Quasiregular Mappings
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Problem Set 10
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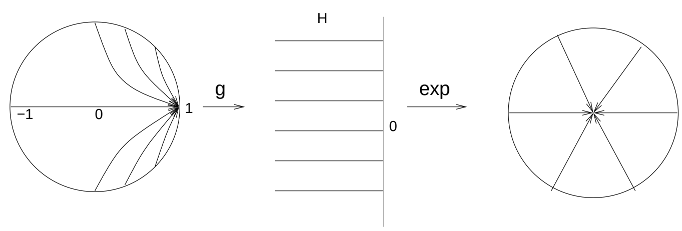
1. Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous. The *cluster set* of f at a point $b \in \partial G$ is the set $C(f,b) = \{b' \in \overline{\mathbf{R}}^n : \exists (b_k) \in G^n, b_k \to b, f(b_k) \to b'\}$. It is clear that $C(f,b) \subset \overline{G'}$, and that for injective maps $C(f,b) \subset \partial G'$. The cluster set $C(f,b)$ is a singleton iff f has a limit at b. The cluster set is connected if there are arbitrarily small numbers $t > 0$ such that $B(b,t) \cap G$ is connected. We say that f is *boundary preserving* if $C(f,b) \subset \partial G'$ for all $b \in \partial G$.

(a) Find for each $b \in S^1$ the cluster set $C(f,b)$ of the analytic function $f : B^2 \to B^2$, with $f(z) = \exp g(z)$ when $g(z) = -(1+z)/(1-z), z \in B^2$.

(b) Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be open and continuous. Show that f is boundary preserving iff f is proper.

Solution

(a)



Denote $H = \{z \in \mathbb{C} \ : \ \mathrm{Re} z < 0\}$. First we find $C(f,1)$. It is clear that

$$C(f,1) = \bigcap_{\epsilon > 0} \overline{f(B^2 \cap B^2(1,\epsilon))}.$$

We will show that $\therefore C(f,1) = \overline{B^2}$ which is equivalent to

$$\forall w \in \overline{B^2},\ \exists z_k \in B^2,\ z_k \to 1 \text{ such that } f(z_k) \to w.$$

If $z \in B^2 \setminus \{0\}$ then there exists sequence (z_k) such that $z_k \to 1$ and $f(z_k) = w$ for all k. Therefore $B^2 \setminus \{0\} \subset C(f,1)$.

If $z \in \{0\} \cup S^1$, then there exists (w_k) such that $w_k \to w$ and $w_k \in B^2 \setminus \{0\}$. There exists $z_k \in B^2 \setminus \{0\}$ such that $|z_k - 1| < 1/k$ and $f(z_k) = w_k$. Clearly $f(z_k) \to w$ and $z_k \to 1$ as $k \to \infty$. Therefore $\{0\} \cup S^1 \subset C(f,1)$.

Let $b = \cos\theta + i\sin\theta$, for $\theta \in (0, 2\pi)$. Since $f(x)$ is continuous is in $\overline{B^2} \setminus \{1\}$, the cluster set is a singleton.

$$C(f,b) = f(b) = \frac{1}{e^{\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}}},$$

where

$$\begin{aligned}
\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} &= \frac{(1+\cos\theta+i\sin\theta)(1-\cos\theta-i\sin\theta)}{(1-\cos\theta)^2-(i\sin\theta)^2} \\
&= \frac{(1+i\sin\theta)^2-\cos^2\theta}{1-2\cos\theta+\cos^2\theta+\sin^2\theta} = \frac{1+2i\sin\theta-\sin^2\theta-\cos^2\theta}{2-2\cos\theta} \\
&= \frac{i\sin\theta}{1-\cos\theta}. \\
\therefore C(f,b) &= \left\{e^{\frac{i\sin\theta}{\cos\theta-1}}\right\},\ \theta = \arg(b)\ \forall b \in S^1 \setminus \{1\}.
\end{aligned}$$

(b) Assume f is boundary preserving. Let $E \subset G'$ be compact. Consider a sequence (b_k) in $f^{-1}E$ with $b_k \to b \in \overline{G}$. We must show that $b \in f^{-1}E$. There are two cases:

Case 1 $b \in \partial G$. By passing to a subsequence we may assume that $f(b_k) \to b'$. Then $b' \in C(f,b) \subset \partial G'$. But $f(b_k) \in E$ for all k, which implies $b' \in E \subset G'$, since E is compact. This is a contradiction.

