Quasiregular Mappings
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Problem Set 4
Winter 2009/ Vuorinen

1. (1) Show that an inversion f in $S^{n-1}(a,r)$, when $a_n = 0$, preserves the upper half-space

$$f(\mathbf{H}^n) = \mathbf{H}^n.$$

(2) Show that the expression

$$\frac{|x-y|^2}{2x_n y_n},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, is invariant under an inversion in $S^{n-1}(a,r)$ when $a_n = 0$.

Solution: (1) Because $a_n = 0$ we see that

$$fx = a + \frac{r^2(x-a)}{|x-a|^2} \implies (fx)_n = \frac{r^2 x_n}{|x-a|^2}$$

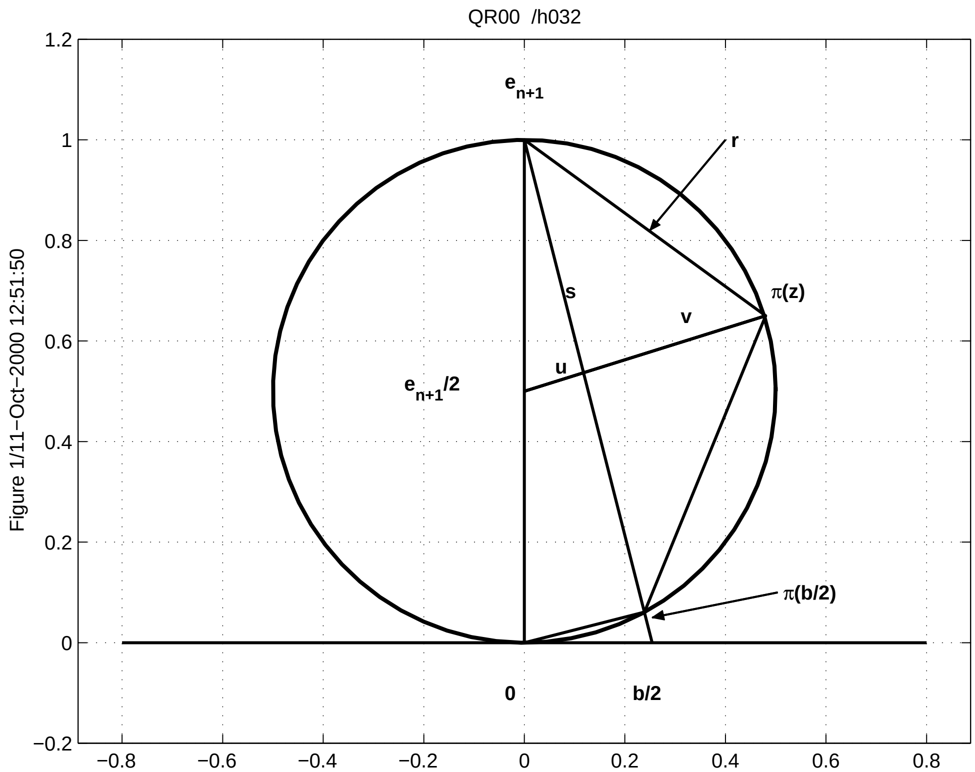
and hence $fH^n \subset H^n$. Because $f = f^{-1}$ the assertion follows.
(2) The proof follows from part (1), because by (1.5)[CGQM]

$$|fx - fy| = \frac{r^2|x-y|}{|x-a||y-a|}.$$

2. Let f be the translation $f : x \mapsto x + b$. Find an upper bound for the Lipschitz constant of f in the spherical metric.

