Quasiregular Mappings
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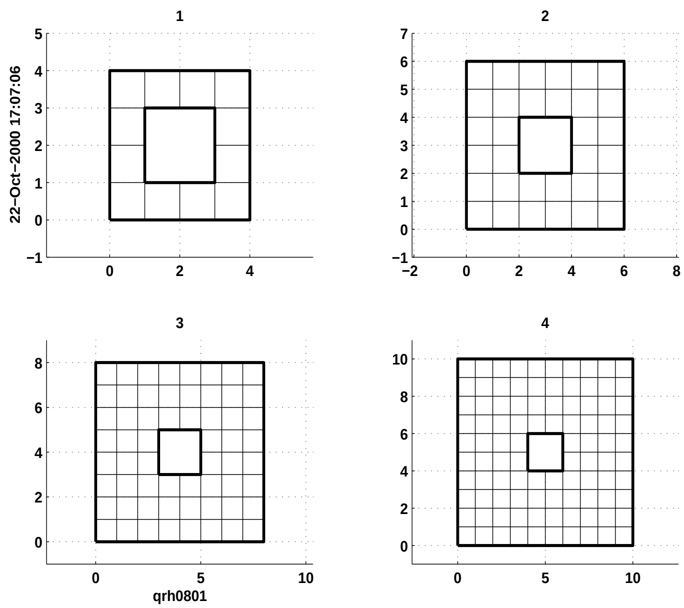
Problem Set 9
Winter 2009 / Vuorinen

1. The enclosed picture displays four ring domains, each of which consists of the region between two squares with parallel sides and the same center. For each ring, compute the following estimates for the modulus of the family of all curves joining the boundary components of the ring:
(a) an upper bound using a separating annulus,
(b) a lower bound using an annulus separated by the ring,
(c) an upper bound using Lemma 5.24 [CGQM]
(d) (optional) other upper and lower bounds.



Solution Denote by Γ the curve family at hand and by Γ' the curve family connecting the boundary components of the annulus at hand.

(a) 1) $\Gamma' < \Gamma \Rightarrow$

$$\begin{aligned} \mathsf{M}(\Gamma) \le \mathsf{M}(\Gamma') &= \omega_{n-1}(\log(2/\sqrt{2}))^{1-n} = \omega_{n-1}/\log\sqrt{2} \\ &= 2\pi/\log\sqrt{2} \approx 18.13 \end{aligned}$$

2) $\mathsf{M}(\Gamma) \le 2\pi/\log(3/\sqrt{2}) \approx 8.35$

3) $\mathsf{M}(\Gamma) \le 2\pi/\log(4/\sqrt{2}) \approx 6.04$

4) $\mathsf{M}(\Gamma) \le 2\pi/\log(5/\sqrt{2}) \approx 4.98$

(b) 1) $\Gamma < \Gamma' \Rightarrow \mathsf{M}(\Gamma) \ge \mathsf{M}(\Gamma') = 2\pi/\log(2\sqrt{2}) \approx 6.04$

2) $\mathsf{M}(\Gamma) \ge 2\pi/\log(3\sqrt{2}) \approx 4.34$

3) $\mathsf{M}(\Gamma) \ge 2\pi/\log(4\sqrt{2}) \approx 3.63$

4) $\mathsf{M}(\Gamma) \ge 2\pi/\log(5\sqrt{2}) \approx 3.21$

(c) 1) Denote by G the ring domain in 1). By 5.5[CGQM], $\mathsf{M}(\Gamma) \le m(G)/1^2 = 4^2 - 2^2 = 12$.

