Quasiregular Mappings
Department of Mathematics and Statistics
University of Helsinki
Problem Set 11
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1. Let $c \geq 1$, and let $G \subset \mathbf{R}^\mathbf{n}$ be an open set. A positive, continuous function $u : G \to R_+ \setminus \{0\}$ is called a *c-Harnack function* if the inequality

$$\sup_{\mathbf{B}^n(x,r)} u(z) \leq c \inf_{\mathbf{B}^n(x,r)} u(z)$$

holds whenever $\mathbf{B}^n(x, 2r) \subset G$. Well known examples of functions satisfying Harnack's inequality are positive harmonic functions in the plane.

(a) Let $u(z) = \arg z$ and $G = \mathbb{C} \setminus \{x \in \mathbf{R} \colon \mathrm{x} \geq \mathbf{0}\}$. Find a constant $c \geq 1$ such that $u(z)$ is c-Harnack in G.

(b) Let $K \subset G$ be compact and $u(z)$ as in (a). Does there exist a constant D depending on $d(K)/d(K, \partial G)$ such that

$$u(z_1) \leq D\, u(z_2)$$

for all $z_1, z_2 \in K$? Hint. Show that in the domain of part (a) there are compact sets K such that the quasihyperbolic diameter of K does not have a majorant in terms of $d(K)/d(K, \partial G)$.

(c) Let $K \subset G$ be compact. Show that $u(x) = \exp(-k_G(x, K))$ satisfies the Harnack inequality.

Solution

(a) We investigate the problem in $G := \mathbb{C} \setminus \{te_1 \mid t \geq 0\}$, where $z \longmapsto \arg z$ is continuous. Let $z = (r\cos\varphi, r\sin\varphi) \in G$, $t > 0$ such that $\mathbf{B}^n(z, 2t) \subset G$. Let ω be as in the figure.

Assume $\varphi \in (0, \pi/2)$. Since $\mathbf{B}^n(z, 2t) \subset G$, we have that $t < (r/2)\sin\varphi$. Hence

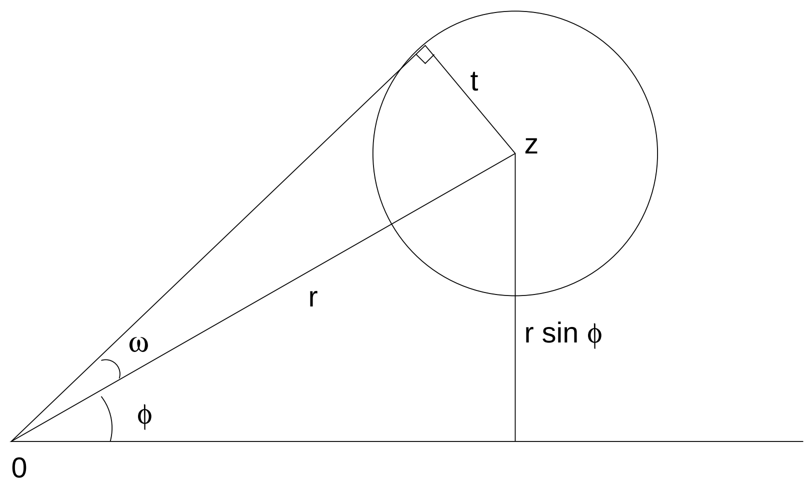
$$\begin{aligned} \sin\omega &= \frac{t}{r} < \frac{\sin\varphi}{2} \in (0, \frac{1}{2}) \\ \Rightarrow \quad \omega &< \arcsin\left(\frac{\sin\varphi}{2}\right) < \frac{\pi}{2}\frac{\sin\varphi}{2} < \frac{\pi\varphi}{4}. \end{aligned}$$

Now

$$\frac{\sup_{\zeta \in B^n(z,t)} \arg\zeta}{\inf_{\zeta \in B^n(z,t)} \arg\zeta} = \frac{\varphi + \omega}{\varphi - \omega}$$

is increasing with respect to ω and

$$\frac{\varphi + \omega}{\varphi - \omega} \leq \frac{\varphi + \frac{\pi\varphi}{4}}{\varphi - \frac{\pi\varphi}{4}} = \frac{1 + \frac{\pi}{4}}{1 - \frac{\pi}{4}} = \frac{4 + \pi}{4 - \pi} \approx 7.66\,.$$