Case 2 $b \in G$. Now $f(b_k) \to f(b)$ by continuity and so $f(b) \in E$ as E is compact. Hence $b \in f^{-1}E$.

For the converse implication assume that f is proper. Let $b \in \partial G$ and $b' \in C(f,b)$. Choose a sequence $(b_k) \in G$ with $b_k \to b$ and $f(b_k) \to b'$. If we had $b' \in G'$, then $E = \{b'\} \cup \{f(b_k)\ :\ k \geq 1\} \subset G'$ would be compact, while $f^{-1}E$ would not, since $b \in \partial G$. This is a contradiction because f is proper. $\therefore b' \in \partial G'$ and f is boundary preserving.

2. Let $f : \mathbf{B}^n \to f(\mathbf{B}^n) \subset \mathbf{R^n}$ be a homeomorphism with the property that there exists a number $K \geq 1$ such that for all $x, y \in \mathbf{B}^n$ $\mu_{f(\mathbf{B}^n)}(f(x), f(y)) \leq K\mu_{\mathbf{B}^n}(x,y)$, and let (b_n) be a sequence of points in $\mathbf{B}^n$ such that $b_k \to b \in \partial\mathbf{B}^n$ and $f(b_k) \to \beta$. (It is known, that $\partial f\mathbf{B}^n$ is connected, cf. **1**.) Let $\rho(a_k, b_k) < M\ \forall k$. Show that $\lim_{k\to\infty} f(a_k) = \beta$ exists. Does the same conclusion hold for noninjective mappings?

Solution Since $\partial f B^n$ is connected, we may use 8.31[CGQM], the assumptions, and 8.6[CGQM] to obtain

$$\begin{aligned} j_{fB^n}(f(a_k), f(b_k)) &\le \frac{1}{c_n}\mu_{fB^n}(f(a_k), f(b_k)) \le \frac{K}{c_n}\mu_{B^n}(a_k, b_k) \\ &= \frac{K}{c_n}\gamma\left(\frac{1}{\operatorname{th}(\rho(a_k, b_k)/2)}\right) \le \frac{K}{c_n}\gamma\left(\frac{1}{\operatorname{th}(M/2)}\right) < \infty. \end{aligned}$$

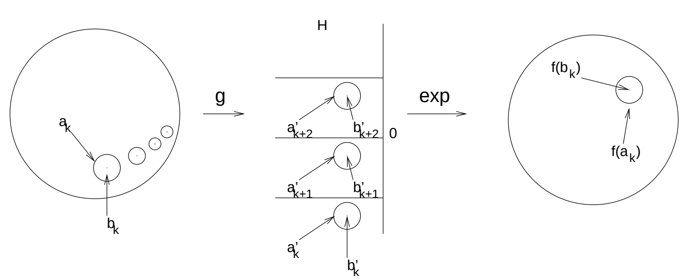
Let α be an accumulation point of the sequence $(f(a_k))$. Assume that $q(\alpha, \beta) =: 3\varepsilon > 0$.