2) The ring domain in 2) can be divided into two ring domains G_1 and G_2, of which G_2 is "inside" G_1. Denote by Γ_1 and Γ_2 the curve families joining the boundary components of G_1 and G_2, respectively. Then $\Gamma_i < \Gamma$ for $i = 1, 2$. By 5.24[CGQM] and 5.5[CGQM],

$$\begin{aligned}
\mathsf{M}(\Gamma)^{1/(1-2)} &\ge \mathsf{M}(\Gamma_1)^{1/(1-2)} + \mathsf{M}(\Gamma_2)^{1/(1-2)} \\
&\ge m(G_1)^{-1} + m(G_2)^{-1} \\
&= (6^2 - 4^2)^{-1} + (4^2 - 2^2)^{-1} = \frac{1}{20} + \frac{1}{12} \\
\Rightarrow \mathsf{M}(\Gamma) &\le \frac{1}{\frac{1}{20} + \frac{1}{12}} = \frac{15}{2} = 7.5.
\end{aligned}$$

3) Similarly $\mathsf{M}(\Gamma) \le ((8^2 - 6^2)^{-1} + (6^2 - 4^2)^{-1} + (4^2 - 2^2)^{-1})^{-1} = 1/(1/28 + 1/20 + 1/12) \approx 5.92$.

4) $\mathsf{M}(\Gamma) \le ((10^2 - 8^2)^{-1} + (8^2 - 6^2)^{-1} + (6^2 - 4^2)^{-1} + (4^2 - 2^2)^{-1})^{-1} = 1/(1/36 + 1/28 + 1/20 + 1/12) \approx 5.08$.

(d) An example of other bounds:

Lower bound: Let $\tilde{\Gamma}$ be the family of line segments parallel to the coordinate axes, joining the boundary components of the ring domain, and denote by A their range i.e. "the ring minus corner squares". Then $\mathsf{M}(\tilde{\Gamma}) = m(A)/t^2$ (Proof is similar to the proof of 5.5.[CGQM]), and since $\tilde{\Gamma} \subset \Gamma$, we get $\mathsf{M}(\Gamma) \ge m(A)/t^2$.

1) $\mathsf{M}(\Gamma) \ge \frac{4^2-2^2-4\cdot 1}{1^2} = 8$

2) $\mathsf{M}(\Gamma) \ge \frac{6^2-2^2-4\cdot 2^2}{2^2} = 4$

3) $\mathsf{M}(\Gamma) \ge \frac{8^2-2^2-4\cdot 3^2}{3^2} = \frac{8}{3} \approx 2.67$

4) $\mathsf{M}(\Gamma) \geq \frac{10^2-2^2-4\cdot 4^2}{4^2} = 2$

Upper bound: Use Lemma 5.5[CGQM]: $\mathsf{M}(\Gamma) \leq m(G)/r^2$.

1) $\mathsf{M}(\Gamma) \leq \frac{4^2-2^2}{1^2} = 12$

2) $\mathsf{M}(\Gamma) \leq \frac{6^2-2^2}{2^2} = 8$

3) $\mathsf{M}(\Gamma) \leq \frac{8^2-2^2}{3^2} = \frac{20}{3} \approx 6.67$

4) $\mathsf{M}(\Gamma) \leq \frac{10^2-2^2}{4^2} = 6$

NOTE: Using a computer, the moduli can be computed:

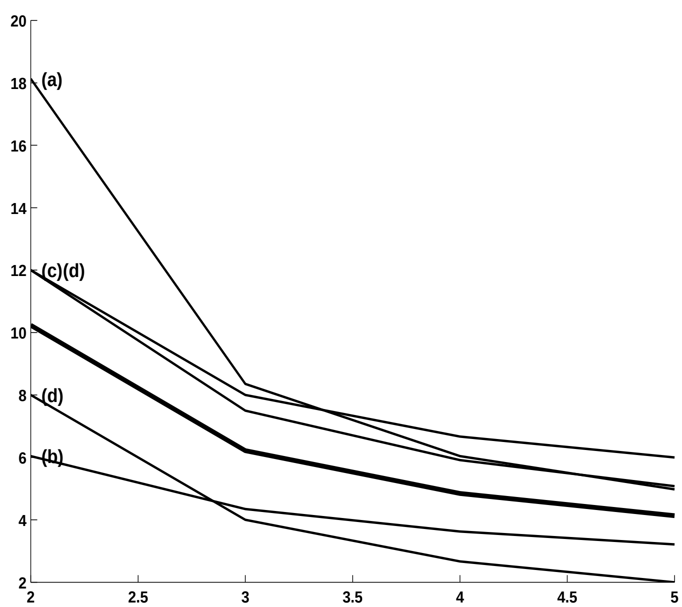
1) 10.2341

2) 6.2155

3) 4.8444

4) 4.1345

The following figure illustrates the computed values (thick line) and the bounds obtained in (a)-(d).



