Quasiregular Mappings
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Problem Set 6
Winter 2009 / Vuorinen

1. Show that for all $x, y \in G \subsetneq \mathbf{R^n}$ the inequality

$$\inf_{\gamma \in \Gamma_{xy}} \log\left(1 + \frac{\ell(\gamma)}{\min\{d(x), d(y)\}}\right) \le k_G(x, y),$$

where Γ_{xy} is the family of all curves connecting the points x and y within G, and $d(x) = \mathrm{dist}(x, \partial G)$. Hint. Recall the normal representation of a curve.

Solution: Let $x, y \in G$, and assume without loss of generality that $\mathrm{d}(x, \partial G) \le \mathrm{d}(y, \partial G)$. Assume that all paths $\gamma \in \Gamma_{xy}$ are parametrized by the normal representation (i.e. with the pathlength as a parameter) $\gamma\colon [0, \ell(\gamma)] \to G$. Then $d(\gamma(t), \partial G) \le d(x, \partial G) + t$ and $|\gamma'(t)| = 1$ a.e. and further

$$\begin{aligned}
k_G(x, y) &= \inf_{\gamma \in \Gamma_{xy}} \int_\gamma \frac{|dz|}{\mathrm{d}(z, \partial G)} = \inf_{\gamma \in \Gamma_{xy}} \int_0^{\ell(\gamma)} \frac{|\gamma'(t)|\, dt}{\mathrm{d}(\gamma(t), \partial G)} \ge \inf_{\gamma \in \Gamma_{xy}} \int_0^{\ell(\gamma)} \frac{dt}{\mathrm{d}(x, \partial G) + t} \\
&= \inf_{\gamma \in \Gamma_{xy}} \Big|_{t=0}^{\ell(\gamma)} \log(\mathrm{d}(x, \partial G) + t) = \inf_{\gamma \in \Gamma_{xy}} (\log[\mathrm{d}(x, \partial G) + \ell(\gamma)) - \log \mathrm{d}(x, \partial G)] \\
&= \inf_{\gamma \in \Gamma_{xy}} \log\left(1 + \frac{\ell(\gamma)}{\mathrm{d}(x, \partial G)}\right) = \inf_{\gamma \in \Gamma_{xy}} \log\left(1 + \frac{\ell(\gamma)}{\min\{\mathrm{d}(x, \partial G), \mathrm{d}(y, \partial G)\}}\right).
\end{aligned}$$

2. Let $D = \mathbf{R^2} \setminus \{\mathbf{te_1 : t \ge 0}\}$. Show that there is no constant $C > 0$ such that

$$k_D(x, y) \le C j_D(x, y),\ \forall x, y \in D.$$

Solution: For $t > 0$ let $x_t = e_1 + te_2$, $y_t = e_1 - te_2$. Let $J_t = J_D[x_t, y_t]$ and let $z_t \in J_t \cap \{se_1 \mid s < 0\}$. Then, by (3.4)[CGQM],

$$k_D(x_t, y_t) \ge k_D(x_t, z_t) \ge j_D(x_t, z_t) = \log(1 + \frac{|x_t - z_t|}{t}) \ge \log(1 + \frac{1}{t}) \to \infty,$$

as $t \to 0$.
On the other hand,

$$j_D(x_t, y_t) = \log(1 + \frac{|x_t - y_t|}{t}) = \log(1 + \frac{2t}{t}) = \log 3.$$

$\therefore$ There is no constant $c > 0$ such that for $t > 0$, $k_D(x_t, y_t) \le C j_D(x_t, y_t)$.

3. Let us define

$$m(x,y)=|x-y|^{1/2}$$

for $x,y\in\mathbf{R}$. Show that m is a metric and $[0,1]$ is not rectifiable with respect to m. (Hint: length of $[0,1]$ is

$$\sup\left\{\sum_{i=1}^{n} m(x_i,x_{i+1})\colon n\in\mathbb{N},\ x_1=0, x_{n+1}=1, x_k<x_{k+1}\right\}.)$$

Solution: By Problem Set 3 Ex. 6 m is a metric.
Consider $x_1=0, x_2=1/n,\dots,x_k=(k-1)/n,\dots,x_{n+1}=1$. Now

$$\sum_{i=1}^{n} m(x_i,x_{i+1})=\sum_{i=1}^{n}\sqrt{x_{i+1}-x_i}=\sum_{i=1}^{n}\sqrt{1/n}=n\frac{1}{\sqrt{n}}=\sqrt{n}$$

and $\sqrt{n}\to\infty$ as $n\to\infty$. Therefore $[0,1]$ is not rectifiable.