Let $k_0 \in \mathbb{N}$ be such that $q(f(a_k), \beta) < \varepsilon$ and $q(f(a_k), \alpha) < \varepsilon$ whenever $k \ge k_0$. Then, for $k \ge k_0$,

$$\begin{aligned} j_{fB^n}(f(a_k), f(b_k)) &= \log\left(1 + \frac{|f(a_k) - f(b_k)|}{\min\{d(f(a_k), \partial f B^n), d(f(b_k), \partial f B^n)\}}\right) \\ &\ge \log\left(1 + \frac{q(f(a_k), f(b_k))}{d(f(b_k), \partial f B^n)}\right) \\ &\overset{\Delta-\text{ineq}}{\ge} \log\left(1 + \frac{q(\alpha, \beta) - q(f(a_k), \alpha) - q(f(b_k), \beta)}{d(f(b_k), \partial f B^n)}\right) \\ &\ge \log\left(1 + \frac{\varepsilon}{d(f(b_k), \partial f B^n)}\right) \to \infty \end{aligned}$$

as $k \to \infty$, since $f(b_k) \to \beta \in \partial f B^n$ and $\varepsilon > 0$. This is a contradiction. Hence $\alpha = \beta$ and the proof is complete.

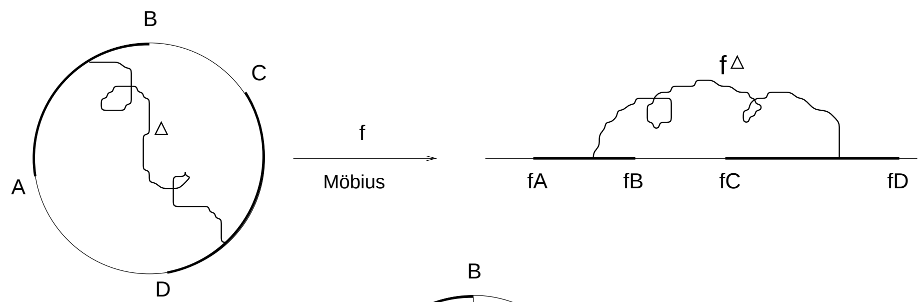
Note The conclusion does not hold for noninjective mappings: Let f be as in 1.(a). Choose a_k,b_k as in the following figure:

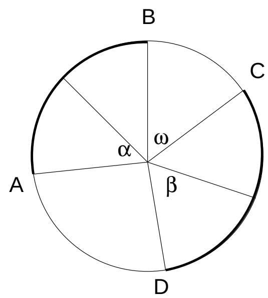


Now $a_k, b_k \to 1 \in \partial B^2$, $\rho(a_k, b_k) < M$, $\beta := f(b_k) = \lim_{k\to\infty} f(b_k)$, but $f(a_k) \not\to \beta$.

3. Let A, B, C, D be distinct points on the unit circle S^1 in the stated order and 2α and 2β the lengths of the arcs AB and CD, respectively. Find the least value of $\mathsf{M}\big(\Delta(AB, CD)\big)$. [Hint: $|A-C||B-D| = |A-B||C-D| + |B-C||A-D|$ by Ptolemy's theorem [CG, p. 42], [BER, 10.9.2].]

Solution Map S^1 onto $\mathbf{R}$ by a Möbius map f, $f(D) = \infty$. Then





$$\begin{aligned} 2\mathsf{M}(\Delta) &= 2\mathsf{M}(f\Delta) = \tau\left(\frac{|fC - fB|}{|fB - fA|}\right) = \tau(|fB, fA, fC, fD|) \\ &= \tau(|B, A, C, D|) = \tau\left(\frac{|B-C||A-D|}{|A-B||C-D|}\right) \end{aligned}$$

by the Möbius invariance of the modulus and the cross ratio. Now

$$\begin{cases} \sin\alpha = \frac{1}{2}|A-B| \\ \sin\beta = \frac{1}{2}|C-D| \end{cases} \Rightarrow \begin{cases} |A-B| = 2\sin\alpha \\ |C-D| = 2\sin\beta \end{cases}$$

and τ is strictly decreasing. Hence minimizing $M(\Delta)$ is equivalent to maximizing $\frac{|B-C||A-D|}{|A-B||C-D|}$ which is equivalent to maximizing $|B-C||A-D|$ which is equivalent, by Ptolemy, to maximizing $|A-C||B-D|$.

