Quasiregular Mappings
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Problem Set 3
Winter 2009 / Vuorinen

1. Show that for all $a, x, y \in B^n$

$$\frac{|T_a x - T_a y|^2}{(1-|T_a x|^2)(1-|T_a y|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}\,.$$

Hint. Lectures, Ahlfors bracket.

Solution: By (1.36)[LN]

$$(1-|T_a x|^2)(1-|T_a y|^2) = \frac{(1-|a|^2)(1-|x|^2)(1-|a|^2)(1-|y|^2)}{A[x,a]^2 A[y,a]^2}$$

and by (1.5)[CGQM] and (1.35)[LN]

$$|T_a x - T_a y|^2 = |\sigma_a x - \sigma_a y|^2 = \frac{r^4 |x-y|^2}{|x-a^*|^2 |y-a^*|^2} = \frac{(1-|a|^2)^2 |x-y|^2}{A[x,a]^2 A[y,a]^2}$$

and the assertion follows.

2. Let $r \in (0,1)$. Find a point $a \in (0, re_1)$ such that $T_a(0) = -T_a(re_1)$.

Solution: Denote $a = te_1$, $0 < t < r$. By (1.34)[LN] and (1.35)[LN]

$$\begin{aligned}|T_a(0)| &= \frac{|0-a|}{\sqrt{1-0+0}} = t\\ |T_a(re_1)| &= \frac{|a-re_1|}{\sqrt{1-2tr-t^2r^2}} = \frac{r-t}{1-rt}.\end{aligned}$$

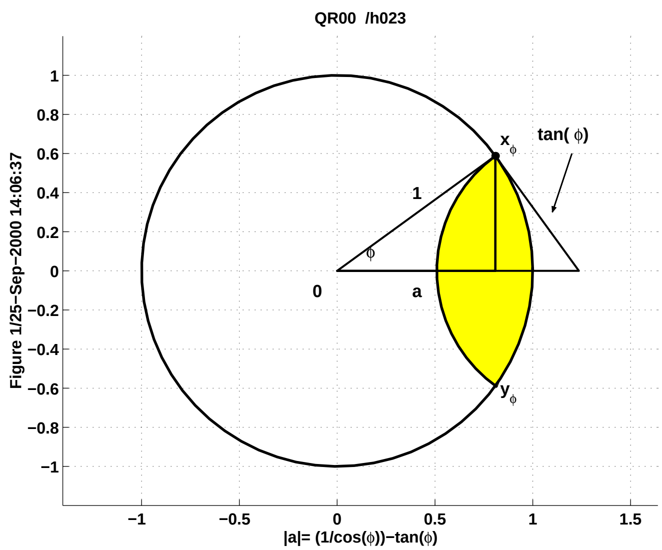
We find t such that

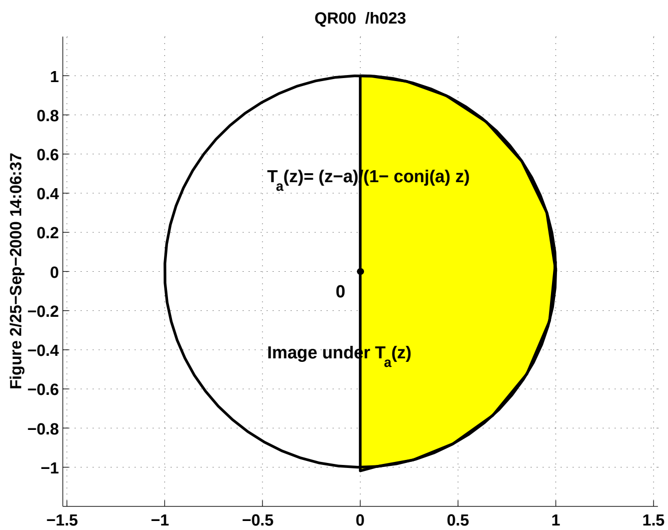
$$\begin{aligned}|T_a(0)| = |T_a(re_1)| \quad &\Leftrightarrow \quad t = \frac{r-t}{1-rt}\\ &\Leftrightarrow \quad t - rt^2 = r - t \Leftrightarrow rt^2 - 2t + r = 0\\ &\Leftrightarrow \quad t = \frac{2 \pm \sqrt{4-4r^2}}{2r} = \frac{1 \pm \sqrt{1-r^2}}{r}\\ &\Leftrightarrow \quad \begin{cases} t = \frac{1+\sqrt{1-r^2}}{r} = \frac{1-(1-r^2)}{r(1-\sqrt{1-r^2})} = \frac{r}{1-\sqrt{1-r^2}} > r, & \text{not possible}\\ \text{OR} & \\ t = \frac{1-\sqrt{1-r^2}}{r} = \frac{1-(1-r^2)}{r(1+\sqrt{1-r^2})} = \frac{r}{1+\sqrt{1-r^2}}, & \text{OK.}\end{cases}\end{aligned}$$

