Show that f is a constant, equal to $f(0)$.

4.
Let $\sigma\colon \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, $\mathbf{R}_+ = (0,\infty)$, be defined by $\rho(x,y) = |\log(x/y)|$. Show that ρ is a metric.
(a) Suppose that $u\colon (\mathbf{R}_+,\rho) \to (\mathbf{R}_+,\rho)$ is uniformly continuous. Show that $u(x) \le Ax^B$ for some constants A and B for all $x \ge 1$.
(b) Suppose that $u\colon (\mathbf{R}_+,d) \to (\mathbf{R}_+,\rho)$ is uniformly continuous, where d is the euclidean metric. Find an inequality of the same type as in (a) [but some other function in place of Ax^B].

5. Let $D = H^2 = \{(x,y) \in \mathbf{R}^2 \colon y > 0\}$. The modulus $M(D;-1,0,s,\infty)$ of a quadrilateral $(D;-1,0,s,\infty)$, $s > 0$, is usually denoted $\tau(s)/2$ [we will define the function $\tau(s)$ later]. Fix $\alpha \in (0,\pi)$ and denote $D_\alpha = \{z \in \mathbf{C} \colon 0 < \arg z < \alpha\}$, $r > s > 0$, $t > u > 0$. Using the above notation, find $M(D_\alpha; re^{i\alpha}, se^{i\alpha}, u, t)$. [Hint: The modulus is a conformal invariant. Apply an auxiliary Möbius transformation to map the example to the previous case.]

6. Are the following mappings Hölder or Lipschitz?
(a) $f\colon \mathbf{R}^2 \to \mathbf{R}^2$, $f(x,y) = \begin{cases} (x,y^2), & y > 0, \\ (x,-y^2), & y \le 0, \end{cases}$
(b) $f\colon \mathbf{R}^2 \to \mathbf{R}^2$, $f(x,y) = \begin{cases} (x,2y), & y > 0, \\ (x,-2y), & y \le 0, \end{cases}$
(c) Let (r,φ,z) be the cylinderical coordinates of $\mathbf{R}^3 (r \ge 0, 0 \le \varphi < 2\pi, z \in \mathbf{R})$. Fix $0 < \alpha < \beta < \pi$ and define $f(r,\varphi,z) = (r, \varphi\beta/\alpha, z)$ if $0 \le \varphi \le \alpha$ and $f(r,\varphi,z) = (r, \beta + (\varphi-\alpha)(2\pi-\beta)/(2\pi-\alpha), z)$ if $\varphi \in (\alpha, 2\pi)$. Then f defines a mapping $f\colon \mathbf{R}^3 \to \mathbf{R}^3$.

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