If $\varphi \in [\pi/2, 2\pi)$, then $\sup_{\zeta \in B^n(z,t)} \arg \zeta \le 2\pi$ and $r \ge 2t$ imply that

$$\begin{aligned} & \sin \omega = \frac{t}{r} \le \frac{t}{2t} = \frac{1}{2} \Rightarrow \omega \le \arcsin \frac{1}{2} = \frac{\pi}{6} \\ \Rightarrow \quad & \inf_{\zeta \in B^n(z,t)} \arg \zeta \ge \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \\ \Rightarrow \quad & \frac{\sup_{\zeta \in B^n(z,t)} \arg \zeta}{\inf_{\zeta \in B^n(z,t)} \arg \zeta} \le \frac{2\pi}{\frac{\pi}{3}} = 6 \end{aligned}$$

Therefore $\arg z$ is $(4+\pi)/(4-\pi)$-Harnack in G.

(b) Let G be as in (a) and choose $K_r = \{z_r, \overline{z_r}\}$ for $z_r = 1 + ir$ and $r > 0$. Now K_r is compact but not connected. Moreover, $d(K_r)/d(K_r, \partial G) = 2$ for all r.

For small r $u(\overline{z_r}) \approx 2\pi - r$ and $u(z_r) \approx r$ and therefore

$$\frac{u(\overline{z_r})}{u(z_r)} \approx \frac{2\pi - r}{r} \to \infty$$

as $r \to 0$.

(c) Let $K \subset G$ be compact; $z \in G$, $r > 0$ such that $\mathbf{B}^n(z, 2r) \subset G$; $x, y \in \mathbf{B}^n(z, r)$. Since by 3.7(1)[CGQM]

$$k_G(\mathbf{B}^n(z,r)) \le 2 \log \left(1 + \frac{r}{2r - r} \right),$$

it follows that

$$\begin{aligned} k_G(y, K) &\le k_G(x, K) + k_G(\overline{\mathbf{B}}^n(z,r)) \\ &\le k_G(x, K) + 2 \log \left(1 + \frac{r}{2r - r} \right) \\ &= k_G(x, K) + 2 \log 2. \end{aligned}$$

Hence

$$u(x) = e^{-k_G(x,K)} \le e^{-k_G(y,K)+\log 4} = 4u(y)$$

and therefore

$$\sup_{\zeta \in \mathbf{B}^n(z,r)} u(\zeta) \le 4 \inf_{\zeta \in \mathbf{B}^n(z,r)} u(\zeta).$$

2. Let $G, G' \subset \mathbf{R^n}$ and $f : (G, k_G) \to (G', k'_G)$ be uniformly continuous, and let $b' \in \partial G'$. Show that $u : G \to R_+, u(x) = |f(x) - b'|$ satisfies Harnack's inequality.

Solution From now on, let $z \in G$, $r > 0$ be such that $\mathbf{B}^n(z, 2r) \subset G$. It follows from 3.7(1)[CGQM], that for all $x, y \in \mathbf{B}^n(z, r)$

$$k_G(x,y) \le k_G(x,z) + k_G(z,y) \le 2\log\left(1 + \frac{r}{2r - r}\right) = \log 4.$$

Furthermore, 2.36(1)[CGQM] and (3.4)[CGQM] imply that for all $x, y \in G$,

$$\left|\log\frac{|f(x) - b'|}{|f(y) - b'|}\right| \le j_{G'}(f(x), f(y)) \le k_{G'}(f(x), f(y)).$$

Since $f \colon (G, k_G) \to (G', k_{G'})$ is uniformly continuous, there exists a homeomorphism (" modulus of continuity ") $\omega_1 \colon (0,\infty) \to (0,\infty)$ such that

$$k_{G'}(f(x), f(y)) \le \omega_1(k_G(x,y))$$

for all $x, y \in G$. Hence

$$\left|\log\frac{|f(x) - b'|}{|f(y) - b'|}\right| \le \omega_1(\log 4)$$
$$\Rightarrow \quad |f(x) - b'| \le e^{\omega_1(\log 4)}|f(y) - b'|$$

for all $x, y \in \mathbf{B}^n(z, r)$. $\therefore u$ satisfies the Harnack inequality with $c = e^{\omega_1(\log 4)}$.