Solution: By 1.3[CGQM] $f = f_1 \circ f_2$, where f_1 is the reflection in $P(b,0)$ and f_2 is the reflection in $P(b, \frac{|b|^2}{2})$. Since $0 \in P(b,0)$, $f_1 \in O(n)$ (orthogonal map) and we have that $\mathrm{Lip}(f_1) = 1$. $P(b, \frac{|b|^2}{2}) = \partial Q(z,r)$, where $\infty \in \partial Q(z,r)$. Hence f_2 is the inversion in $\partial Q(z,r)$ and by Lemma 1.54 (3)[CGQM], $\mathrm{Lip}(f_2) \le \frac{1}{r^2} - 1$.
Now $2s = q(\frac{b}{2}, \infty) = \frac{1}{\sqrt{1+\frac{|b|^2}{4}}}$, and it follows that $s = \frac{1}{\sqrt{4+|b|^2}}$. Since $u^2 + s^2 = \frac{1}{4}$, it follows that

$$u = \sqrt{\frac{1}{4} - s^2} = \frac{|b|}{2\sqrt{4+|b|^2}}.$$



Then

$$v = \frac{1}{2} - u = \frac{\sqrt{4+|b|^2} - |b|}{2\sqrt{4+|b|^2}}.$$

Thus

$$\begin{aligned} r^2 &= s^2 + v^2 = \frac{1}{4+|b|^2} + \frac{4+|b|^2 - 2|b|\sqrt{4+|b|^2} + |b|^2}{4(4+|b|^2)} \\ &= \frac{4+|b|^2 - |b|\sqrt{4+|b|^2}}{2(4+|b|^2)}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} \frac{1}{r^2} - 1 &= \frac{2(4+|b|^2)}{4+|b|^2 - |b|\sqrt{4+|b|^2}} - 1 = \frac{2(4+|b|^2)(4+|b|^2+|b|\sqrt{4+|b|^2})}{(4+|b|^2)^2 - |b|^2(4+|b|^2)} \\ &= \frac{1}{2}(4+|b|^2+|b|\sqrt{4+|b|^2}) - 1 = 1 + \frac{|b|^2}{2} + |b|\sqrt{1+\frac{|b|^2}{4}}. \end{aligned}$$

$\therefore \mathrm{Lip}(f) \le 1 + \frac{|b|^2}{2} + |b|\sqrt{1+\frac{|b|^2}{4}}$.

3. Verify the following elementary relations.
(1) $1 - e^{-s} \le \operatorname{th} s \le 1 - e^{-2s}$ for $s \ge 0$.
(2) If $s \ge 0$, then

$$\operatorname{th} s = \frac{\operatorname{th} 2s}{1+\sqrt{1-\operatorname{th}^2 2s}} \ .$$

Further, if $u \in [0,1]$ and $2s = \operatorname{arth} u$, then

$$\operatorname{th} s = \frac{u}{1+\sqrt{1-u^2}} \le \tfrac{1}{2}(u+u^2) \ .$$

(3) $\log \operatorname{th} s = -2 \operatorname{arth} e^{-2s}, \ s > 0$.
(4) $\log(1+x) \le \operatorname{arsh} x \le 2\log(1+x) \ , \ x \ge 0$.
(5) $2\log\Big(1+\sqrt{\tfrac{1}{2}(x-1)}\Big) \le \operatorname{arch} x \le 2\log\Big(1+\sqrt{2(x-1)}\Big) \ , \ x \ge 1$.

Solution:

(1)

$$\begin{aligned}
\operatorname{th} s &= \frac{e^s - e^{-s}}{e^s + e^{-s}} = \frac{1-e^{-2s}}{1+e^{-2s}} \le 1 - e^{-2s} \\
\operatorname{th} s \ge 1 - e^{-s} &\Leftrightarrow \frac{1-y^2}{1+y^2} \ge 1 - y \Leftrightarrow y^3 - 2y^2 + y \ge 0 \\
&\Leftrightarrow y(y-1)^2 \ge 0 \Leftrightarrow e^{-s}(e^{-s}-1)^2 \ge 0, \ \text{OK}
\end{aligned}$$

where we denoted e^{-s} by y.