NOTE THAT THE ESTIMATE FOR 4 (A) WAS GIVEN IN THE LECTURE ON THE SAME DAY WHEN THE PROBLEM SESSION WAS HELD.

4. Let $r_k\in(0,2^{-k-3})$, let $B_k=B^n(2^{-k}e_n,r_k)$, and let $E=\cup B_k$.
(a) Show that the numbers r_k can be chosen so that $\mathsf{M}(\Delta(E,\partial\mathbf{H}^n))=\infty$.
(b) Show that, for every $\varepsilon>0$, the numbers r_k can be chosen so that

$$\mathsf{M}(\Delta(E,\partial\mathbf{H}^n))<\varepsilon.$$

(c) Assume that the numbers r_k have been chosen as in (b) with $\varepsilon=1$. Show that

$$\mathsf{M}(\Delta(E_r,\partial\mathbf{H}^n))\to 0$$

when $r\to 0$ where $E_r=E\cap\overline{\mathbf{B}^n}(r)$.

Solution:

(a) Choose $r_k=2^{-k-4}$ and denote

$$\Delta_k=\Delta(B_k,\partial\mathbf{H}^n;R(2^{-k}+r_k,2^{-k}-r_k)).$$

Since Δ_k are separate, we get by [CGQM, 5.4] and [CGQM, 5.32],

$$\begin{aligned}\mathsf{M}(\Delta(E,\partial\mathbf{H}^n)) &\geq \sum_{k=0}^{\infty}\mathsf{M}(\Delta_k)\geq\sum_{k=0}^{\infty}c_n\log\frac{2^{-k}+r_k}{2^{-k}-r_k}\\ &= \sum_{k=0}^{\infty}c_n\log(1+\frac{2r_k}{2^{-k}-r_k}).\end{aligned}$$

Hence $M(\Delta(E,\partial\mathbf{H}^n))\geq\sum_{k=0}^{\infty}c_n\frac{4}{15}\log\frac{3}{2}=\infty$.

(b) Let $\varepsilon > 0$. Denote $\Gamma_k = \Delta(B_k, \partial\mathbf{H}^n)$, $\tilde{\Gamma}_k = \Delta(B_k, S^{n-1}(2^{-k}e_n, 2^{-k}))$ for all $k = 0, 1, \ldots$. Then $\tilde{\Gamma}_k < \Gamma_k$ for all k and we get by [CGQM, 5.2.(3), 5.3, (5.14)],

$$\mathsf{M}(\Delta(E, \partial\mathbf{H}^n)) \le \mathsf{M}(\bigcup_{k=0}^{\infty}\Gamma_k) \le \sum_{k=0}^{\infty}\mathsf{M}(\Gamma_k) \le \sum_{k=0}^{\infty}\mathsf{M}(\tilde{\Gamma}_k) = \sum_{k=0}^{\infty}\omega_{n-1}\left(\log\frac{2^{-k}}{r_k}\right)^{1-n}.$$

Choose $r_k = 2^{-k}e^{-k^p}$ to get

$$\mathsf{M}(\Delta(E, \partial\mathbf{H}^n)) \le \omega_{n-1}\sum_{k=0}^{\infty}k^{-p(n-1)}.$$

Making $p > 1$ so big that $\sum_{k=0}^{\infty}k^{-p(n-1)} < \varepsilon/\omega_{n-1}$, we get the inequality $\mathsf{M}(\Delta(E, \partial\mathbf{H}^n)) < \varepsilon$.

(c) For $r > 0$, let $k_r \in \mathbb{N}\cup\{0\}$ be such that $\overline{\mathbf{B}}^n(r)\cap B_{k_r} \neq \emptyset$ $\forall k \ge k_r$ and $\overline{\mathbf{B}}^n(r)\cap B_{k_r-1} = \emptyset$. It is clear that $k_r \to \infty$ as $r \to 0$. We have that

$$\mathsf{M}(\Delta(E_r, \partial\mathbf{H}^n)) \le \mathsf{M}(\bigcup_{k=k_r}^{\infty}\Gamma_k) \le \omega_{n-1}\sum_{k=k_r}^{\infty}\left(\log\frac{2^{-k}}{r_k}\right)^{1-n}.$$

By the choice of r_k in (b), the series $\sum_{k=0}^{\infty}\left(\log\frac{2^{-k}}{r_k}\right)^{1-n}$ converges. Now $\sum_{k=k_r}^{\infty}\left(\log\frac{2^{-k}}{r_k}\right)^{1-n}$ is a remainder term of the series and hence

$$\sum_{k=k_r}^{\infty}\left(\log\frac{2^{-k}}{r_k}\right)^{1-n} \to 0$$

as $r \to 0$, since then $k_r \to \infty$. The claim follows.

