Quasiregular Mappings
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Problem Set 7
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1. Let $t > r > s > 0$, $E \subset \mathbb{B}(s)$ and $\Delta_a = \Delta(E, S^{n-1}(a))$. Show that $\mathsf{M}(\Delta_r) \le c\mathsf{M}(\Delta_t)$, where c is only dependent on n, r, s and t.

Solution: Denote $F_1 = E$, $F_2 = S^{n-1}(t)$, $F_3 = S^{n-1}(r)$, $F_4 = S^{n-1}(s)$ and $\Gamma_{ij} = \Delta(F_i, F_j)$ for all $i, j \in \{1,2,3,4\}$. Then $\mathsf{M}(\Gamma_{24}) = \omega_{n-1}(\log(t/s))^{1-n}$ and $\mathsf{M}(\Delta_r) = \mathsf{M}(\Gamma_{13}) \le \omega_{n-1}(\log(r/s))^{1-n}$, since $\Gamma_{13} > \Delta(S^{n-1}(r), S^{n-1}(s))$. If $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$, then by [CGQM,5.32],

$$\begin{aligned}\mathsf{M}(\Delta(|\gamma_{13}|, |\gamma_{24}|)) &\ge c_n \log\frac{r}{s} \ge c_n \log\frac{r}{s}\frac{\mathsf{M}(\Delta_r)}{\omega_{n-1}(\log\frac{r}{s})^{1-n}}\\ &= \frac{c_n}{\omega_{n-1}}(\log\frac{r}{s})^n\mathsf{M}(\Delta_r).\end{aligned}$$

Using [CGQM, 5.35], we get

$$\begin{aligned}\mathsf{M}(\Delta_t) &= \mathsf{M}(\Gamma_{12}) \ge 3^{-n}\min\{\mathsf{M}(\Gamma_{24}), \mathsf{M}(\Gamma_{13}), \inf \mathsf{M}(|\gamma_{13}|, |\gamma_{24}|)\}\\ &\ge 3^{-n}\min\{\omega_{n-1}(\log\frac{t}{s})^{1-n}, \mathsf{M}(\Delta_r), \frac{c_n}{\omega_{n-1}}(\log\frac{r}{s})^n\mathsf{M}(\Delta_r)\}\\ &\ge 3^{-n}\min\{\omega_{n-1}(\log\frac{t}{s})^{1-n}\frac{\mathsf{M}(\Delta_r)}{\omega_{n-1}(\log r/s)^{1-n}}, \mathsf{M}(\Delta_r), \frac{c_n}{\omega_{n-1}}(\log\frac{r}{s})^n\mathsf{M}(\Delta_r)\}\\ &= \tilde{c}\,\mathsf{M}(\Delta_r)\end{aligned}$$

where the infimum was taken over all rectifiable curves $\gamma_{13} \in \Gamma_{13}$, $\gamma_{24} \in \Gamma_{24}$ and $\tilde{c} = 3^{-n}\min\{1, (\frac{\log(r/s)}{\log(t/s)})^{n-1}, c_n(\log(r/s))^n/\omega_{n-1}\}$. Hence $\mathsf{M}(\Delta_r) \le c\mathsf{M}(\Delta_t)$, where $c = 1/\tilde{c}$ depends only on n,r,s and t.