(2)

$$\begin{aligned}
\frac{2\operatorname{th} s}{1+\operatorname{th}^2 s} &= \frac{2\frac{\operatorname{sh} s}{\operatorname{ch} s}}{1+\frac{\operatorname{sh}^2 s}{\operatorname{ch}^2 s}} = \frac{2\operatorname{sh} s \operatorname{ch} s}{\operatorname{ch}^2 s + \operatorname{sh}^2 s} \\
&= \frac{2\operatorname{sh} s\operatorname{ch} s}{2\operatorname{ch}^2 - 1} = \frac{2\frac{1}{2}(e^s - e^{-s})\frac{1}{2}(e^s+e^{-s})}{2(\frac{1}{2}(e^s+e^{-s}))^2 - 1} \\
&= \frac{e^{2s} - e^{-2s}}{e^{2s}+e^{-2s}} = \operatorname{th} 2s
\end{aligned}$$

Thus $\operatorname{th}(2s)\operatorname{th}^2(s) - 2\operatorname{th}(s) + \operatorname{th}(2s) = 0$ and

$$\begin{aligned}
\operatorname{th} s &= \frac{2 \pm \sqrt{4 - 4\operatorname{th}^2(2s)}}{2\operatorname{th}(2s)} = \frac{1 \pm \sqrt{1-\operatorname{th}^2(2s)}}{\operatorname{th}(2s)} \\
&= \frac{1 - \sqrt{1-\operatorname{th}^2(2s)}}{\operatorname{th}(2s)} = \frac{\operatorname{th}(2s)}{1+\sqrt{1-\operatorname{th}^2(2s)}}.
\end{aligned}$$

Now it follows that

$$\mathrm{th}(\frac{1}{2}\,\mathrm{arth}\,u) = \frac{\mathrm{th}(\mathrm{arth}\,u)}{1+\sqrt{1-(\mathrm{th}\,\mathrm{arth}\,u)^2}} = \frac{u}{1+\sqrt{1-u^2}}.$$

We also have

$$\begin{aligned}
& \frac{u}{1+\sqrt{1-u^2}} \le \frac{1}{2}(u+u^2) \\
\Leftrightarrow\quad & u \le \frac{1}{2}(u+u^2)(1+\sqrt{1-u^2}) = \frac{1}{2}(u+u^2+u\sqrt{1-u^2}+u^2\sqrt{1-u^2}) \\
\Leftrightarrow\quad & \frac{1}{2}u(u+u\sqrt{1-u^2}+\sqrt{1-u^2}-1) \ge 0 \\
\Leftrightarrow\quad & \sqrt{1-u^2}(u+1)+u-1 \ge 0,
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{1-u^2}(u+1)+u-1 &\ge \sqrt{1-u^2}(u+1)+\sqrt{1-u^2}(u-1) \\
&= 2u\sqrt{1-u^2} \ge 0, \text{ OK}
\end{aligned}$$

(3) For all $s>0$ we have

$$\log \mathrm{th}\, s = \log\frac{e^s-e^{-s}}{e^s+e^{-s}} = -\log\frac{1+e^{-2s}}{1-e^{-2s}} = -2\,\mathrm{arth}\,e^{-2s}.$$

(4) For all $x\ge 0$ we have

$$1+x \le x+\sqrt{x^2+1} \Rightarrow \log(1+x) \le \log(x+\sqrt{x^2+1}) = \mathrm{arsh}\,x.$$

We also have

$$\begin{aligned}
& x+\sqrt{x^2+1} \le x+\sqrt{x^2+2x+1} = x+x+1 \le (1+x)^2 \\
\Rightarrow\quad & \mathrm{arsh}\,x \le \log(1+x^2) = 2\log(1+x)
\end{aligned}$$

(5) If $x\ge 1$ we have

$$\begin{aligned}
& (1+\sqrt{\frac{x-1}{2}})^2 \le x+\sqrt{x^2-1} \\
\Leftrightarrow\quad & 1+\frac{x-1}{2}+\sqrt{2(x-1)} \le x+\sqrt{x^2-1} \\
\Leftrightarrow\quad & \sqrt{2(x-1)} \le \frac{x-1}{2}+\sqrt{(x-1)(x+1)} \\
\Leftrightarrow\quad & \sqrt{2} \le \frac{\sqrt{x-1}}{2}+\sqrt{x+1}, \text{ OK}
\end{aligned}$$

