Quasiregular Mappings
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Problem Set 8
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1. Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous. Then f is said to be *open* if it maps all open subsets onto open subsets of G', *closed* if it maps all closed subsets onto closed subsets of G', and *proper* if for every compact $K \subset G'$ also $f^{-1}K$ is compact. Note the condition $fG = G'$ above, i.e. f is a surjective map.

(a) Show that the map $f\colon H \to B^2 \setminus \{0\}$, $H = \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$, $f(z) = \exp(z)$, is open but neither proper nor closed.

(b) Prove: Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous, open, and closed. If $y \in G'$, then $f^{-1}(y)$ is compact.

(c) Prove: Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous, open, and closed. If $y \in G'$ and U is an open neighborhood of $f^{-1}(y)$ in G, then there is an open neighborhood V of y in G' such that $f^{-1}V \subset U$.

Solution:

(a) It is clear that f is locally injective, hence open, and not proper (the point $z = 1/2$ has preimages $-\log 2 + i2\pi k$, $k \in \mathbb{Z}$.) It is less obvious that it is not closed (consider the closed set $C = \{k_p = \log(1/2+1/p), p \in \mathbb{N}, p > 2\}$ whose image is not closed in $B^2 \setminus \{0\}$).

(b) Suppose the opposite. Then there is a sequence (b_k) in $f^{-1}(y)$ such that $b_k \to b \in \partial G$. Since f is continuous, for each k there is $r_k \in (0, 1/k)$ such that if $U_k = B^n(b_k, r_k)$, then $U_k \subset G$ and $fU_k \subset B^n(y, 1/k)$. Since f is open, fU_k is an open neighborhood of y in G'. Hence, there is $a_k \in U_k$ such that $f(a_k) \neq y$. Now $a_k \to b$ and $f(a_k) \to y$. Hence, $\{a_k \mid k \geq 1\}$ is closed in G, but $\{f(a_k) \mid k \geq 1\}$ is not closed in G', a contradiction as f is closed.

(c) Suppose the opposite. Then there is a sequence (b_k) in $F = G \setminus U$ with $f(b_k) \to y$. Now F is closed in G and hence fF closed in G'. Thus, $y \in fF$. This is a contradiction as $f^{-1}(y) \subset G \setminus F$.

2. Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous, open, and closed. Then f is proper, i.e., for every compact $E \subset G'$, also $f^{-1}E$ is compact.

Solution: Suppose that $E \subset G'$ is compact. We must prove that $f^{-1}E$ is compact. Suppose the opposite. By continuity, the set $f^{-1}E$ is closed in G. Hence there is a sequence (x_j) in $f^{-1}E$ such that $x_j \to x \in \partial G$. The set $\{x_j : j \geq 1\}$ is closed in G, and thus $\{f(x_j) : j \geq 1\}$ is closed in G', and hence also in E, i.e., it is compact. The sequence (x_j) has a subsequence (x_{j_k}) such that $f(x_{j_k}) \to f(z)$ for some $z \in \partial G$. According to exercises 1b and 1c there is a neighborhood V of $f(z)$ in G' such that $A = \overline{f^{-1}V} \subset G$. By passing to a subsequence we may assume that $f(x_{j_k}) \in V$ for all k. Now $x_{j_k} \in A$ for all k. The set A is compact and hence $x = \lim x_{j_k} \in A \subset G$, a contradiction.

3. For $\alpha > 0$ we denote by $I(\alpha)$ the class of compact subsets E of $\overline{\mathbf{B}^n}$ with

$$A = \int_{B^n(2)\setminus E} \frac{dm}{d(x,E)^\alpha} < \infty.$$

Then, for example, $\{0\} \in I(\alpha)$ when $\alpha < n$, and $S^{n-1} \in I(\alpha)$ when $\alpha < 1$. Fix $E \in I(\alpha)$, denote $E_k = \{x \in \mathbf{R^n} : \mathbf{2^{-k-1}} \leq \mathrm{d(x,E)} \leq \mathbf{2^{-k}}\}$, $k = 1, 2, \ldots$, and for $p > 0$ let Γ_p be the family of all curves in $\Delta(E, S^{n-1}(2); \mathbf{R}^n)$ with $\ell(\gamma \cap E_k) \geq 2^{-kp}$. Show that $\mathsf{M}(\Gamma_p) = 0$ for $p < \alpha/n$.

