Quasiregular Mappings
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Problem Set 2
Winter 2009 / Vuorinen

1. Let f be an inversion in $S^{n-1}(a,r)$ as defined in 1.2(2). Show that $f^{-1}=f$ and that $|x-a||f(x)-a|=r^2$ for all $x\in\mathbb{R}^n\setminus\{a\}$. By considering similar triangles show that the following identity holds for $x,y\in\mathbb{R}^n\setminus\{a\}$:

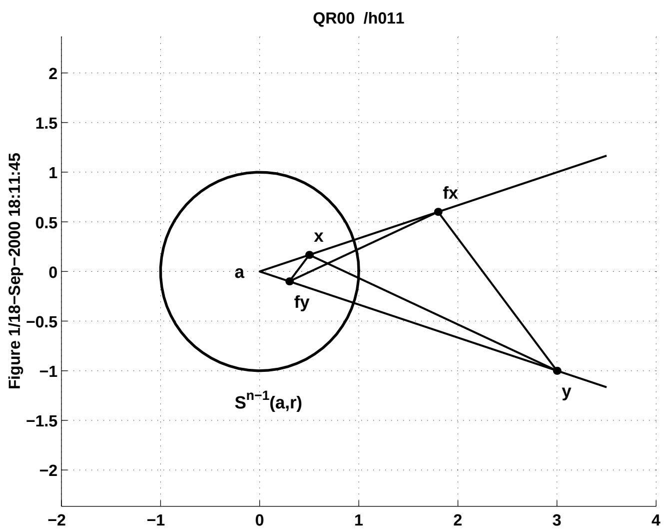
$$|f(x)-f(y)|=\frac{r^2|x-y|}{|x-a||y-a|}\ .$$

Solution: Let $f:\overline{\mathbb{R}}^n\to\overline{\mathbb{R}}^n$ be the map $f(x)=a+\frac{r^2(x-a)}{|x-a|^2}$, $f(a)=\infty$, $f(\infty)=a$. Now

$$f(f(x))=a+\frac{r^2\left(a+\frac{r^2(x-a)}{|x-a|^2}-a\right)}{\left|a+\frac{r^2(x-a)}{|x-a|^2}-a\right|^2}=a+\frac{\frac{r^4(x-a)}{|x-a|^2}}{\frac{r^4}{|x-a|^2}}=x$$

$\therefore\ f^{-1}=f.$

$$|x-a||f(x)-a|=|x-a|\frac{r^2(x-a)}{|x-a|^2}=r^2,\quad \forall x\in\mathbb{R}^n\backslash\{a\}.$$



Furthermore, $\forall x,y\in\mathbb{R}^n\backslash\{a\}$:

$$\frac{|f(x)-a|}{|f(y)-a|}=\frac{\frac{r^2(x-a)}{|x-a|^2}}{\frac{r^2(y-a)}{|y-a|^2}}=\frac{|y-a|}{|x-a|}.$$

It follows that the triangles $f(y), a, f(x)$ and x, a, y are similar. Hence

$$\frac{|f(x)-f(y)|}{|x-y|} = \frac{|f(y)-a|}{|x-a|} = \frac{r^2}{|x-a||y-a|}.$$

2. (a) For $0 < t < 1$ let $w(t) = t/\sqrt{1-t^2}$. Show that $q(0, w(t)e_1) = t$ and that

$$\frac{t}{s} < \frac{w(t)}{w(s)} < \frac{2t}{s}$$

for $0 < s < t < \frac{1}{2}\sqrt{3}$.
(b) Show that

$$q(Q(z,r)) = q(\partial Q(z,r)) = 2r\sqrt{1-r^2}$$

for $0 < r \le 1/\sqrt{2}$.
Solution: a) $q(0, w(t)e_1) = \frac{w(t)}{\sqrt{1+w(t)^2}} = \frac{t}{\sqrt{1-t^2}} \frac{1}{\sqrt{1+\frac{t^2}{1-t^2}}} = t.$
Since $0 < s < t < 1$, we have $\sqrt{1-s^2} > \sqrt{1-t^2}$ and