$\therefore 2\log(1+\sqrt{\frac{x-1}{2}}) \le \operatorname{arch} x$. In the case $x \ge 1$ we also have that

$$\begin{aligned} & x+\sqrt{x^2-1} \le (1+\sqrt{2(x-1)})^2 = 2x-1+2\sqrt{2(x-1)} \\ \Leftrightarrow\ & \sqrt{x^2-1} \le x-1+2\sqrt{2(x-1)} \\ \Leftrightarrow\ & \sqrt{x+1} \le \sqrt{x-1}+2\sqrt{2} \\ \Leftrightarrow\ & \sqrt{x+1}-\sqrt{x-1} \le 2\sqrt{2} \\ \Leftrightarrow\ & \frac{2}{\sqrt{x+1}+\sqrt{x-1}} \le 2\sqrt{2},\ \text{OK} \end{aligned}$$

$\therefore \operatorname{arch} x \le 2\log(1+\sqrt{2(x-1)})$.

4. Observe first that, for $t \in (0,1)$,

$$\rho_{\mathbb{H}^n}(te_n, e_n) = \rho_{\mathbb{H}^n}\Big(te_n,\ S^{n-1}(\tfrac{1}{2}e_n, \tfrac{1}{2})\Big).$$

Making use of this observation and the formula for ρ-balls in terms of euclidean balls show that

$$B^n(\tfrac{1}{2}e_n, \tfrac{1}{2}) = \bigcup_{t\in(0,1)} D(te_n,\ \log\frac{1}{t})\ .$$

Solution 1: Let $y \in S^{n-1}(\frac{1}{2}e_n, \frac{1}{2})$ and $x \in S^{n-1}(te_n, 1-t) \cap J[te_n, y]$. Then from (2.9)[CGQM] it follows that

$$\operatorname{ch}\rho_{\mathbb{H}^n}(te_n, x) = 1+\frac{(1-t)^2}{2tx_n} \ge 1+\frac{(1-t)^2}{2t} = \operatorname{ch}\rho_{\mathbb{H}^n}(te_n, e_n).$$

Thus $\rho_{\mathbb{H}^n}(te_n, x) \ge \rho_{\mathbb{H}^n}(te_n, e_n)$, since $\operatorname{ch} x$, $x \ge 0$, is strictly increasing. Hence

$$\rho_{\mathbb{H}^n}(te_n, y) = \rho_{\mathbb{H}^n}(te_n, x) + \rho_{\mathbb{H}^n}(x, y) \ge \rho_{\mathbb{H}^n}(te_n, x) \ge \rho_{\mathbb{H}^n}(te_n, e_n).$$

Since $e_n \in S^{n-1}(\frac{1}{2}e_n, \frac{1}{2})$ the first claim follows. (2.6)[CGQM] implies that $\rho_{\mathbb{H}^n}(te_n, e_n) = \log\frac{1}{t}$, so the first claim implies

$$D(te_n, \log\frac{1}{t}) = D(te_n, \rho_{\mathbb{H}^n}(te_n, S^{n-1}(\frac{1}{2}e_n, \frac{1}{2}))) \subset B^n(\frac{1}{2}e_n, \frac{1}{2})$$

for all $t \in]0,1[$.

Let $z \in B^n(\frac{1}{2}e_n, \frac{1}{2})$. We have, according to (2.11)[CGQM], that

$$\begin{aligned} D(te_n, \log\frac{1}{t}) &= B^n((t\,\mathrm{ch}\log(\frac{1}{t}))e_n, t\,\mathrm{sh}\log(\frac{1}{t})) \\ &= B^n\left(t\frac{e^{\log\frac{1}{t}} + e^{-\log\frac{1}{t}}}{2}e_n, t\frac{e^{\log\frac{1}{t}} - e^{-\log\frac{1}{t}}}{2}\right) \\ &= B^n\left(\frac{1+t^2}{2}e_n, \frac{1-t^2}{2}\right). \end{aligned}$$

Now we can choose $t \in]0,1[$ to be so small that $z \in B^n\left(\frac{1+t^2}{2}e_n, \frac{1-t^2}{2}\right) = D(te_n, \log\frac{1}{t})$.
$\therefore B^n(\frac{1}{2}e_n, \frac{1}{2}) = \bigcup_{t\in]0,1[} D(te_n, \log\frac{1}{t})$.