Let w be the length of the arc BC. Then

$$\begin{aligned} |A-C| &= 2\sin\left(\frac{2\alpha+\omega}{2}\right) \\ |B-D| &= 2\sin\left(\frac{2\beta+\omega}{2}\right). \end{aligned}$$

It follows that

$$\begin{aligned}|A-C||B-D| &= 4\sin(\alpha+\omega/2)\sin(\beta+\omega/2)\\ &= 2(\cos(\alpha+\omega/2-\beta-\omega/2)-\cos(\alpha+\beta+\omega))\\ &= 2(\cos(\alpha-\beta)-\cos(\alpha+\beta+\omega))\end{aligned}$$

where we used the formula $\sin(a)\sin(b)=(1/2)(\cos(a-b)-\cos(a+b))$. This is maximized when $\cos(\alpha+\beta+\omega)=-1$ which is equivalent to $\alpha+\beta+\omega=\pi$ and furthermore equivalent to $\omega=\pi-\alpha-\beta$.

Then, by Ptolemy's theorem,

$$\begin{aligned}|B,A,C,D| &= \frac{|A-C||B-D|-|A-B||C-D|}{|A-B||C-D|}\\ &= \frac{2(\cos(\alpha-\beta)+1)-4\sin\alpha\sin\beta}{4\sin\alpha\sin\beta}\\ &= \frac{\cos(\alpha-\beta)+1}{2\sin\alpha\sin\beta}-1.\\ \therefore \min \mathsf{M}(\Delta(AB,CD)) &= \tau\left(\frac{\cos(\alpha-\beta)+1}{2\sin\alpha\sin\beta}-1\right)\frac{1}{2}.\end{aligned}$$

4. Let $E\subset\mathbf{R^n}$ be compact, $\operatorname{cap} E>0$ and $E(t)=\cup_{x\in E}\mathbf{B}^n(x,t)$. It follows from Ziemer's theorem that for a fixed $t>0$ $\operatorname{cap}(E(t),\overline{E(s)})\to\operatorname{cap}(E(t),E)$, $s\to 0$. Show that $\operatorname{cap}(E(t),E)\to\infty$, when $t\to 0$. [Hint: Ziemer's theorem and 5.24[CGQM] may be helpful here.]

Solution The function $h(t)=\operatorname{cap}(\mathrm{E(t),E})$ is decreasing by 5.3[CGQM]. Hence there exists the limit $\lim_{t\to 0+}h(t)=a\in(0,\infty]$. Assume that $a<\infty$. Denote for $t>0$, $0<s<r$, $\Gamma_t=\Delta(E,\partial E(t))$ and $\Gamma_{sr}=\Delta(\partial E(s),\partial E(r);\overline{E(r)}\setminus E(s))$. By Lemma 5.24[CGQM], we have for $0<s<r$,

$$\mathsf{M}(\Gamma_r)^{1/(1-n)}\geq \mathsf{M}(\Gamma_s)^{1/(1-n)}+\mathsf{M}(\Gamma_{sr})^{1/(1-n)},$$

where $\mathsf{M}(\Gamma_s)\to a$, and $\mathsf{M}(\Gamma_{sr})\to\mathsf{M}(\Gamma_r)$, as $s\to 0$ (Ziemer). This implies that $a^{1/(1-n)}\leq 0$ and therefore $a\leq 0$, which is a contradiction. $\therefore a=\infty$.

5. Let $f:\mathbf{B}^n\to\mathbf{B}^n$ be a homeomorphism with $f(0)=0$ and assume that there is $K\geq 1$ such that for all distinct $x,y\in\mathbf{B}^n$

$$\lambda_{\mathbf{B}^n}(x,y)/K\leq\lambda_{f\mathbf{B}^n}(f(x),f(y))\leq K\lambda_{\mathbf{B}^n}(x,y).$$

Prove that there are $a,b,c,d>0$ such that $a|x|^b\leq|f(x)|\leq c|x|^d$ for all $x\in\mathbf{B}^n$.