3. Let $E \subset \mathbf{B}^n$ be compact. Suppose that

$$m_n(E_k) = a_k,\ E_k = \{x \in \mathbf{R^n} \setminus \mathbf{E} : \mathbf{2^{-k-1}} < \mathbf{d(x,E)} < \mathbf{2^{-k}}\}, \mathbf{k} = \mathbf{1}, \mathbf{2}, \ldots.$$

Use Lemma 5.24[CGQM] to find an upper bound for $\mathsf{M}(\Delta(E, S^{n-1}(2)))$. Apply your bound to give a sufficient condition for $\operatorname{cap} E = 0$ in terms of the numbers (a_k).

Solution Denote $\Gamma = \Delta(E, S^{n-1}(2))$ and by $\Gamma_k = \{\gamma \in E_k \colon l(\gamma \geq 2^{-k-1})\}$. Then Γ_i, Γ_j are separate for $i \neq j$. Since $E \subset \mathbf{B}^n$,

$$E \cup \bigcup_{k=0}^{\infty} E_k \subset \mathbf{B}^n(2).$$

Hence $\Gamma_k < \Gamma$ for all k. By 5.5[CGQM],

$$\mathsf{M}(\Gamma_k) \leq \frac{m(E_k)}{(2^{-k-1})^n} = a_k 2^{n(k+1)}.$$

Using 5.24[CGQM], we get

$$\begin{aligned} \mathsf{M}(\Gamma)^{1/(1-n)} &\geq \sum_{k=0}^{\infty} \mathsf{M}(\Gamma_k)^{1/(1-n)} \geq \sum_{k=0}^{\infty} \frac{1}{a_k^{1/(n-1)} 2^{n(k+1)/(n-1)}} \\ \therefore \quad \mathsf{M}(\Gamma) &\leq \left(\sum_{k=0}^{\infty} \frac{1}{a_k^{1/(n-1)} 2^{n(k+1)/(n-1)}} \right)^{1-n}. \end{aligned}$$

A sufficient condition for cap $E = 0$ is that the series above diverges. Assume $a_k/a_{k+1} \to c > 2^n$, as $k \to \infty$. Then

$$\begin{aligned} \frac{\left|a_{k+1}^{1/(1-n)} 2^{n(k+2)/(1-n)}\right|}{\left|a_k^{1/(1-n)} 2^{n(k+1)/(1-n)}\right|} &= \left| \left(\frac{a_k}{a_{k+1}}\right)^{1/(n-1)} 2^{n/(1-n)} \right| \\ &= \left| \left(\frac{1}{2^n}\frac{a_k}{a_{k+1}}\right)^{1/(n-1)} \right| \to \left(\frac{c}{2^n}\right)^{1/(n-1)} > 1 \end{aligned}$$

as $k \to \infty$. Hence, by the ratio test,

$$\sum_{k=0}^{\infty} a^{1/(1-n)} 2^{n(k+1)/(1-n)} = \infty$$

which implies $\mathsf{M}(\Gamma) = 0$ and cap $E = 0$.

$\therefore$ $\lim_{k\to\infty} a_k/a_{k+1} \to c > 2^n$ is a sufficient condition.

4. Let $G = \mathbf{B}^n \setminus \{0\}, f : G \to G' = f(G)$, be a homeomorphism with the property that there exist curves $\alpha_j : [0,1) \to G, j = 1,2$, such that $\alpha_j(t) \to 0, f(\alpha_j(t)) \to \beta_j \in \partial G', t \to 1$. Show that $\beta_1 = \beta_2$ if there exists $C \geq 1$ with $k_{G'}(f(x), f(y)) \leq C k_G(x,y)$ for all $x, y \in G$. Show that $\beta_1 = \beta_2$ also holds if there exists $K \geq 1$ such that $\mathsf{M}(\Gamma) \leq K\mathsf{M}(f\Gamma) \leq K^2\mathsf{M}(\Gamma)$ for all curve families Γ in G.

Solution

Case k_G: Assume $\beta_1 \neq \beta_2$, $|\beta_1 - \beta_2| = a > 0$. For t, $0 < t < t_0 = (1/2)\min\{|\alpha_1(0)|, |\alpha_2(0)|\}$, choose $x_t \in |\alpha_1|$, $y_t \in |\alpha_2|$ such that $|x_t| = |y_t| = t$, $f(x_t) \to \beta_1$, $f(y_t) \to \beta_2$. Then, by (3.4)[CGQM],

$$\begin{aligned} k_{G'}(f(x_t), f(y_t)) &\geq j_{G'}(f(x_t), f(y_t)) \\ &= \log\left(1 + \frac{|f(x_t) - f(y_t)|}{\min\{d(f(x_t)), d(f(y_t))\}}\right) \to \infty \end{aligned}$$

as $t \to 1$, since $|f(x_t) - f(y_t)| \to a > 0$ and $d(f(x_t)), d(f(y_t)) \to 0$.