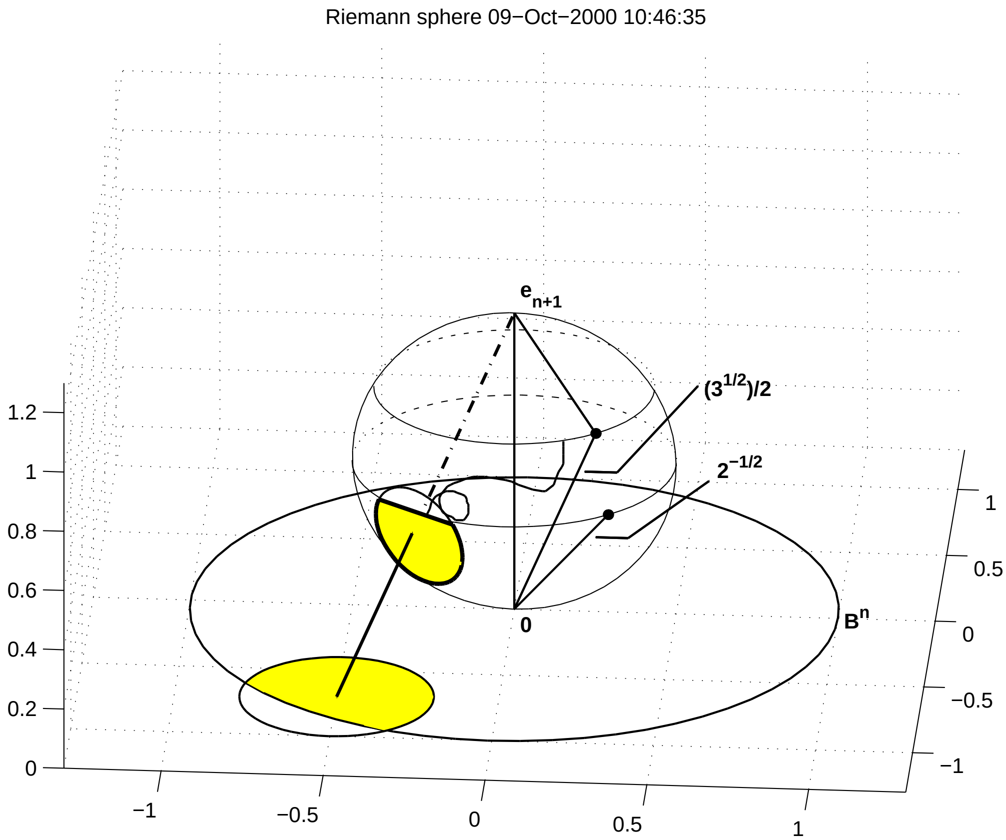
2. Prove Theorem 6.1 (3) [CGQM].
(2. Let $t \in (0, \frac{1}{\sqrt{2}}]$ and $z \in \overline{\mathbf{R}^n}$. Without using Theorem 3.42/lectures, find an upper bound for $c(\overline{Q}(z,t))$ in terms of n and t. Your bound should tend to 0 when $t \to 0$.)

Solution: Let us recall the definition of c.

$$c(\overline{Q}(z,t)) = \inf_{x \in \overline{\mathbf{R}}^n} \max\{m(\overline{Q}(z,t), x), m(\overline{Q}(z,t), \tilde{x})\}$$

where

$$m(\overline{Q}(z,t), x) = \mathsf{M}(\Delta(\partial Q(x, \frac{\sqrt{3}}{2}), \overline{Q}(z,t) \cap \overline{Q}(x, \frac{1}{\sqrt{2}}))).$$



Let $x \in \overline{\mathbf{R}}^n$. We may assume that $z \in Q(x, \frac{1}{\sqrt{2}})$. (Otherwise $z \in Q(\tilde{x}, \frac{1}{\sqrt{2}})$, which case is handled similarly). Let t_x be the spherical isometry in (1.46)[CGQM]. Then $t_x(x) = 0$, $t_x Q(x, \frac{1}{\sqrt{2}}) = \mathbb{B}$, $t_x Q(x, \frac{\sqrt{3}}{2}) = \mathbb{B}(\sqrt{3})$ by 1.25 (1) [CGQM]. Hence $t_x z \in \mathbb{B}$. Now

$$Q(t_x z, q(S^{n-1}, S^{n-1}(\sqrt{3})) \subset \mathbb{B}(\sqrt{3}).$$

We use [CQGM, 6.1] to obtain

$$\begin{aligned} m(\overline{Q}(z,t), x) &= m(\overline{Q}(t_x z, t), 0) \\ &= \mathsf{M}(\Delta(S^{n-1}(\sqrt{3}), \overline{Q}(t_x z, t) \cap \mathbb{B})) \\ &\leq \mathsf{M}(\Delta(\partial Q(t_x z, q(S^{n-1}, S^{n-1}(\sqrt{3}))), \partial Q(t_x z, t))) \\ &= \mathsf{M}(\Delta(\partial Q(0, q(S^{n-1}, S^{n-1}(\sqrt{3}))), \partial Q(0, t))). \end{aligned}$$

For the last equation we applied the map $t_{t_x z}$.