Solution By 8.6(2)[CGQM],

$$\lambda_{B^n}(x,0) = \frac{1}{2}\tau_n\left(\mathrm{sh}^2\left(\frac{1}{2}\rho(x,0)\right)\right),$$

and by 8.7[CGQM],

$$c_n \log\frac{1}{\mathrm{th}\,\frac{1}{4}\rho(x,0)} \le \frac{1}{2}\tau_n\left(\mathrm{sh}^2\left(\frac{1}{2}\rho(x,0)\right)\right) \le c_n\log\frac{2}{\mathrm{th}\,\frac{1}{4}\rho(x,0)}.$$

Using 2.29(2)[CGQM] and (2.18)[CGQM], we get

$$\mathrm{th}\,\frac{1}{4}\rho(x,0) = \frac{\mathrm{th}\,\frac{1}{2}\rho(x,0)}{1+\sqrt{1-\mathrm{th}^2\,\frac{1}{2}\rho(x,0)}}$$

and

$$\begin{aligned}\mathrm{th}^2\,\frac{1}{2}\rho(x,0) &= \frac{\mathrm{sh}^2\,\frac{1}{2}\rho(x,0)}{\mathrm{ch}^2\,\frac{1}{2}\rho(x,0)} = \frac{\mathrm{sh}^2\,\frac{1}{2}\rho(x,0)}{1+\mathrm{sh}^2\,\frac{1}{2}\rho(x,0)}\\ &= \frac{\frac{|x|^2}{1-|x|^2}}{1+\frac{|x|^2}{1-|x|^2}} = \frac{|x|^2}{1-|x|^2+|x|^2} = |x|^2.\end{aligned}$$

Hence $\mathrm{th}\,\frac{1}{4}\rho(x,0) = |x|/(1+\sqrt{1-|x|^2})$ and

$$c_n\log\frac{1+\sqrt{1-|x|^2}}{|x|} \le \lambda_{B^n}(x,0) \le c_n\log\frac{2(1+\sqrt{1-|x|^2})}{|x|}.$$

Since $fB^n = B^n$, $f(0)=0$, we get $\lambda_{fB^n}(f(x),f(0)) = \lambda_{B^n}(f(x),0)$. Consequently,

$$\begin{aligned} & c_n\log\frac{1+\sqrt{1-|f(x)|^2}}{|f(x)|} \le Kc_n\log\frac{2(1+\sqrt{1-|x|^2})}{|x|}\\ \Rightarrow\quad & \frac{|f(x)|}{1+\sqrt{1-|f(x)|^2}} \ge \frac{1}{2^K}\frac{|x|^K}{(1+\sqrt{1-|x|^2})^K}\\ \Rightarrow\quad & |f(x)| \ge \frac{1}{2^K}\frac{|x|^K}{2^K} = \frac{1}{2^{2K}}|x|^K\end{aligned}$$

and

$$\begin{aligned}
& c_n \log \frac{1+\sqrt{1-|x|^2}}{|x|} \leq K c_n \log \frac{2(1+\sqrt{1-|f(x)|^2})}{|f(x)|} \\
\Rightarrow \quad & \frac{|x|}{1+\sqrt{1-|x|^2}} \geq \frac{|f(x)|^K}{2^K(1+\sqrt{1-|f(x)|^2})^K} \\
\Rightarrow \quad & |x| \geq \frac{|f(x)|^K}{2^{2K}} \\
\Rightarrow \quad & |f(x)| \leq (2^{2K}|x|)^{1/K} = 4|x|^{1/K} \\
\therefore \quad & \frac{1}{2^{2K}}|x|^K \leq |f(x)| \leq 4|x|^{1/K} \quad \forall x \in B^n.
\end{aligned}$$

6. In complex notation, Möbius transformations are defined by $T(z) = \frac{az+b}{cz+d}$ with $\Delta = ad - bc \neq 0$. These mappings generate a group.

(a) Prove that $T(z_1) - T(z_2) = \frac{\Delta(z_1-z_2)}{(cz_1+d)(cz_2+d)}$.

(b) Prove that the cross ratio $[z_1, z_2, z_3, z_4] = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}$ is invariant under T.

