Quasiregular Mappings
Department of Mathematics and Statistics
University of Helsinki
Problem Set 5
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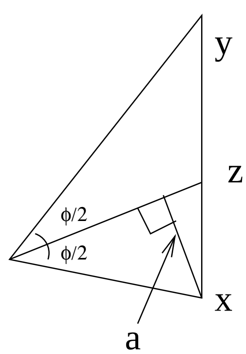
1. Let $G = \mathbb{R}^n \setminus \{0\}$, $x, y \in G$, and let $\varphi \in [0, \pi]$ be the angle between the segments $[0, x]$ and $[0, y]$.
(a) Show that $\sin \frac{1}{2}\varphi \leq \frac{|x-y|}{|x|+|y|}$.
(b) Show that $|x - y| \leq ||x| - |y|| + 2\min\{|x|, |y|\} \sin(\varphi/2)$.
(c) It is known (by [MOS]) that $k_G(x, y) = \sqrt{\varphi^2 + \log^2 \frac{|x|}{|y|}}$. Using this result show that there is constant A such that that $k_G(x, y) \leq A\, j_G(x, y)$ for all $x, y \in G$ i.e. that G is a uniform domain.

Solution: Let z and a be as in the picture. Now



$$\frac{|y-z|}{|x-z|} = \frac{|y|}{|x|} \Leftrightarrow |y-z| = \frac{|y|}{|x|}|x-z| = \frac{|y|}{|x|}(|x-y| - |y-z|)$$

$$\Leftrightarrow |y-z| = \frac{\frac{|y|}{|x|}|x-y|}{1 + \frac{|y|}{|x|}} = \frac{|x-y||y|}{|x|+|y|}.$$

Similarly $|x - z| = \frac{|x-y||x|}{|x|+|y|}$. Now $a \leq |x - z|$ implies

$$\sin\frac{\phi}{2} = \frac{a}{|x|} \leq \frac{|x-z|}{|x|} = \frac{|x-y|}{|x|+|y|}.$$

Hence

$$\begin{aligned}\phi &\le 2\arcsin\frac{|x-y|}{|x|+|y|} \overset{(*)}{\le} 2\Big(\frac{\pi}{2}\frac{|x-y|}{|x|+|y|}\Big)\\ &= \pi\frac{|x-y|}{|x|+|y|} \overset{(**)}{\le} \frac{\pi\log(1+\frac{|x-y|}{|x|+|y|})}{\log 2} \le \frac{\pi}{\log 2} j_G(x,y).\end{aligned}$$

In (*) we used the fact that for $z\in[0,1]$ $\arcsin z \le \frac{\pi}{2}z$ and in (**) the Bernoulli inequality: $\log 2 = \log(1+\frac{1}{z}z) \le \frac{1}{z}\log(1+z)$ $(z\in]0,1])$.
It follows from lemma 2.36(1)[CGQM] that

$$\begin{aligned}k_G(x,y)^2 &= \phi^2 + \log^2\frac{|x|}{|y|} \le \frac{\pi^2}{\log^2 2} j_G(x,y)^2 + j_G(x,y)^2\\ &= \left(1+\left(\frac{\pi}{\log 2}\right)^2\right) j_G(x,y)^2.\end{aligned}$$

Therefore $k_G(x,y) \le A j_G(x,y)$ where $A = \sqrt{1+(\frac{\pi}{\log 2})^2}$.
It remains to prove (b). We may assume $|x|\le|y|$. Choose $z'\in[0,y]$ such that $|z'|=|x|$. Then

$$\begin{aligned}|x-y| &\le |x-z'|+|z'-y| = 2|x|\sin\frac{\phi}{2} + ||y|-|x||\\ &= ||x|-|y|| + 2\min\{|x|,|y|\}\sin\frac{\phi}{2}.\end{aligned}$$

2. Let $x,y\in\mathbb{R}^n\setminus\{0\}$ and $|y|\ge|x|$. Show that $d(y,[0,x]) \ge \frac{|x-y|}{2}$.

Solution: Let π_0, π_x be the hyperplanes orthogonal $[0,x]$ such that $0\in\pi_0$ and $x\in\pi_x$. There are 3 cases:

i) $y\in\pi_x$ or y is on the opposite side of π_x from 0. Then $d(y,[0,x]) = |x-y| \ge \frac{|x-y|}{2}$.