$$\frac{w(t)}{w(s)} = \frac{t\sqrt{1-s^2}}{s\sqrt{1-t^2}} > \frac{t}{s}.$$

Since $t < \frac{\sqrt{3}}{2}$ we have $1 - t^2 \ge \frac{1}{4}$ and

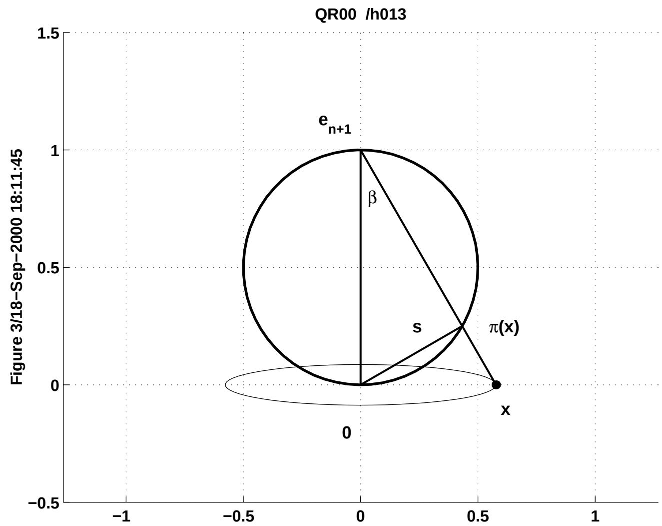
$$\frac{w(t)}{w(s)} = \frac{t\sqrt{1-s^2}}{s\sqrt{1-t^2}} \le \frac{2t\sqrt{1-s^2}}{s} < \frac{2t}{s}.$$

b) Using a q-isometry t_z we see that

$$\begin{aligned} q(Q(z,r)) &= q(Q(0,r)) \\ &= q(-w(r)e_1, w(r)e_1) \\ &= \frac{2|w(r)|}{1+w(r)^2} \\ &= \frac{2\frac{r}{\sqrt{1-r^2}}}{\frac{1-r^2+r^2}{1-r^2}} \\ &= 2r\sqrt{1-r^2}. \end{aligned}$$

3.(a) Let $x, y \in \mathbf{B}^n$ with $s = q(0,x)$, $t = q(0,y)$. Show that

$$q(x,y) \le s\sqrt{1-t^2} + t\sqrt{1-s^2} \le t + s\ .$$



(b) Let $x, y \in \mathbb{R}^n \backslash \{0\}$ with $q(0,x) > q(0,y)$. Show that the strict inequality $q(x,y) > q(0,x) - q(0,y)$ holds.

Solution: a) Let $x, y \neq 0$.

$$
\begin{aligned}
s\sqrt{1-t^2} \;&+\; t\sqrt{1-s^2} \\
&= \frac{|x|}{\sqrt{1+|x|^2}}\sqrt{1-\frac{|y|^2}{1+|y|^2}} + \frac{|y|}{\sqrt{1-|y|^2}}\sqrt{1-\frac{|x|^2}{1+|x|^2}} \\
&= \frac{|x|+|y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \geq \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \\
&= q(x,y).
\end{aligned}
$$

$$
\begin{aligned}
s\sqrt{1-t^2} + t\sqrt{1-s^2} &= \frac{|x|+|y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \leq \frac{|x|}{\sqrt{1+|x|^2}} + \frac{|y|}{\sqrt{1+|y|^2}} \\
&= s+t.
\end{aligned}
$$

b)

$$
q(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \geq \frac{|x|-|y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}
$$

and

$$\begin{aligned} q(0,x)-q(0,y) &= \frac{|x|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \le \frac{|x|}{\sqrt{1+|x|^2}} - \frac{|y|}{\sqrt{1+|y|^2}} \\ &= \frac{|x|\sqrt{1+|y|^2}-|y|\sqrt{1+|x|^2}}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \end{aligned}$$

so it suffices to show that

$$\begin{aligned} & |x|-|y| > |x|\sqrt{1+|y|^2}-|y|\sqrt{1+|x|^2} \\ \Leftrightarrow\quad & |x|(\sqrt{1+|y|^2}-1) < |y|(\sqrt{1+|x|^2}-1) \\ \Leftrightarrow\quad & \frac{|y|}{\sqrt{1+|y|^2}+1} < \frac{|x|}{\sqrt{1+|x|^2}+1} \end{aligned}$$

Now $q(0,x) > q(0,y)$, hence we have $|x| > |y|$ and the mapping $r \mapsto \frac{r}{\sqrt{1+r^2}+1}$ is strictly increasing. Hence the inequality above holds.

4. For $x, y \in \mathbb{R}^n$ prove the following:

$$q(x,y) = \frac{|x-y|}{\sqrt{(1+|x||y|)^2+(|x|-|y|)^2}}.$$

$$\frac{|x-y|}{\sqrt{|x-y|^2+(1+|x||y|)^2}} \le q(x,y) \le \frac{|x-y|}{\sqrt{|x-y|^2+(1-|x||y|)^2}}.$$

$$q(x,y) \le \frac{|x-y|}{2}$$

for $|x||y| \ge 1$, with equality for $x, y \in S^{n-1}$ or $x = y$.

$$q(x,y) \le |x-y|/(|x|+|y|),$$

with equality iff $|x||y| = 1$ or $x = y$.

