Quasiregular Mappings
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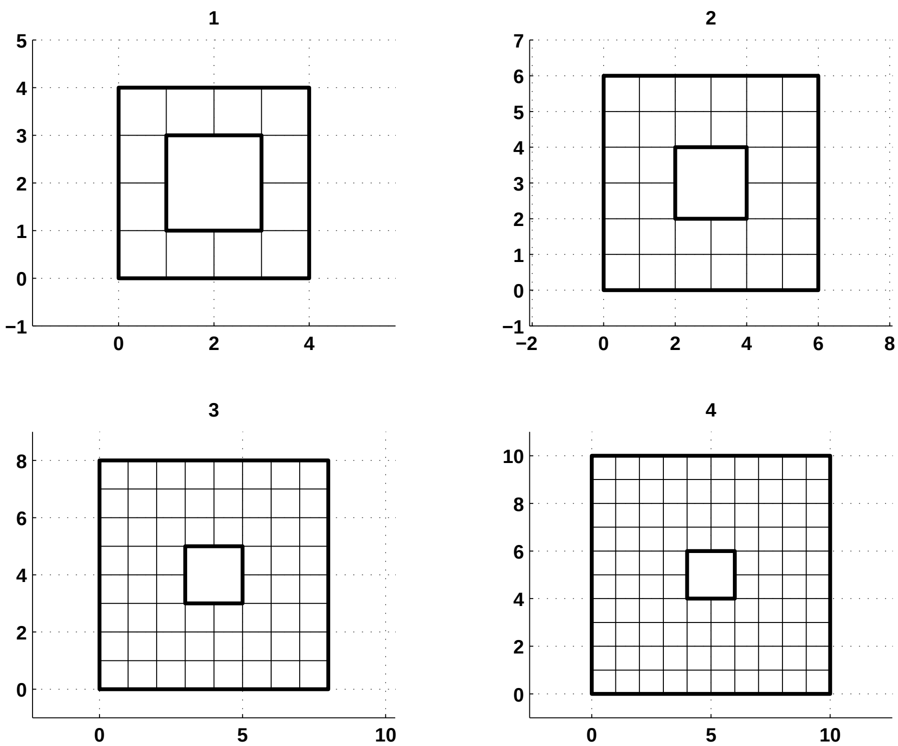
Problem Set 9
Winter 2009 / Vuorinen

1. The enclosed picture displays four ring domains, each of which consists of the region between two squares with parallel sides and the same center. For each ring, compute the following estimates for the modulus of the family of all curves joining the boundary components of the ring:
(a) an upper bound using a separating annulus,
(b) a lower bound using an annulus separated by the ring,
(c) an upper bound using Lemma 5.24 [CGQM]
(d) (optional) other upper and lower bounds.



2. Recall first that

$$\omega_{n-1}(\log \lambda_n s)^{1-n} \le \gamma_n(s) \le \omega_{n-1}(\log s)^{1-n}$$

for $s > 1$. Derive from this the following inequality

$$t^\alpha/\lambda_n \le \gamma_n^{-1}(K\gamma_n(t)) \le \lambda_n^\alpha t^\alpha$$

for all $t > 1$ and $K > 0$, where $\alpha = K^{1/(1-n)}$.

3. Let D be a c-QED domain in $\mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}^n$ $K-$qc (i.e. $M(\Gamma)/K \le M(f\Gamma) \le KM(\Gamma)$). Show that fD is also c'-QED. Does the claim hold if f is only defined in D? A domain D is c-QED if there exists a constant $c > 0$ such that for all compact connected sets $E, F \subset D$, $\mathsf{M}(\Delta(E,F;D)) \ge c\mathsf{M}(\Delta(E,F;\mathbf{R}^n))$.

4. Verify the following identities for $K, t > 0,\ r \in (0,1)$:

$$\tau_2(t) = \frac{\pi}{\mu(1/\sqrt{1+t}\,)} = \frac{2\pi}{\mu(\,(\sqrt{1+t}-\sqrt{t}\,)^2\,)}\ , \tag{1}$$

$$\tau_2(t) = 2\,\tau_2\big(4\big[t+\sqrt{t(1+t)}\,\big]\big[1+t+\sqrt{t(1+t)}\,\big]\big)\ , \tag{2}$$

$$\mu\,(r^2)\;\mu\Big(\Big(\tfrac{1-r}{1+r}\Big)^2\Big) = \pi^2\ . \tag{3}$$

5. In the study of distortion theory of quasiconformal mappings the following special function will be useful

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}$$

for $0 < r < 1$, $K > 0$. Show that $\varphi_{AB,n}(r) = \varphi_{A,n}(\varphi_{B,n}(r))$ and $\varphi^{-1}_{A,n}(r) = \varphi_{1/A,n}(r)$ and that

$$\varphi_{K,2}(r) = \varphi_K(r) = \mu^{-1}\big(\tfrac{1}{K}\mu(r)\big)\ .$$

Verify also that

$$\varphi_2(r) = \tfrac{2\sqrt{r}}{1+r}\ , \tag{4}$$

$$\varphi_K(r)^2 + \varphi_{1/K}\big(\sqrt{1-r^2}\,\big)^2 = 1\ . \tag{5}$$

Exploiting (6) and (7) find $\varphi_{1/2}(r)$. Show also that

$$\varphi_{1/K}\Big(\frac{1-r}{1+r}\Big) = \frac{1-\varphi_K(r)}{1+\varphi_K(r)}\ , \tag{6}$$

$$\varphi_K\Big(\frac{2\sqrt{r}}{1+r}\Big) = \frac{2\sqrt{\varphi_K(r)}}{1+\varphi_K(r)}\ . \tag{7}$$

6. Verify the following identities for $K,\ t > 0$:

$$\tau_2^{-1}\big(\tau_2(t)/K\big) = \frac{1}{\tau_2^{-1}(K\tau_2(1/t))} \tag{8}$$

$$\tau_2(t) = \frac{4}{\tau_2(1/t)}\ . \tag{9}$$

(Hint: The identity (5) from exercise 5 may be useful.)
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