ii) $y\in\pi_0$ or y is on the opposite side of π_0 from x. Then $|x-y| \le |x|+|y| \le 2|y| = 2d(y,[0,x])$.

iii) y is between the planes π_0 and π_x. Choose $u\in[0,x]$ such that $|y-u| = d(y,[0,x])$. We denote $d=d(y,[0,x])$. Now

$$\begin{aligned}d^2 &= |y|^2-|u|^2 = |y-x|^2-|u-x|^2\\ \Rightarrow 2d^2 &= |y|^2+|y-x|^2-(|u|^2+|u-x|^2)\\ &\ge |y|^2+|y-x|^2-|x|^2 \ge |y-x|^2\\ \Rightarrow d &\ge \frac{|x-y|}{\sqrt{2}} \ge \frac{|x-y|}{2}.\end{aligned}$$

Second solution: For $z \in (0,x)$ we have $|y-z| \geq |x-z|$ because $y \notin B^n(z,|z-x|)$. On the other hand $|y-z|+|x-z| \geq |x-y|$ and hence $2|y-z| \geq |x-y|$.

3. Let $f \in \mathcal{GM}(\mathbf{B}^n)$ and $r \in (0,1)$. Show that

$$|f(x)-f(y)| \leq \frac{1}{1-r^2}|x-y|,$$

for $|x|,|y| \leq r$. [Hint: $\operatorname{sh}^2 \frac{\rho(x,y)}{2} = \ldots$]

Solution: Recall the formulas (2.18)[CGQM] and (2.20)[CGQM]:

$$\operatorname{sh}^2 \frac{\rho(x,y)}{2} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

and

$$\rho(f(x),f(y)) = \rho(x,y)$$

for all $x,y \in B^n$. Hence

$$|f(x)-f(y)|^2 \leq \frac{|f(x)-f(y)|^2}{(1-|f(x)|^2)(1-|f(y)|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} \leq \frac{|x-y|^2}{(1-r^2)^2}$$

because $|x|,|y| \leq r < 1$.

4. For an open set D in $\mathbb{R}^n$, $D \neq \mathbb{R}^n$, let

$$\varphi_D(x,y) = \log\left(1+\max\left\{\frac{|x-y|}{\sqrt{d(x)d(y)}}\ ,\ \frac{|x-y|^2}{d(x)d(y)}\right\}\right)\ ;\ \ x,\,y \in D\,.$$

Show that $j_D(x,y)/2 \leq \varphi_D(x,y) \leq 2\,j_D(x,y)$.

Solution: Exercise 4.5 gives us

$$\phi_D(x,y) \leq \operatorname{arch}\left(1+\frac{|x-y|^2}{2d(x)d(y)}\right) \leq 2\phi_D(x,y)$$

and from exercise 4.3.(5) we obtain

$$\operatorname{arch}(1+\frac{|x-y|^2}{2d(x)d(y)}) \leq 2\log(1+\frac{|x-y|}{\sqrt{d(x)d(y)}}) \leq 2j_D(x,y)$$

$\therefore \phi_D(x,y) \leq 2j_D(x,y)$.
For the proof of the first inequality we may assume $0 < d(x) \leq d(y)$.

Case 1: $|x-y| > d(x)$. Then by using the $\triangle$-inequality we have

$$\begin{aligned}\frac{|x-y|^2}{d(x)d(y)} - \frac{|x-y|}{2d(x)} &\geq \frac{|x-y|^2}{d(x)(d(x)+|x-y|)} - \frac{|x-y|}{2d(x)} \\ &= \frac{2|x-y|^2 - |x-y|^2 - d(x)|x-y|}{2d(x)(d(x)+|x-y|)} \\ &= \frac{|x-y|(|x-y|-d(x))}{2d(x)(d(x)+|x-y|)} > 0\end{aligned}$$

Now by using the Bernoulli inequality and the approximation above in this order we have

$$\begin{aligned} j_D(x,y) &= \log(1+\frac{|x-y|}{d(x)}) \leq 2\log(1+\frac{|x-y|}{2d(x)}) \\ &\leq 2\log(1+\frac{|x-y|^2}{d(x)d(y)}) \leq 2\phi_D(x,y).\end{aligned}$$