Solution 2: Letting $A = \cup_{t\in(0,1)} D(te_n, \log\frac{1}{t})$ and $B = B^n(\frac{e_n}{2}, \frac{1}{2})$, we claim that $A = B$. We know by (2.11)[CGQM] that

$$D(te_n, \log\frac{1}{t}) = D(te_n, \rho_{\mathbb{H}^n}(te_n, S^{n-1}(\frac{1}{2}e_n, \frac{1}{2}))) \subset B^n(\frac{1}{2}e_n, \frac{1}{2}) = B,$$

hence $A \subset B$. To prove $B \subset A$, let $x \in B$. Then there exists some $r \in (0, 1/2)$ such that

$$x \in B^n(\frac{e_n}{2}, r) = D(se_n, M),$$

for some $s \in (0,1)$ and $M < \log\frac{1}{s}$. According with Diagram 2.4, p.22 in CGQM, we can choose M such that

$$\frac{r}{\sqrt{\frac{1}{4} - r^2}} = \mathrm{sh}\, M \Longrightarrow M = \mathrm{arsh}\,\frac{2r}{\sqrt{1-4r^2}},$$

(whereas $\mathrm{ch}\, M = 1/\sqrt{1-4r^2}$.)

5. Assume that $a \geq 0$ and define b by $\mathrm{ch}\, b = 1 + \frac{1}{2}a$. Show that

$$\begin{aligned} \log\left(1 + \max\left\{a, \sqrt{a}\right\}\right) &\leq b \leq \log\left(1 + a + \sqrt{a}\right) \\ &\leq 2\log\left(1 + \max\left\{a, \sqrt{a}\right\}\right) . \end{aligned}$$

Solution:

$$\mathrm{ch}\, b = 1 + \frac{a}{2} \Rightarrow b = \mathrm{arch}(1 + \frac{a}{2}) = \log(1 + \frac{a}{2} + \sqrt{a + \frac{a^2}{4}}).$$
$$b \geq \log(1 + \frac{a}{2} + \frac{a}{2}) = \log(1+a)$$
$$b \geq \log(1 + \frac{a}{2} + \sqrt{a}) \geq \log(1 + \sqrt{a})$$
$$\therefore b \geq \log(1 + \max\{a, \sqrt{a}\}).$$

For all $a > 0$ we have that

$$\begin{aligned} b &= \log(1+\frac{a}{2}+\sqrt{a+\frac{a^2}{4}}) \\ &\leq \log(1+\frac{a}{2}+\sqrt{a}+\sqrt{\frac{a^2}{4}}) = \log(1+a+\sqrt{a}) \end{aligned}$$

and in the case $a \in]0,1[$ it follows that

$$\log(1+a+\sqrt{a}) \leq \log(1+2\sqrt{a}) \leq \log(1+2\sqrt{a}+a) = \log(1+\sqrt{a})^2$$

and in the case $a \geq 1$ it follows that

$$\log(1+a+\sqrt{a}) \leq \log(1+2a) \leq \log(1+2a+a^2) = \log(1+a)^2.$$

$\therefore \log(1+a+\sqrt{a}) \leq 2\log(1+\max\{a,\sqrt{a}\})$.