$\therefore a = \frac{re_1}{1+\sqrt{1-r^2}}$.

3. For $\varphi \in (0, \frac{1}{2}\pi)$, let $x_\varphi = (\cos\varphi, \sin\varphi)$ and $y_\varphi = (\cos\varphi, -\sin\varphi)$. Then there exists a Möbius transformation $T_a\colon \mathbf{B}^2 \to \mathbf{B}^2$ with $T_a e_1 = e_1$, $T_a(-e_1) = -e_1$, $T_a(x_\varphi) = e_2 = -T_a(y_\varphi)$. Find $|a|$.

Solution:





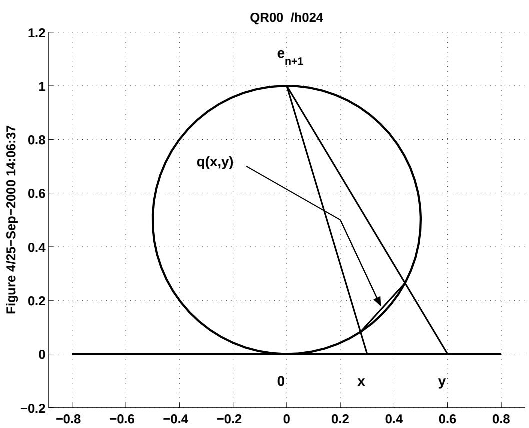
By geometry

$$|a| = \frac{1}{\cos\phi} - \tan\phi = \frac{1-\sin\phi}{\cos\phi}.$$

4. Let $x, y \in \mathbb{R}^n$ and let t_x be a spherical isometry with $t_x(x) = 0$. Show

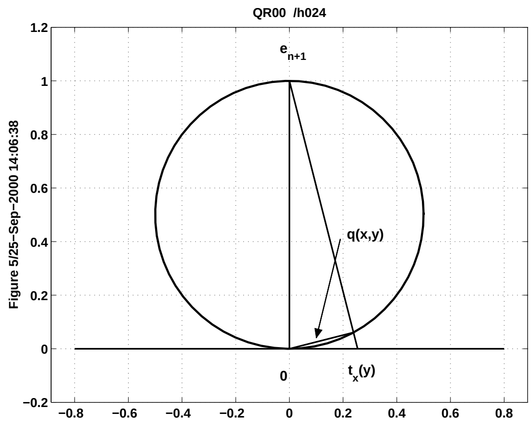
that
$$|t_x y| = \frac{|x-y|}{\sqrt{(1+|x|^2)(1+|y|^2) - |x-y|^2}} \ .$$
Let $\alpha \in [0, \frac{1}{2}\pi]$ be such that $\sin\alpha = q(x,y)$. Then α is the angle between the segments $[e_{n+1},\, t_x x] = [e_{n+1},\, 0]$ and $[e_{n+1},\, t_x y]$ at e_{n+1} . Show that the above formula can be rewritten as $|t_x y| = \tan\alpha$.