5. Let $E, F \subset R(2,1)$, $R(a,b) = \mathbf{B}^n(a)\setminus\overline{\mathbf{B}^n(b)}$, $a > b > 0$, and

$$\delta = \mathsf{M}(\Delta(E, F, \mathbf{R}^n)) > 0.$$

Find a number $c > 1$, $c = c(n,\delta)$, such that $\mathsf{M}(\Delta(E, F, R(2c, 1/c))) \ge \delta/2$.

Solution: Denote $\Gamma = \Delta(E, F; \mathbf{R}^n)$, $\Gamma_c = \Delta(E, F; R(2c, 1/c)))$,

$$\Gamma'_c = \{\gamma\in\Gamma \mid |\gamma|\cap\overline{\mathbf{B}}^n(1/c)\neq\emptyset \text{ or } |\gamma|\cap(\mathbf{R}^n\setminus\mathbf{B}^n(2c))\neq\emptyset\}$$

for $c > 1$. Then $\Gamma = \Gamma_c\cup\Gamma'_c$. It follows that

$$\begin{aligned}\delta = \mathsf{M}(\Gamma) &\le \mathsf{M}(\Gamma_c) + \mathsf{M}(\Gamma'_c)\\ &\le \mathsf{M}(\Gamma_c) + 2\omega_{n-1}(\log c)^{1-n}.\end{aligned}$$

Choose $c \geq e^{(4\omega_{n-1}/\delta)^{1/(n-1)}}$ to get

$$\begin{aligned}\mathsf{M}(\Delta(E,F;R(2c,1/c))) &= M(\Gamma_c) \geq \delta - 2\omega_{n-1}(\log c)^{1-n} \\ &\geq \delta - 2\omega_{n-1}(\log e^{(4\omega_{n-1}/\delta)^{1/(n-1)}})^{1-n} \\ &= \delta - \frac{\delta}{2} = \frac{\delta}{2}.\end{aligned}$$

6. Let $E \subset \mathbf{B}^n$ and $\delta = \mathsf{M}(\Delta(S^{n-1}(2), E; \mathbf{R}^n)) > \mathbf{0}$. Find a number $c > 1, c = c(n,\delta)$, such that

$$\mathsf{M}(\Delta(S^{n-1}(2), E; R(2c, 1/c))) \geq \delta/2.$$

Solution: Denote

$$\begin{aligned}\Gamma &= \Delta(S^{n-1}(2), E; \mathbf{R}^\mathbf{n}) \\ \Gamma_c &= \Delta(S^{n-1}(2), E; R(2c, 1/c)) \\ \Gamma'_c &= \{\gamma \in \Gamma \mid |\gamma| \cap \mathbf{B}^n(1/c) \neq \emptyset\}\end{aligned}$$

for $c > 1$. By [CGQM, (5.10)],

$$\mathsf{M}(\Gamma) = \mathsf{M}(\Delta(E, S^{n-1}(2); \mathbf{B}^n(2)) = \mathsf{M}(\Gamma_c \cup \Gamma'_c),$$

so we have that

$$\begin{aligned}\delta = \mathsf{M}(\Gamma) &\leq \mathsf{M}(\Gamma_c) + \mathsf{M}(\Gamma'_c) \\ &= \mathsf{M}(\Gamma_c) + \omega_{n-1}(\log \frac{2}{1/c})^{1-n} \\ &= \mathsf{M}(\Gamma_c) + \omega_{n-1}(\log 2c)^{1-n}.\end{aligned}$$

Choose $c \geq e^{(2\omega_{n-1}/\delta)^{1/(n-1)}}/2$ $(c > 1)$ to get

$$\mathsf{M}(\Delta(S^{n-1}(2), E; R(2c, 1/c))) = \mathsf{M}(\Gamma_c) \geq \delta - \omega_{n-1}(\log 2\frac{e^{(2\omega_{n-1}/\delta)^{1/(n-1)}}}{2})^{1-n} = \frac{\delta}{2}.$$