6. (1) Show that for distinct points a, b, c, u, v in $\mathbb{R}^n$,

$$|u,a,b,v| = |u,a,c,v||u,c,b,v|,$$
$$|u,a,b,v||u,b,a,v| = 1 = |u,a,b,v||v,a,b,u|.$$

(2) Conclude from (1) that, for a proper subdomain domain G of $\mathbb{R}^n$ and for $x, y \in G$, the quantity

$$m_G(x,y) \equiv \log\sup\{|u,x,y,v| : u,v \in \partial G\}$$

is nonnegative and symmetric, and that it satisfies the triangle inequality

$$m_G(x,y) \leq m_G(x,z) + m_G(z,y).$$

Observe also that $m_G(x,y) = m_{h(G)}(h(x),h(y))$ for $h \in \mathcal{GM}(G)$ and $x, y \in G$.
(3) Show that, for $x \in \mathbf{B}^n \setminus \{0\}, e_x = x/|x|$,

$$m_{\mathbf{B}^n}(0,x) = \log|-e_x,0,x,e_x| = \log\left(\frac{1+|x|}{1-|x|}\right).$$

Conclude that $m_{\mathbf{B}^n}(x,y) = \rho_{\mathbf{B}^n}(x,y)$ for all x, y of points in $\mathbf{B}^n$.
(4) Show that m_G is not a metric for $G = \mathbb{R}^n \setminus \{0\}$.

Solution:

(1)

$$\begin{aligned}
|u,a,c,v||u,c,b,v| &= \frac{|u-c||a-v||u-b||c-v|}{|u-a||c-v||u-c||b-v|} = |u,a,b,v| \\
|u,a,b,v||u,b,a,v| &= \frac{|u-b||a-v||u-a||b-v|}{|u-a||b-v||u-b||a-v|} = 1 \\
|u,a,b,v||v,a,b,u| &= \frac{|u-b||a-v||v-b||a-u|}{|u-a||b-v||v-a||b-u|} = 1.
\end{aligned}$$

(2) Since $|u,x,y,v||v,x,y,u| = 1$, one of the cross ratios is greater than or equal to 1. From this it follows that $m_G(x,y) \geq 0$.
Since $|u,x,y,v||u,y,x,v| = 1$, we have that

$$|u,x,y,v| = \frac{1}{|u,y,x,v|} = |v,y,x,u|,$$

which means that $m_G(x,y) = m_G(y,x)$.
Let $x, y \in G$, then fix $u, v \in \partial G$ such that $m_G(x,y) = \log|u,x,y,v|$. Then for $z \in G$ we have

$$\begin{aligned}
m_G(x,y) &= \log|u,x,y,v| = \log(|u,x,z,v||u,z,y,v|) \\
&\leq m_G(x,z) + m_G(z,y).
\end{aligned}$$

The $GM(G)$-invariance follows directly from the invariance of the cross ratio.

(3) Fix $x \in B^n \setminus \{0\}$ and let $e_x = \frac{x}{|x|}$. If $u \in S^{n-1}$, the triangle inequality yields

$$1 - |x| \leq |u - x| \leq 1 + |x|$$

with equality on the left iff $u = e_x$ and with equality on the right iff $u = -e_x$. Hence

$$\sup\{|u,0,x,v|\,|u,v \in \partial B^n\} = |-e_x, 0, x, e_x| = \frac{1+|x|}{1-|x|}$$

and the formula for $m_{B^n}(0,x)$ follows.
Let $x, y \in B^n \setminus \{0\}$, $x \neq y$, and $h \in GM(B^n)$ such that $h(y) = 0$ and $h(x) \in]0, e_1[$. By the reasoning above, $\sup\{|u,x,y,v|\,|u,v \in \partial B^n\} = |-e_1, h(y), h(x), e_1|$. Hence it follows from the $GM(B^n)$-invariance that $\sup\{|u,x,y,v|\,|u,v \in \partial B^n\} = |x_*, x, y, y_*|$ where $x_* = h^{-1}(e_1)$ and $y_* = h^{-1}(-e_1)$. It follows from (2.9)[CGQM] that $m_{B^n}(x,y) = \rho_{B^n}(x,y)$.

(4) Now $\partial G = \{0, \infty\}$ and

$$m_G(x,y) = \max\{\log|0,x,y,\infty|, \log|\infty,x,y,0|\} = |\log(\frac{|x|}{|y|})|$$

for all $x, y \in G$. Hence $m_G(x,-x) = 0$ for all $x \neq 0$ and m_G cannot be a metric.