Solution: By (1.47)[CGQM] and (1.15)[CGQM] in the lecture notes we have that
$$\begin{aligned} |t_x y|^2 &= \frac{q(x,y)^2}{1-q(x,y)^2} = \frac{\frac{|x-y|^2}{(1+|x|^2)(1+|y|^2)}}{1-\frac{|x-y|^2}{(1+|x|^2)(1+|y|^2)}} \\ &= \frac{|x-y|^2}{(1+|x|^2)(1+|y|^2)-|x-y|^2}. \end{aligned}$$
We take square roots to get the first equality.



If we assume that $\sin\alpha = q(x,y)$ and $\alpha \in [0, \frac{\pi}{2}]$, we have
$$\begin{aligned} |t_x y|^2 &= \frac{q(x,y)^2}{1-q(x,y)^2} = \frac{\sin^2\alpha}{1-\sin^2\alpha} \\ &= \frac{\sin^2\alpha}{\cos^2\alpha} = \tan^2\alpha. \end{aligned}$$
Now $\tan\alpha \in [0,\infty]$ and thus $|t_x y| = \tan\alpha$.

5. For $x, y \in \mathbf{B}^n$ and $T_x \in \mathcal{M}(\mathbf{B}^n)$ show that
$$|T_x y| = \frac{|x-y|}{\sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)}} = \frac{s}{\sqrt{1+s^2}} \ ,$$
where $s^2 = |x-y|^2/((1-|x|^2)(1-|y|^2))$.



Solution: By (2.25)[CGQM], (2.18)[CGQM] and (2.12)[CGQM] we have

$$\begin{aligned}
|T_x y| &= \operatorname{th}\frac{\rho(x,y)}{2} = \frac{\operatorname{sh}\frac{\rho(x,y)}{2}}{\operatorname{ch}\frac{\rho(x,y)}{2}} \\
&= \frac{\operatorname{sh}\frac{\rho(x,y)}{2}}{\sqrt{1+\operatorname{sh}^2\frac{\rho(x,y)}{2}}} = \frac{\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}}{\sqrt{1+|x-y|^2/((1-|x|^2)(1-|y|^2))}} \\
&= \frac{s}{\sqrt{1+s^2}} = \frac{|x-y|}{\sqrt{|x-y|^2+(1-|x|^2)(1-|y|^2)}},
\end{aligned}$$

with $s^2 = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$.

6. Let $h:[0,\infty)\to[0,\infty)$ be strictly increasing with $h(0)=0$ such that $h(t)/t$ is decreasing. Show that $h(x+y)\le h(x)+h(y)$ for all $x>0$. With $h(t)=t^\alpha$, $\alpha\in(0,1)$, apply this result to show that if $d(x,y)$ is a metric then $d^\alpha(x,y)=d(x,y)^\alpha$ also is a metric.

Solution: Let $x,y>0$. Then

$$\left\{\begin{array}{l} \frac{h(x+y)}{x+y} \le \frac{h(x)}{x} \\ \frac{h(x+y)}{x+y} \le \frac{h(y)}{y} \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{l} xh(x+y) \le (x+y)h(x) \\ yh(x+y) \le (x+y)h(y) \end{array}\right.$$

After addition we have

$$(x+y)h(x+y) \le (x+y)(h(x)+h(y)).$$

From this follows the claim $h(x+y)\le h(x)+h(y)$. Let $h(t)=t^\alpha$, $\alpha\in]0,1[$. Then h is strictly increasing, $h(0)=0$ and $\frac{h(t)}{t}=t^{\alpha-1}$ is decreasing, since

$\alpha - 1 \in]-1, 0[$. Let d be a metric. For the application it suffices to prove the triangle inequality:

$$\begin{aligned} d^\alpha(x,y) &= h(d(x,y)) \leq h(d(x,z) + d(z,y)) \\ &\leq h(d(x,z)) + h(d(z,y)) = d^\alpha(x,z) + d^\alpha(z,y) \end{aligned}$$

for all x,y and z in the metric space. In the calculation the first inequality follows from the $\triangle$-inequality for the metric d and from the fact that h is increasing. The second inequality follows from the first part of the exercise.