

Quasiregular Mappings
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Problem Set 7
Winter 2009 / Vuorinen

1. Let $t > r > s > 0$, $E \subset \mathbf{B}^n(s)$ and $\Delta_a = \Delta(E, S^{n-1}(a))$. Show that $M(\Delta_r) \leq cM(\Delta_t)$, where c is only dependent on n, r, s and t .
2. Prove Theorem 6.1 (3) [CGQM].
3. Let $G = \mathbf{R}^n \setminus \{0\}$ and let s_G be defined as

$$s_G(x, y) = \frac{|x - y|^2}{2|x||y|}, \quad x, y \in G.$$

Define ρ_G by $\text{ch } \rho_G(x, y) = 1 + s_G(x, y)$. Show that

$$j_G(x, y)/2 \leq \rho_G(x, y) \leq 4j_G(x, y)$$

for $x, y \in G$. Hint: Use the inequality from h0405 to the effect that for $a \geq 0$

$$\begin{aligned} \log(1 + \max\{a, \sqrt{a}\}) &\leq b \leq \log(1 + a + \sqrt{a}) \\ &\leq 2 \log(1 + \max\{a, \sqrt{a}\}) \end{aligned}$$

if $\text{ch } b = 1 + \frac{1}{2}a$.

4. Show that for $0 < r < 1$ and $M > 0$, $m_h(\bigcup_{|x| \leq r} D(x, M)) \leq d_2(n, M)(1 - r)^{1-n}$, where m_h is the hyperbolic measure of $(\mathbf{B}, \rho_{\mathbf{B}})$.
5. Show that for given $\varepsilon > 0$ there are numbers $r_1 > s_1 > r_2 > s_2 > \dots$ such that $M(\Delta(E, F, \mathbf{R}^n)) < \varepsilon$, when $E = \cup S^{n-1}(r_j)$ and $F = \cup S^{n-1}(s_j)$.
6. Let $G, G' \subset \mathbf{R}^n$, $n \geq 2$, and let $f : G \rightarrow G'$ be a homeomorphism with the following property: There exists $c_1 \in (0, \infty)$ such that for every subdomain $D \subset G$ and for all $x, y \in D$,

$$(\star) \quad j_{fD}(f(x), f(y)) \leq c_1 j_D(x, y).$$

Show that for each $z \in G$,

$$H(f, z) = \limsup_{r \rightarrow 0} \left\{ \frac{|f(x) - f(z)|}{|f(y) - f(z)|} : |x - z| = |y - z| = r \right\} \leq c_2,$$

where $c_2 \in (1, \infty)$. Show that this inequality holds (possibly with a different constant c_2) also if in (\star) j_D and j_{fD} are replaced by k_D and k_{fD} .

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