Solution: The first equality follows from the definition of q and the identity $(1+|x|^2)(1+|y|^2) = (|x|-|y|)^2 + (1+|x||y|)^2$. The *Cauchy-Schwartz* inequality implies $-|x||y| \le x \cdot y \le |x||y|$ with equality iff $y = -tx$ or $x = -ty$ on the left and $y = tx$ or $x = ty$ on the right, where $t \ge 0$. We have

$$|x-y|^2 + (1+|x||y|)^2 \ge (|x|-|y|)^2 + (1+|x||y|)^2$$

(use $\triangle$-inequality) and

$$|x-y|^2+(1-|x||y|)^2 \le (|x|-|y|)^2+(1+|x||y|)^2.$$
$$(\Leftrightarrow |x-y|^2 \le |x|^2+|y|^2+2|x||y| \quad \text{true by the } \triangle\text{−ineq.})$$

Hence the second inequality follows from the first equality.
Third inequality: If $x=y$, equality is clear. Assume $x \neq y$, $|x||y| \ge 1$. Then

$$q(x,y)=\frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \le \frac{|x-y|}{1+|x||y|} \le \frac{|x-y|}{2},$$

(where in the first inequality the fact: $(1+|x|^2)(1+|y|^2) \ge (1+|x||y|)^2$ was used) with equality iff $|x|=|y|$ (the first approx.) and $|x||y|=1$ (the second approx.) i.e. iff |x|=|y|=1.
Fourth inequality:

$$\begin{aligned} q(x,y) &= \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} = \frac{|x-y|}{\sqrt{1+|x|^2+|y|^2+|x|^2|y|^2}} \\ &\le \frac{|x-y|}{\sqrt{|x|^2+|y|^2+2|x||y|}} = \frac{|x-y|}{|x|+|y|}, \end{aligned}$$

with equality iff $x=y$ or $|x||y|=1$.
5. Show that $B^n(a,r)$ and $B^n(v)$, where $r^2<1+|a|^2$,

$$v=\frac{2r}{\sqrt{(1+(|a|+r)^2)(1+(|a|-r)^2)}+1+|a|^2-r^2}$$

have equal spherical diameters. Note that $v<1$. Conclusion: The inversion f_1 in 1.52 is in fact the inversion in a euclidean sphere with radius v and center 0.
Solution: First we compute $q(B^n(a,r))$, $a \in \mathbb{R}^n$, $r>0$:

$$\begin{aligned} q(B^n(a,r)) &= \frac{|a+r\frac{a}{|a|}-(a-r\frac{a}{|a|})|}{\sqrt{1+|a-r\frac{a}{|a|}|^2}\sqrt{1+|a+r\frac{a}{|a|}|^2}} \\ &= \frac{2r}{\sqrt{(1+(|a|-r)^2)(1+(|a|+r)^2)}} \end{aligned}$$

In the following we denote $q(B^n(a,r))$ by u. Hence $q(B^n(v))=\frac{2v}{1+v^2}$ for $v>0$. Then

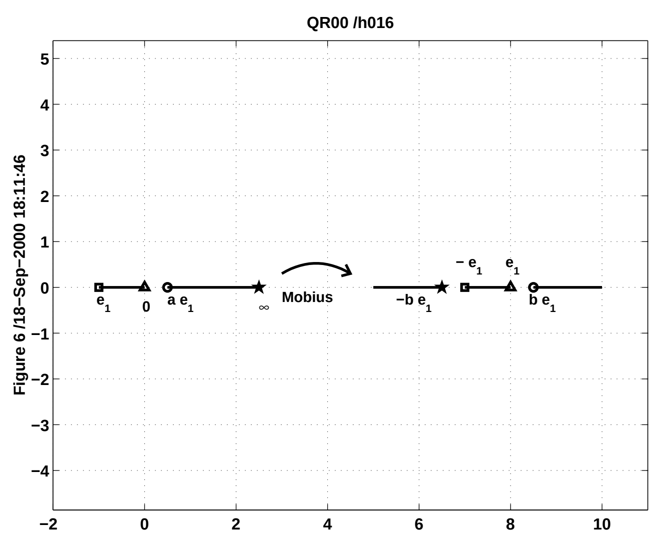
$$\begin{aligned} q(B^n(v))=q(B^n(a,r)) &\Leftrightarrow \frac{2v}{1+v^2}=u \\ &\Leftrightarrow uv^2-2v+u=0 \\ &\Leftrightarrow v=\frac{2\pm\sqrt{4-4u^2}}{2u}=\frac{1\pm\sqrt{1-u^2}}{u} \end{aligned}$$