2. Recall first that

$$\omega_{n-1}(\log \lambda_n s)^{1-n} \leq \gamma_n(s) \leq \omega_{n-1}(\log s)^{1-n}$$

for $s > 1$. Derive from this the following inequality

$$t^\alpha/\lambda_n \le \gamma_n^{-1}(K\gamma_n(t)) \le \lambda_n^\alpha t^\alpha$$

for all $t > 1$ and $K > 0$, where $\alpha = K^{1/(1-n)}$.

Solution Denote $u = \omega_{n-1}(\log \lambda_n s)^{1-n}$ and $v = \omega_{n-1}(\log s)^{1-n}$. Then $u \le \gamma_n(s)$ and $v \ge \gamma_n(s)$. Hence

$$\gamma_n^{-1}(u) \ge s = \frac{1}{\lambda_n} e^{\left[\left(\frac{\omega_{n-1}}{u}\right)^{1/(n-1)}\right]},$$

$$\gamma_n^{-1}(v) \le s = e^{\left[\left(\frac{\omega_{n-1}}{v}\right)^{1/(n-1)}\right]}.$$

Let $t > 1$, $K > 0$, $\alpha = K^{1/(1-n)}$. Then

$$\begin{aligned}\gamma_n^{-1}(K\gamma_n(t)) &\le \gamma_n^{-1}(K\omega_{n-1}(\log \lambda_n t)^{1-n}) \\ &\le e^{\left(\frac{\log \lambda_n t}{K^{1/(n-1)}}\right)} = \lambda_n^\alpha t^\alpha\end{aligned}$$

and

$$\begin{aligned}\gamma_n^{-1}(K\gamma_n(t)) &\ge \gamma_n^{-1}(K\omega_{n-1}(\log t)^{1-n}) \\ &\ge \frac{1}{\lambda_n} e^{\left(\frac{\log t}{K^{1/(n-1)}}\right)} = \frac{1}{\lambda_n} t^\alpha.\end{aligned}$$

Perhaps problem 3 was too difficult! Qc maps not yet defined!

3. Let D be a c-QED domain in $\mathbf{R^n}$ and $f : \mathbf{R^n} \to \mathbf{R^n}$ $K-$qc. Show that fD is also c'-QED. Does the claim hold if f is only defined in D? A domain D is c-QED if there exists a constant $c > 0$ such that for all compact connected sets $E, F \subset D$, $\mathsf{M}(\Delta(E, F; D)) \ge c\mathsf{M}(\Delta(E, F; \mathbf{R}^n))$.

Solution Choose continua $E', F' \subset D'$ and denote

$$E = f^{-1}E', \; F = f^{-1}F'.$$

Then

$$\begin{aligned}\mathsf{M}(\Delta(E', F'; D')) &\ge \frac{1}{K}\mathsf{M}(\Delta(E, F; D)) \\ &\ge \frac{c}{K}\mathsf{M}(\Delta(E, F; \mathbf{R^n})) \ge \frac{\mathbf{c}}{\mathbf{K^2}}\mathsf{M}(\boldsymbol{\Delta}(\mathbf{E'}, \mathbf{F'}; \mathbf{R^n})).\end{aligned}$$

Hence D' is $c' := c/K^2$-QED.

Let $f(z) = z^2$, $B^2_+ \to fB^2_+ = B^2 \setminus [0, e_1)$, where $B^2_+ = B^2 \cap H^2$. Then f is conformal in B^2_+ and B^2_+ is QED, since it can be mapped onto B^2 by a qc-mapping $g\colon \mathbb{C} \to \mathbb{C}$ and B^2 is (1/2)-QED (8.28(1)[CGQM]). But $fB^2_+ = B^2 \setminus [0, e_1)$ is not QED (8.28(3)[CGQM]). Therefore the claim doesn't hold for qc maps $D \to D'$.

