Quasiregular Mappings
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Problem Set 6
Winter 2009 / Vuorinen

1. Show that for all $x, y \in G \subsetneq \mathbf{R}^n$ the inequality

$$\inf_{\gamma \in \Gamma_{xy}} \log\left(1 + \frac{\ell(\gamma)}{\min\{d(x), d(y)\}}\right) \le k_G(x, y),$$

where Γ_{xy} is the family of all curves connecting the points x and y within G, and $d(x) = \mathrm{dist}(x, \partial G)$.

2. Let $D = \mathbf{R}^2 \setminus \{te_1 : t \ge 0\}$. Show that there is no constant $C > 0$ such that

$$k_D(x, y) \le C j_D(x, y),\ \forall x, y \in D.$$

3. Let us define

$$m(x, y) = |x - y|^{1/2}$$

for $x, y \in \mathbf{R}$. Show that m is a metric and $[0, 1]$ is not rectifiable with respect to m. (Hint: length of $[0, 1]$ is

$$\sup\left\{\sum_{i=1}^{n} m(x_i, x_{i+1})\colon n \in \mathbb{N},\ x_1 = 0, x_{n+1} = 1, x_k < x_{k+1}\right\}.)$$

4. Let $r_k \in (0, 2^{-k-3})$, let $B_k = B^n(2^{-k} e_n, r_k)$, and let $E = \cup B_k$.
(a) Show that the numbers r_k can be chosen so that $\mathsf{M}(\Delta(E, \partial \mathbf{H})) = \infty$.
(b) Show that, for every $\varepsilon > 0$, the numbers r_k can be chosen so that

$$\mathsf{M}(\Delta(E, \partial \mathbf{H})) < \varepsilon.$$

(c) Assume that the numbers r_k have been chosen as in (b) with $\varepsilon = 1$. Show that

$$\mathsf{M}(\Delta(E_r, \partial \mathbf{H})) \to 0$$

when $r \to 0$ where $E_r = E \cap \overline{\mathbf{B}^n(r)}$.

5. Let $E, F \subset R(2, 1), R(a, b) = \mathbf{B}^n(a) \setminus \overline{\mathbf{B}^n(b)}, a > b > 0$, and

$$\delta = \mathsf{M}(\Delta(E, F, \mathbf{R}^n)) > 0.$$

Find a number $c > 1, c = c(n, \delta)$, such that $\mathsf{M}(\Delta(E, F, R(2c, 1/c))) \ge \delta/2$.

6. Let $E \subset \mathbf{B}^n$ and $\delta = \mathsf{M}(\Delta(S^{n-1}(2), E, \mathbf{R}^n)) > 0$. Find a number $c > 1, c = c(n, \delta)$, such that

$$\mathsf{M}(\Delta(S^{n-1}(2), E, R(2c, 1/c))) \ge \delta/2.$$

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