(c) Prove that $\frac{T''(z)}{T'(z)} = -\frac{2c}{cz+d}$, $D(\frac{T'(z)}{T''(z)}) = -\frac{1}{2}$, and $S_T = 0$,

$$S_T = \frac{T'''(z)}{T'(z)} - \frac{3}{2}\left(\frac{T''(z)}{T'(z)}\right)^2 = \left(\frac{T''(z)}{T'(z)}\right)' - \frac{1}{2}\left(\frac{T''(z)}{T'(z)}\right)^2.$$

Solution

(a)

$$\begin{aligned}
T(z_1) - T(z_2) &= \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d} \\
&= \frac{(az_1+b)(cz_2+d) - (az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)} \\
&= \frac{acz_1z_2 + bcz_2 + adz_1 + bd - acz_1z_2 - bcz_1 - adz_2 - bd}{(cz_1+d)(cz_2+d)} \\
&= \frac{(ad-bc)z_1 + (bc-ad)z_2}{(cz_1+d)(cz_2+d)} \\
&= \frac{\Delta(z_1-z_2)}{(cz_1+d)(cz_2+d)}.
\end{aligned}$$

(b)

$$\begin{aligned}
[T(z_1), T(z_2), T(z_3), T(z_4)] &= \frac{(T(z_1)-T(z_3))(T(z_2)-T(z_4))}{(T(z_1)-T(z_2))(T(z_3)-T(z_4))} \\
&\overset{(a)}{=} \frac{\frac{\Delta(z_1-z_3)}{(cz_1+d)(cz_3+d)}\frac{\Delta(z_2-z_4)}{(cz_2+d)(cz_4+d)}}{\frac{\Delta(z_1-z_2)}{(cz_1+d)(cz_2+d)}\frac{\Delta(z_3-z_4)}{(cz_3+d)(cz_4+d)}} \\
&= \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)} = [z_1, z_2, z_3, z_4].
\end{aligned}$$

(c)

$$\begin{aligned}
T'(z) &= \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = \frac{\Delta}{(cz+d)^2} \\
T''(z) &= \frac{-2c(cz+d)\Delta}{(cz+d)^4} = \frac{-2c\Delta}{(cz+d)^3} \\
T'''(z) &= \frac{-3c(cz+d)^2(-2c\Delta)}{(cz+d)^6} = \frac{6c^2\Delta}{(cz+d)^4}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{T''(z)}{T'(z)} &= \frac{\frac{-2c\Delta}{(cz+d)^3}}{\frac{\Delta}{(cz+d)^2}} = \frac{-2c}{cz+d}\ ; \\
D\left(\frac{T'(z)}{T''(z)}\right) &= D\left(\frac{\frac{\Delta}{(cz+d)^2}}{\frac{-2c\Delta}{(cz+d)^3}}\right) \\
&= D\left(\frac{cz+d}{-2c}\right) = D\left(-\frac{1}{2}z - \frac{d}{2c}\right) = -\frac{1}{2}\ ; \\
\frac{T'''(z)}{T'(z)} - \frac{3}{2}\left(\frac{T''(z)}{T'(z)}\right)^2 &= \frac{\frac{6c^2\Delta}{(cz+d)^4}}{\frac{\Delta}{(cz+d)^2}} - \frac{3}{2}\left(\frac{-2c}{cz+d}\right)^2 \\
&= \frac{6c^2}{(cz+d)^2} - \frac{3}{2}\frac{4c^2}{(cz+d)^2} = 0\ ; \\
\left(\frac{T''(z)}{T'(z)}\right)' &= \left(\frac{-2c}{cz+d}\right)' = \frac{-c(-2c)}{(cz+d)^2} = \frac{2c^2}{(cz+d)^2} \\
\Rightarrow \left(\frac{T''(z)}{T'(z)}\right)' - \frac{1}{2}\left(\frac{T''(z)}{T'(z)}\right)^2 &= \frac{2c^2}{(cz+d)^2} - \frac{1}{2}\frac{4c^2}{(cz+d)^2} = 0.
\end{aligned}$$