4. Verify the following identities for $K,\, t>0,\ r\in(0,1)$:

$$\tau_2(t) \;=\; \frac{\pi}{\mu(1/\sqrt{1+t})} = \frac{2\pi}{\mu(\,(\sqrt{1+t}-\sqrt{t}\,)^2\,)}\,, \tag{1}$$

$$\tau_2(t) \;=\; 2\,\tau_2\Big(4\Big[t+\sqrt{t(1+t)}\Big]\Big[1+t+\sqrt{t(1+t)}\Big]\Big)\,, \tag{2}$$

$$\mu\left(r^2\right)\,\mu\left(\left(\frac{1-r}{1+r}\right)^2\right) = \pi^2\,. \tag{3}$$

Solution

(1) By Lemma 5.53[CGQM] and (5.56)[CGQM],

$$\tau_2(t) = \frac{\gamma_2(\sqrt{t+1})}{2^{2-1}} = \frac{2\pi}{2\mu\left(\frac{1}{\sqrt{1+t}}\right)} = \frac{\pi}{\mu\left(\frac{1}{\sqrt{t+1}}\right)}.$$

Furthermore, by (5.57)[CGQM],

$$\begin{aligned}
\mu\left((\sqrt{1+t}-\sqrt{t})^2\right) &= 2\mu\left(\frac{2\sqrt{(\sqrt{1+t}-\sqrt{t})^2}}{1+(\sqrt{1+t}-\sqrt{t})^2}\right)\\
&= 2\mu\left(\frac{2(\sqrt{1+t}-\sqrt{t})}{2+2t-2\sqrt{t}\sqrt{1+t}}\right)\\
&= 2\mu\left(\frac{1}{\sqrt{1+t}}\right).
\end{aligned}$$

$$\therefore\ \frac{\pi}{\mu\left(\frac{1}{\sqrt{1+t}}\right)} = \frac{2\pi}{\mu\left((\sqrt{1+t}-\sqrt{t})^2\right)}.$$

(2) Solve:

$$(\sqrt{t}-\sqrt{t+1})^2 = \frac{1}{\sqrt{u+1}} \Leftrightarrow (\sqrt{t}-\sqrt{t+1})^4 = \frac{1}{u+1}$$

$$\begin{aligned}
\Leftrightarrow\quad u &= \frac{1}{(\sqrt{t}-\sqrt{t+1})^4} - 1 = (\sqrt{t}+\sqrt{t+1})^4 - 1\\
&= (t+2\sqrt{t(t+1)}+t+1)^2 - 1^2\\
&= (2t+2+2\sqrt{t(t+1)})(2t+2\sqrt{t(t+1)})\\
&= 4(t+\sqrt{t(t+1)})(1+t+\sqrt{t(t+1)}).
\end{aligned}$$

It follows from (1) that

$$\begin{aligned}
\tau_2(t) &= \frac{2\pi}{\mu((\sqrt{1+t}-\sqrt{t})^2)} = \frac{2\pi}{\mu\left(\frac{1}{\sqrt{u+1}}\right)}\\
&= 2\tau_2(u) = 2\tau_2\left(4(t+\sqrt{t(t+1)})(1+t+\sqrt{t(t+1)})\right).
\end{aligned}$$

(3) Using (5.57)[CGQM], we get

$$\begin{aligned}
\mu(r^2)\mu\left(\left(\frac{1-r}{1+r}\right)^2\right) &= 2\mu(r^2)\mu\left(\frac{2\frac{1-r}{1+r}}{1+\left(\frac{1-r}{1+r}\right)^2}\right)\\
&= 2\mu(r^2)\mu\left(\frac{2(1-r)(1+r)}{(1+r)^2+(1-r)^2}\right)\\
&= 2\mu(r^2)\mu\left(\frac{1-r^2}{1+r^2}\right) = \pi^2\,.
\end{aligned}$$