Case 2: $|x-y| < d(x)$. Then we have the inequality

$$\frac{|x-y|^2}{d(x)d(y)} \geq \frac{|x-y|^2}{d(x)(d(x)+|x-y|)} \geq \frac{|x-y|^2}{2d(x)^2}$$

and from the Bernoulli inequality and the estimate above we obtain that

$$\begin{aligned} j_D(x,y) &= \log(1+\frac{|x-y|}{d(x)}) \leq \sqrt{2}\log(1+\frac{|x-y|}{\sqrt{2}d(x)}) \\ &\leq \sqrt{2}\log(1+\frac{|x-y|}{\sqrt{d(x)d(y)}}) \leq \sqrt{2}\phi_D(x,y).\end{aligned}$$

$\therefore j_D(x,y) \leq 2\phi_D(x,y)$.

5. Let $G = \mathbb{R}^n \setminus \{0\}$ and $f(x) = a^2x/|x|^2$ for $x \in G$, where $a > 0$. Show that $k_G\big(f(x), f(y)\big) = k_G(x,y)$ and $j_G\big(f(x), f(y)\big) = j_G(x,y)$ for $x, y \in G$.

Solution: Since f is a Möbius map and hence conformal,

$$\measuredangle([0,f(x)],[0,f(y)]) = \measuredangle([0,x],[0,y])$$

for all $x, y \in G$. Denote $\phi = \measuredangle([0,x],[0,y])$. Then, using exercise 1, we get

$$\begin{aligned} k_G(f(x), f(y)) &= \sqrt{\phi^2 + \log^2 \frac{|f(x)|}{|f(y)|}} \\ &= \sqrt{\phi^2 + \log^2 \left(\frac{\frac{a^2}{|x|}}{\frac{a^2}{|y|}} \right)} \\ &= \sqrt{\phi^2 + \log^2 \frac{|x|}{|y|}} = k_G(x,y). \end{aligned}$$

We use (1.4)[CGQM] to get

$$\begin{aligned} j_G(f(x), f(y)) &= \log(1 + \frac{|f(x) - f(y)|}{\min\{|f(x)|, |f(y)|\}}) \\ &= \log(1 + \frac{\frac{a^2|x-y|}{|x||y|}}{\min\{\frac{a^2}{|x|}, \frac{a^2}{|y|}\}}) \\ &= \begin{cases} \log(1 + \frac{|x-y|}{|y|}), & \text{if } |x| \geq |y| \\ \log(1 + \frac{|x-y|}{|x|}), & \text{if } |x| \leq |y| \end{cases} \\ &= j_G(x,y). \end{aligned}$$

6. Let $f : G \to G' = f(G)\,, G, G' \subset \mathbb{R}^n$, be a homeomorphism such that for some $C > 0$ and all $x, y \in G$, $k_{G'}(f(x), f(y)) \leq C k_G(x,y)$. Suppose that $b \in \partial G$ and that $b_k \in G$ with $b_k \to b, f(b_k) \to \beta, k \to \infty$, and let $E = \cup D(b_k, 1)$. Here $D(x, M)$ stands for the quasihyperbolic ball. Prove that $f(x) \to \beta$ when $x \to b, x \in E$. Note: By topology, for each sequence (b_k) tending to a boundary point b of G such that the image sequence also has a limit γ, it follows that $\gamma \in \partial G'$.

Solution: Suppose, on the contrary, that there exists a sequence $(a_k) \in E$ with $a_k \to b$ and $f(a_k) \to \alpha \neq \beta$. Then, by topology, $\alpha \in \partial G'$. Denote $\delta = |\beta - \alpha| > 0$. By passing onto a subsequence and relabeling if necessary, we may assume that $k_G(a_k, b_k) < 1$ for all k. Since $\alpha, \beta \in \partial G'$, it follows that

$$j_{G'}(f(a_k), f(b_k)) = \log(1 + \frac{|f(a_k) - f(b_k)|}{\min\{d(f(a_k)), d(f(b_k))\}}) \to \infty$$

when $k \to \infty$. But by (3.4)[CGQM] we have

$$j_{G'}(f(a_k), f(b_k)) \leq k_{G'}(f(a_k), f(b_k)) \leq C k_G(a_k, b_k) < C < \infty$$

This is a contradiction.