On the other hand, $|x_t|, |y_t| < 1/2$ for all $t \in (0, t_0)$, so $d(x_t, \partial G) = d(y_t, \partial G) = t$. Let J_t be the shortest subarc of $S^{n-1}(t)$ joining x_t and y_t. Then

$$k_G(x_t, y_t) \leq \int_{J_t} \frac{|dz|}{d(z, \partial G)} \leq \frac{\pi t}{t} = \pi.$$

This yields a contradiction since $k_{G'}(f(x_t), f(y_t)) \leq C k_G(x_t, y_t)$.

Case $\mathsf{M}(\Gamma)$: Assume $\beta_1 \neq \beta_2$, $|\beta_1 - \beta_2| = a > 0$. Denote

$$\begin{aligned} E_1' &= |f \circ \alpha_1| \cap \mathbf{B}^n(\beta_1, \tfrac{a}{3}), \\ E_2' &= |f \circ \alpha_2| \cap \mathbf{B}^n(\beta_2, \tfrac{a}{3}), \\ \Gamma' &= \Delta(E_1', E_2'; G'), \\ E_1 = f^{-1}E_1',\ E_2 &= f^{-1}E_2',\ \Gamma = \Delta(E_1, E_2; G). \end{aligned}$$

Let $b = \max\{t \mid S^{n-1}(t) \cap E_1 \neq \emptyset \neq S^{n-1}(t) \cap E_2\}$. Then, by 5.2[CGQM] and 5.32[CGQM], for all $\delta > 0$,

$$\begin{aligned} \mathsf{M}(\Delta(E_1, E_2; G)) &\geq \mathsf{M}(\Delta(E_1, E_2; \mathbf{B}^n(b) \setminus \overline{\mathbf{B}}^n(\delta))) \\ &\geq c_n \log\frac{b}{\delta} \to \infty \end{aligned}$$

as $\delta \to 0$. Hence $\mathsf{M}(\Delta(E_1, E_2; G)) = \infty$.

On the other hand, the Euclidean balls $\mathbf{B}^n(\beta_1, \frac{a}{3})$ and $\mathbf{B}^n(\beta_2, \frac{a}{3})$ can be mapped by a Möbius transformation onto $\mathbf{B}^n(0, R)$ and $\mathbf{B}^n(0, r)$ respectively. Since r and R depend only on a we have $M(\Delta(E_1', E_2')) < \infty$. Hence

$$\infty = \mathsf{M}(\Delta(E_1, E_2; G)) \leq K\mathsf{M}(\Delta(E_1', E_2'; G')) \leq KM(\Delta(E_1', E_2')) < \infty,$$

a contradiction.

5. Let $D = \{z \in \mathbb{C} \colon 0 < \arg z < \theta, 0 < |z| < 1\}$, $z_1 = 1$, $z_2 = e^{i\alpha}$ for $0 < \alpha < \theta$, $z_3 = e^{i\theta}$ and $z_4 = 0$. Find the modulus of the quadrilateral $(D; z_1, z_2, z_3, z_4)$.

In other words, find a conformal map of D onto $\{z \in \mathbb{C} \colon \mathrm{Im} z > 0\}$ such that the points z_k are mapped onto the real axis and compute the cross ratio of these points.

Solution Let us first map D on to the upper half of the unit disk by $f_1(z) = z^{\pi/\theta}$. Then map the upper half of the unit disk onto the first quadrant by $f_2(z) = (1+z)/(1-z)$. Finally map the first quadrant to a halfplane by $f_3(z) = z^2$. Now the mapping $f(z) = f_3(f_2(f_1(z)))$ maps D to halfplane so that $f(z_1) = \infty$, $f(z_3) = 0$, $f(z_4) = 1$ and

$$f(z_2) = \left(\frac{1 + e^{(i\alpha\pi)/\theta}}{1 - e^{(i\alpha\pi)/\theta}} \right)^2 .$$

Now $M(D; z_1, z_2, z_3, z_4) = \tau(|z_1, z_2, z_3, z_4|)/2 = \tau(|f(z_2) - 1|)/2$.