5. In the study of distortion theory of quasiconformal mappings the following special function will be useful

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}$$

for $0 < r < 1$, $K > 0$. Show that $\varphi_{AB,n}(r) = \varphi_{A,n}(\varphi_{B,n}(r))$ and $\varphi_{A,n}^{-1}(r) = \varphi_{1/A,n}(r)$ and that

$$\varphi_{K,2}(r) = \varphi_K(r) = \mu^{-1}\left(\tfrac{1}{K}\mu(r)\right)\,.$$

Verify also that

(1) $$\varphi_2(r) = \tfrac{2\sqrt{r}}{1+r}\,,$$

(2) $$\varphi_K(r)^2 + \varphi_{1/K}\left(\sqrt{1-r^2}\right)^2 = 1\,.$$

Exploiting (1) and (2) find $\varphi_{1/2}(r)$. Show also that

(3) $$\varphi_{1/K}\left(\tfrac{1-r}{1+r}\right) = \tfrac{1-\varphi_K(r)}{1+\varphi_K(r)}\,,$$

(4) $$\varphi_K\left(\tfrac{2\sqrt{r}}{1+r}\right) = \tfrac{2\sqrt{\varphi_K(r)}}{1+\varphi_K(r)}\,.$$

Solution First,

$$\begin{aligned}
\varphi_{A,n}(\varphi_{B,n}(r)) &= \frac{1}{\gamma_n^{-1}(A\gamma_n(\frac{1}{\varphi_{B,n}(r)}))}\\
&= \frac{1}{\gamma_n^{-1}(A\gamma_n(\gamma_n^{-1}(B\gamma_n(\frac{1}{r}))))}\\
&= \frac{1}{\gamma_n^{-1}(AB\gamma_n(\frac{1}{r}))} = \varphi_{AB,n}(r).
\end{aligned}$$

Hence we obtain

$$\varphi_{A,n}(\varphi_{1/A,n}(r)) = \varphi_{A\cdot 1/A,n}(r) = \varphi_{1,n}(r) = r$$

and

$$\varphi_{1/A,n}(\varphi_{A,n}(r)) = \varphi_{1,n}(r) = r.$$

Therefore $\varphi_{A,n}^{-1}(r) = \varphi_{1/A,n}(r)$.

By (5.56)[CGQM] we obtain the following

$$\begin{aligned} x := \varphi_{K,2}(r) = 1/\gamma_2^{-1}(K(\gamma_2(1/r))) \quad &\Leftrightarrow \quad \gamma_2\left(\frac{1}{x}\right) = K\gamma_2\left(\frac{1}{r}\right) \\ &\Leftrightarrow \quad \frac{2\pi}{\mu(x)} = \frac{2\pi K}{\mu(r)} \\ &\Leftrightarrow \quad \varphi_{K,2}(r) = x = \mu^{-1}\left(\frac{1}{K}\mu(r)\right). \end{aligned}$$

(1) By (5.57)[CGQM] we obtain

$$y := \varphi_{2,2}(r) = \mu^{-1}\left(\frac{1}{2}\mu(r)\right) \quad \Leftrightarrow \quad \mu(y) = \frac{1}{2}\mu(r) \Leftrightarrow y = \frac{2\sqrt{r}}{1+r}$$

(2) Denote $r' = \sqrt{1-r^2}$, $s = \varphi_K(r)$, $t = \varphi_{1/K}(r')$. Then $\mu(s) = (1/K)\mu(r)$ and $\mu(t) = K\mu(r')$. By (5.57)[CGQM] it follows that $\mu(s)\mu(t) = \mu(r)\mu(r') = \pi^2/4 = \mu(s)\mu(s')$. Since $0 < \mu(s) < \infty$, we have that $\mu(t) = \mu(s')$.

Because μ is a bijection, we get

$$t = s' \Leftrightarrow \varphi_{1/K}(r') = \sqrt{1-\varphi_K(r)^2}.$$

Therefore $\varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1$.

Then

$$\begin{aligned} \varphi_{1/2}(r) &= \sqrt{1-\varphi_2(r')^2} = \sqrt{1-\left(\frac{2\sqrt{r'}}{1+r'}\right)^2} \\ &= \frac{\sqrt{(1+r')^2-4r'}}{1+r'} = \frac{1-r'}{1+r'}. \end{aligned}$$