Solution: Fix $\varepsilon > 0$. Define $\rho(x) = c d(x,E)^{-\alpha/n}, c = (\varepsilon/A)^{1/n}$ for $|x| \leq 2$ and $\rho(x) = 0$ for $|x| > 2$. Then $\int_{\mathbf{R^n}\setminus \mathbf{E}} \rho^n dm < \varepsilon$, since $E \in I(\alpha)$. Fix $p \in (0, \alpha/n)$. For $\gamma \in \Gamma_p$ we have

$$\int_\gamma \rho\, ds \geq c \sum_{k=0}^{\infty} \int_{\gamma \cap E_k} \rho\, ds \geq \sum_{k=0}^{\infty} 2^{-pk} 2^{k\alpha/n} = \infty.$$

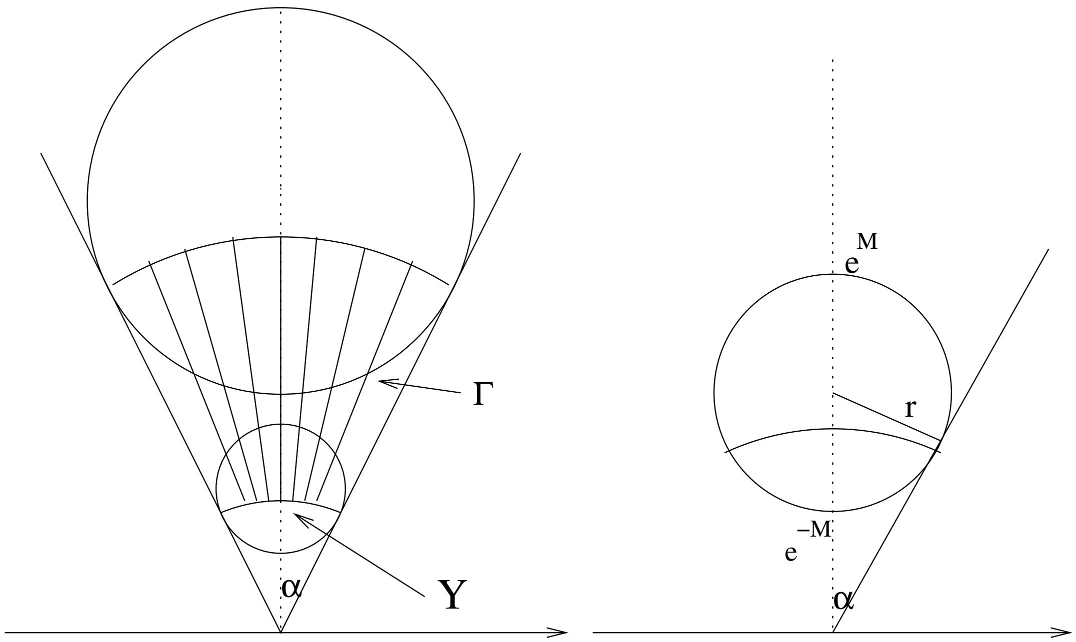
Hence $\rho \in F(\Gamma_p)$. Because $\varepsilon > 0$ is arbitrary $M(\Gamma_p) = 0$. (Cf. also Lemma 5.7[CGQM].)

4. Let $x, y \in \mathbf{B}^n$, $x \neq y$ and $M \in (0, \frac{\rho(x,y)}{2})$. Show that

$$\mathsf{M}(\Delta(D(x,M), D(y,M); \mathbf{B}^n)) \geq d_1(n,M)\rho(x,y)^{1-n},$$

where $d_1 > 0$.

Solution: By the Möbius invariance of the modulus, we may assume that $x = e_n$, $y = e^{\rho(x,y)}e_n$ in $\mathbf{H}^n$. Let Γ be the curve family which consists of line segments $\{ty \mid 1 \leq t \leq e^{\rho(x,y)}\}$, $y \in Y$, and let $\triangle = \triangle(D(x,M), D(y,M), \mathbf{H}^n)$. Then $\Gamma \subset \Delta$.