Because we are interested in the case $v < 1$ it follows that

$$
\begin{aligned}
v &= \frac{1-\sqrt{1-u^2}}{u} = \frac{u}{1+\sqrt{1-u^2}} \\
&= \frac{\frac{2r}{\sqrt{(1+(|a|-r)^2)(1+(|a|+r)^2)}}}{1+\sqrt{\frac{(1+(|a|+r)^2)(1+(|a|-r)^2)-4r^2}{(1+(|a|+r)^2)(1+(|a|-r)^2)}}} \\
&= \frac{2r}{\sqrt{(1+(|a|-r)^2)(1+(|a|+r)^2)}+\sqrt{(1+|a|^2-r^2)^2}} \\
&= \frac{2r}{\sqrt{(1+(|a|-r)^2)(1+(|a|+r)^2)}+1+|a|^2-r^2}.
\end{aligned}
$$

6. The lines $[-e_1, 0]$ and $[ae_1, \infty]$, $a > 0$, can be mapped onto $[-e_1, e_1]$ and $[be_1, \infty] \cup [-be_1, \infty]$ by a Möbius transformation. Give a definition for b in terms of a. Notice that $[x, \infty] = \{xt : t \geq 1\}$, if $x \in \mathbb{R}^n \setminus \{0\}$.

Solution:



Now because of the Möbius invariance it follows that

$$\begin{aligned}
& |-e_1, 0, ae_1, \infty| = |-e_1, e_1, be_1, -be_1| \\
\Leftrightarrow \quad & \frac{|-e_1 - ae_1|}{|-e_1|} = \frac{|-e_1 - be_1||e_1 + be_1|}{|-e_1 - e_1||be_1 + be_1|} \\
\Leftrightarrow \quad & a + 1 = \frac{(b+1)^2}{4b} \\
\Leftrightarrow \quad & 4ab + 4b = b^2 + 2b + 1 \\
\Leftrightarrow \quad & b^2 + (2 - 4(a+1))b + 1 = 0 \\
\Leftrightarrow \quad & b = \frac{4(a+1) - 2 \pm \sqrt{(2 - 4(a+1))^2 - 4}}{2} = 2a + 1 \pm 2\sqrt{a^2 + a}
\end{aligned}$$

By Topology we have that $b > 1$ and thus it follows that $b = 2a + 1 + 2\sqrt{a^2 + a}$.
From the second possible Möbius map we get the absolute ratio equation

$$|-e_1, 0, ae_1, \infty| = |e_1, -e_1, -be_1, be_1|$$

where

$$|e_1, -e_1, -be_1, be_1| = \frac{q(e_1, -be_1)q(-e_1, be_1)}{q(e_1, -e_1)q(-be_1, be_1)} = |-e_1, e_1, be_1, -be_1|$$

and this case reduces to the one above.

7. Are the following mappings Hölder or Lipschitz?
(a) $f\colon B^2 \to \mathbb{R}^2$, $f(x,y) = \begin{cases} (x, y^2), & y > 0, \\ (x, -y^2), & y \le 0, \end{cases}$
(b) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = \begin{cases} (x, 2y), & y > 0, \\ (x, y), & y \le 0, \end{cases}$
Solution: (a) Denote $z_i = (x_i, y_i)$, $i = 1, 2$ and $z_3 = (x_2, y_1)$. If $y_1 y_2 > 0$, then

$$\begin{aligned}
|f(z_1) - f(z_2)| &\le |f(z_1) - f(z_3)| + |f(z_3) - f(z_2)| \\
&= |x_1 - x_2| + |y_1^2 - y_2^2| \\
&= |x_1 - x_2| + |(y_1 - y_2)(y_1 + y_2)| \\
&\le 3|z_1 - z_2|.
\end{aligned}$$

If $y_1 y_2 \le 0$, then

$$\begin{aligned}
|f(z_1) - f(z_2)| &\le |f(z_1) - f(z_3)| + |f(z_3) - f(z_2)| \\
&= |x_1 - x_2| + |y_1^2 + y_2^2| \\
&= |x_1 - x_2| + y_1^2 + y_2^2 \\
&\le |x_1 - x_2| + |y_1| + |y_2| \\
&\le 2|z_1 - z_2|
\end{aligned}$$

and f is Lipschitz.
(b) Let $g(t) = \begin{cases} 2t, & t > 0, \\ t, & t \leq 0, \end{cases}$. Now $|g'(t)| = \begin{cases} 2, & t > 0, \\ 1, & t \leq 0, \end{cases} \leq 2$ and therefore g and also f are Lipschitz (and also Hölder).