Here

$$\begin{aligned}
\partial Q(0,t) &= S^{n-1}\Big(\frac{t}{\sqrt{1-t^2}}\Big), \\
q(S^{n-1}, S^{n-1}(\sqrt{3})) &= \frac{\sqrt{3}-1}{\sqrt{(1+3)(1+1)}} = \frac{\sqrt{3}-1}{2\sqrt{2}}, \\
\frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{\sqrt{1-\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}} &= \frac{\sqrt{3}-1}{2\sqrt{2}}\sqrt{\frac{(2\sqrt{2})^2}{8-(3-2\sqrt{3}+1)}} \\
&= \frac{\sqrt{3}-1}{\sqrt{4+2\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{2}\sqrt{\sqrt{3}+1}}
\end{aligned}$$

$$\therefore c(\overline{Q}(z,t)) \le m(\overline{Q}(z,t),x) \le \omega_{n-1}\left(\log \frac{\sqrt{3}-1}{\sqrt{2}\sqrt{\sqrt{3}+1}}\frac{\sqrt{1-t^2}}{t}\right)^{1-n}.$$

3. Let $G = \mathbf{R^n} \setminus \{0\}$ and let s_G be defined as

$$s_G(x,y) = \frac{|x-y|^2}{2|x||y|}\ , \quad x,\, y \in G\ .$$

Define ρ_G by $\operatorname{ch}\rho_G(x,y) = 1 + s_G(x,y)$. Show that

$$j_G(x,y)/2 \le \rho_G(x,y) \le 4\, j_G(x,y)$$

for $x,y \in G$. Hint: Use the inequality from h0405 to the effect that for $a \ge 0$

$$\log\left(1+\max\left\{a,\sqrt{a}\right\}\right) \le b \le \log\left(1+a+\sqrt{a}\right) \le 2\log\left(1+\max\left\{a,\sqrt{a}\right\}\right)$$

if $\operatorname{ch} b = 1 + \frac{1}{2}a$.

Solution: By exercises 0405 and 0504 we get

$$\begin{aligned}
\rho_G(x,y) &\ge \log(1+\max\{2s_G(x,y), \sqrt{2s_G(x,y)}\}) \\
&= \varphi_G(x,y) \ge j_G(x,y)/2
\end{aligned}$$

and

$$\begin{aligned}
\rho_G(x,y) &\le 2\log(1+\max\{2s_G(x,y), \sqrt{2s_G(x,y)}\}) \\
&= 2\varphi_G(x,y) \le 4 j_G(x,y).
\end{aligned}$$

4. Show that for $0 < r < 1$ and $M > 0$, $m_h\Big(\bigcup_{|x|\le r} D(x,M)\Big) \le d_2(n,M)(1-r)^{1-n}$, where m_h is the hyperbolic measure of $(\mathbb{B}, \rho_{\mathbb{B}})$.

Solution: Let $0 < r < 1$. We have that

$$\bigcup_{|z|\le r} D(z,M) = \mathbb{B}(R)$$

for some $R > 0$. By the 2 first lines of page 25 [CGQM],

$$\log\frac{1+r}{1-r} + M = \log\frac{1+R}{1-R}$$

which implies

$$\frac{1+R}{1-R} = e^M\frac{1+r}{1-r}.$$

We use (4.15)[CGQM] to obtain

$$\begin{aligned} m_h(\bigcup_{|z|\le r} D(z,M)) &= m_h(\mathbb{B}(R)) < \frac{2^n\omega_{n-1}}{n-1}\left(\frac{R}{1-R}\right)^{n-1} \\ &\le \frac{2^n\omega_{n-1}}{n-1}\left(\frac{1+R}{1-R}\right)^{n-1} = \frac{2^n\omega_{n-1}}{n-1}\left(e^M\frac{1+r}{1-r}\right)^{n-1} \\ &\le \frac{2^n2^{n-1}\omega_{n-1}}{n-1}e^{M(n-1)}(1-r)^{1-n} \\ &= \frac{2^{2n-1}\omega_{n-1}}{n-1}e^{M(n-1)}(1-r)^{1-n}. \end{aligned}$$

5. Show that for given $\varepsilon > 0$ there are numbers $r_1 > s_1 > r_2 > s_2 > \dots$ such that $\mathsf{M}(\Delta(E,F,\mathbf{R^n})) < \varepsilon$, when $E = \cup S^{n-1}(r_j)$ and $F = \cup S^{n-1}(s_j)$.