(3)

$$\begin{aligned} \varphi_{1/K}\left(\frac{1-r}{1+r}\right) &= \varphi_{1/K}(\varphi_{1/2}(r')) \\ &= \varphi_{1/K\cdot 1/2}(r') \\ &= \varphi_{1/2}(\varphi_{1/K}(r')) \\ &= \frac{1-\sqrt{1-\varphi_{1/K}(r')^2}}{1+\sqrt{1-\varphi_{1/K}(r')^2}} = \frac{1-\varphi_K(r)}{1+\varphi_K(r)}. \end{aligned}$$

(4)

$$\begin{aligned}\varphi_K\left(\frac{2\sqrt{r}}{1+r}\right) &= \varphi_K(\varphi_2(r)) = \varphi_{K\cdot 2}(r)\\ &= \varphi_2(\varphi_K(r)) = \frac{2\sqrt{\varphi_K(r)}}{1+\varphi_K(r)}.\end{aligned}$$

6. Verify the following identities for $K,\, t>0$:

$$\tau_2^{-1}\Big(\tau_2(t)/K\Big) = \frac{1}{\tau_2^{-1}(K\tau_2(1/t))}\,, \tag{1}$$

$$\tau_2(t) = \frac{4}{\tau_2(1/t)}\,. \tag{2}$$

(Hint: The identity (5) from exercise 5 may be useful.)

Solution

(1) Denote

$$\begin{aligned}\frac{1}{y} = \frac{1}{\tau_2^{-1}(K\tau_2(1/t))} \quad &\Leftrightarrow \quad \tau_2(y) = K\tau_2(1/t)\\ &\overset{CGQM,5.53}{\Leftrightarrow} \quad 2^{1-2}\gamma_2\left(\sqrt{1+y}\right) = K2^{1-2}\gamma_2\left(\sqrt{1+\frac{1}{t}}\right)\\ &\Leftrightarrow \quad y = \left(\gamma_2^{-1}\left(K\gamma_2\left(\sqrt{1+\frac{1}{t}}\right)\right)\right)^2 - 1\\ &\qquad = \frac{1-\varphi_{K,2}(\sqrt{\frac{t}{1+t}})^2}{\varphi_{K,2}(\sqrt{\frac{t}{1+t}})^2}.\end{aligned}$$

Denote

$$\begin{aligned}x = \tau_2^{-1}\left(\frac{1}{K}\tau_2(t)\right) \quad &\overset{CGQM,5.53}{\Leftrightarrow} \quad \gamma_2\left(\sqrt{1+x}\right) = \frac{1}{K}\gamma_2\left(\sqrt{1+t}\right)\\ &\Leftrightarrow \quad x = \left(\gamma_2^{-1}\left(\frac{1}{K}\gamma_2\left(\sqrt{1+t}\right)\right)\right)^2 - 1\\ &\qquad = \frac{1-\varphi_{1/K,2}\left(\frac{1}{\sqrt{1+t}}\right)^2}{\varphi_{1/K,2}\left(\frac{1}{\sqrt{1+t}}\right)^2}.\end{aligned}$$

Then, using $\varphi_K(r)^2 + \varphi_{1/K}(\sqrt{1-r^2})^2 = 1$ we obtain

$$\tau_2^{-1}\left(\frac{1}{K}\tau_2(t)\right) = \frac{1-\varphi_{1/K,2}\left(\frac{1}{\sqrt{1+t}}\right)^2}{\varphi_{1/K,2}\left(\frac{1}{\sqrt{1+t}}\right)^2} = \frac{\varphi_{K,2}\left(\sqrt{\frac{t}{t+1}}\right)^2}{1-\varphi_{K,2}\left(\sqrt{\frac{t}{t+1}}\right)^2} = \frac{1}{\tau_2^{-1}(K\tau_2(1/t))}.$$

(2) By exercise 5.(1) and (5.57)[CGQM],

$$\tau_2(t)\tau_2\left(\frac{1}{t}\right) = \frac{\pi}{\mu\left(\frac{1}{\sqrt{1+t}}\right)} \cdot \frac{\pi}{\mu\left(\sqrt{\frac{t}{1+t}}\right)} = \frac{\pi}{\mu\left(\frac{1}{\sqrt{1+t}}\right)} \cdot \frac{\pi}{\mu\left(\sqrt{1-\left(\frac{1}{\sqrt{1+t}}\right)^2}\right)} = 4.$$