By (5.12)[CGQM],

$$\begin{aligned} \mathsf{M}(\Gamma) &= m_{n-1}(Y)(\log \frac{e^{\rho(x,y)}}{1})^{1-n} \\ &= m_{n-1}(Y)\rho(x,y)^{1-n} \end{aligned}$$

where $m_{n-1}(Y) = \omega_{n-1}\alpha/\pi = \omega_{n-1}\arcsin(\operatorname{th} M)/\pi$. Hence we have

$$\mathsf{M}(\triangle) \geq \mathsf{M}(\Gamma) = d_1(n, M)\rho(x,y)^{1-n},$$

where $d_1(n, M) = \omega_{n-1}\arcsin(\operatorname{th} M)/\pi > 0$.

5. Let $f : \mathbf{B}^n \to \mathbf{B}^n$ be a homeomorphism mapping each sphere centered at 0 onto another sphere centered at 0 (such a mapping is called a *radial mapping*) and with the property that for some $K \geq 1, \mathsf{M}(\Gamma)/K \leq \mathsf{M}(f(\Gamma)) \leq K\mathsf{M}(\Gamma)$ whenever Γ is the family of all curves connecting the boundary components of a spherical annulus centered at 0. Show that for all $x \in \mathbf{B}^n$

$$|x|^{1/\alpha} \leq |f(x)| \leq |x|^{\alpha}, \alpha = K^{1/(1-n)}.$$

Solution:

Denote by $h\colon [0,1) \to [0,1)$ the map $h(t) = |f(te_1)|$. Let $0 < s < r < 1$ and $\Gamma = \Delta(S^{n-1}(r), S^{n-1}(s))$. Now $\mathsf{M}(\Gamma) = \omega_{n-1}\left(\log \frac{r}{s}\right)^{1-n}$ and since f is a radial mapping we have

$$\text{(1)} \qquad \mathsf{M}(f\Gamma) \;=\; \omega_{n-1}\left(\log \frac{h(r)}{h(s)}\right)^{1-n}.$$

Denote $\alpha = K^{1/(1-n)}$. By (1) and the modulus property of f we have

$$\begin{aligned}
& \frac{1}{K}\mathsf{M}(\Gamma) \le \mathsf{M}(f\Gamma) \le K\mathsf{M}(\Gamma) \\
\Leftrightarrow \quad & \left(\frac{1}{\alpha}\right)^{1-n}\left(\log\frac{r}{s}\right)^{1-n} \le \left(\log\frac{h(r)}{h(s)}\right)^{1-n} \le \alpha^{1-n}\left(\log\frac{r}{s}\right)^{1-n} \\
\Leftrightarrow \quad & \alpha\log\frac{r}{s} \le \log\frac{h(r)}{h(s)} \le \frac{1}{\alpha}\log\frac{r}{s} \\
\Leftrightarrow \quad & \left(\frac{r}{s}\right)^{\alpha} \le \frac{h(r)}{h(s)} \le \left(\frac{r}{s}\right)^{1/\alpha}.
\end{aligned}$$

Hence $h(r) \ge (r/s)^{\alpha}h(s)$ and h is increasing. From this we can conclude that $h(r) \to 1$ as $r \to 1$. (This conclusion can also be obtained by using purely topological arguments.) Letting $r \to 1$, we obtain the inequality

$$\left(\frac{1}{s}\right)^{\alpha} \le \frac{1}{h(s)} \le \left(\frac{1}{s}\right)^{1/\alpha}$$

which implies that

$$s^{1/\alpha} \le h(s) \le s^{\alpha}$$

and furthermore that

$$|x|^{1/\alpha} \le |f(x)| \le |x|^{\alpha}.$$

6. Let $G = \mathbb{B}^2 \setminus \{0\}$.

(a) For $0 < r < 1/2$ compute the quasihyperbolic area w.r.t. k_G of the annulus $\{z : r < |z| < 1/2\}$.

(b) For $1/2 < r < 1$ compute the quasihyperbolic area w.r.t. k_G of the annulus $\{z : 1/2 < |z| < r\}$.

Solution: Let us denote $A(a,b) = \{z \in G\colon a < |z| < b\}$.

(a)

$$m_k(A(r_0, 1/2)) = \int_0^{2\pi}\int_{1/2}^{r_0}\frac{dr}{r}d\alpha = 2\pi(\log(1/2) - \log(r)) = 2\pi\log\frac{1}{2r}.$$

(b)

$$m_k(A(1/2, r_1)) = \int_0^{2\pi} \int_{r_1}^{1/2} \frac{r\,dr}{(1-r)^2} d\alpha = 2\pi \left(\frac{1}{1-r_1} + \log(1-r_1) - 2 - \log(1/2) \right).$$