Solution: Let $\varepsilon > 0$. Denote $k_i = e^{-e^{p^i}}$, $i = 1,2,\dots$, $p > 1$. Choose $r_i = k_{2i-1}$, $s_i = k_{2i}$, $i = 1,2,\dots$. Then $r_1 > s_1 > r_2 > s_2 > \cdots$. Denote for $i = 1,2,\dots$

$$\Gamma_i = \Delta(S^{n-1}(k_i), S^{n-1}(k_{i+1})).$$

Then $\bigcup_{i=1}^\infty \Gamma_i < \Delta(E,F;\mathbf{R^n})$, which implies with (5.14)[CGQM], that

$$\begin{aligned} \mathsf{M}(\Delta(E,F;\mathbf{R^n})) &\le \mathsf{M}(\bigcup_{i=1}^\infty \Gamma_i) \le \sum_{i=1}^\infty \mathsf{M}(\Gamma_i) \\ &= \omega_{n-1}\sum_{i=1}^\infty\left(\log\frac{k_i}{k_{i+1}}\right)^{1-n} \\ &= \omega_{n-1}\sum_{i=1}^\infty(-e^{p^i} - (-e^{p^i+p}))^{1-n} \\ &= \omega_{n-1}\sum_{i=1}^\infty(e^{p^i}(e^p-1))^{1-n} \\ &\le \omega_{n-1}\sum_{i=1}^\infty e^{-(n-1)p^i} < \varepsilon, \end{aligned}$$

if $p > 1$ is so big that $\sum_{i=1}^{\infty} e^{-(n-1)p^i} < \varepsilon/\omega_{n-1}$.

6. Let $G, G' \subset \mathbf{R^n}$, $n \geq 2$, and let $f : G \to G'$ be a homeomorphism with the following property: There exists $c_1 \in (0, \infty)$ such that for every subdomain $D \subset G$ and for all $x, y \in D$,

$$(\star) \qquad j_{fD}(f(x), f(y)) \leq c_1 j_D(x, y).$$

Show that for each $z \in G$,

$$H(f, z) = \limsup_{r \to 0} \left\{ \frac{|f(x) - f(z)|}{|f(y) - f(z)|} \ : \ |x - z| = |y - z| = r \right\} \leq c_2,$$

where $c_2 \in (1, \infty)$. Show that this inequality holds (possibly with a different constant c_2) also if in $(\star)$ j_D and j_{fD} are replaced by k_D and k_{fD}.

Solution: Fix $z \in G$ and $r \in (0, d(z, \partial G)/2)$. Let $D = G \setminus \{z\}$. Then

$$j_D(x, y) = \log(1 + \frac{|x - y|}{r}) \leq \log(1 + \frac{2r}{r}) = \log 3$$

for all $x, y \in S^{n-1}(z, r)$, while by lemma 2.36(1)[CGQM],

$$j_{fD}(f(x), f(y)) \geq \left| \log \frac{|f(x) - f(z)|}{|f(y) - f(z)|} \right| .$$

These inequalities with (*) show that $H(f, z) \leq c_2$, where $c_2 = 3^{c_1} \in (1, \infty)$.
In the case of k_D and k_{fD}, we use (3.4)[CGQM] and (*) to obtain

$$j_{fD}(f(x), f(y)) \ \leq \ k_{fD}(f(x), f(y)) \leq c_1 k_D(x, y)$$

for all $x, y \in D$. With the above argument this yields $H(f, z) \leq e^{c_1 k_D(x,y)}$. By the choice of r, $d(\zeta, \partial D) = r$ for all $\zeta \in S^{n-1}(z, r)$. Let γ be the shortest arc contained in $S^{n-1}(z, r)$ joining $x, y \in S^{n-1}(z, r)$. Then

$$k_D(x, y) \leq \int_\gamma \frac{|d\zeta|}{d(\zeta, \partial D)} \leq \frac{1}{r} \pi r = \pi.$$

$\therefore H(f, z) \leq e^{c_1 \pi}$.