Quasiregular Mappings
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Problem Set 1
Winter 2009 / Vuorinen

1. Perhaps the simplest non-injective map is $f\colon z \mapsto z^2$. Find a domain $D \subset \mathbb{C}$ such that $f|_D\colon D \to fD$ is not closed.

Solution: Let $D = \{z \in \mathbb{C}\colon 1/2 < |z| < 3/2, 0 < \arg z < 3\pi/2\}$ and $G = \{e^{i\pi/(2n)}\}_{n=1}^{\infty}$. Now G is closed in D because $D \setminus G$ is open in D. We have $fD = \{z \in \mathbb{C}\colon 1/4 < |z| < 9/4\}$ and $fG = \{e^{i\pi/n}\}_{n=1}^{\infty}$, which is not closed in fD since $1 \in fD \setminus fG$ and $fD \setminus fG$ is not open at 1.

2. Find a Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}, \quad ad-bc \neq 0,$$

which maps $H^2 = \{(x,y) \in \mathbb{R}^2\colon y > 0\}$ onto $B^2 = \{(x,y) \in \mathbb{R}^2\colon x^2+y^2 < 1\}$ such that $(-1,0,1) \mapsto (1,i,-1)$ $[i=(0,1)]$.

Solution: We will find a Möbius transformation f such that $f(-1)=1$, $f(0)=i$ and $f(1)=-1$. Since f preserves cross-ratios

$$\begin{aligned}
\frac{-1-1}{-1-0}\frac{0-z}{1-z} &= \frac{1-(-1)}{1-i}\frac{i-f(z)}{-1-f(z)} \\
\iff \frac{2z}{z-1} &= \frac{2}{1-i}\frac{f(z)-1}{1+f(z)} \\
\iff f(z)\left(\frac{2z}{z-1}-\frac{2}{1-i}\right) &= -\frac{2i}{1-i}-\frac{2z}{z-1} \\
\iff f(z)\frac{2(1-zi)}{(z-1)(1-i)} &= \frac{2(i-z)}{(z-1)(1-i)} \\
\iff f(z) &= \frac{i-z}{1-zi} \\
\iff f(z) &= -i\frac{z-i}{z+i}.
\end{aligned}$$

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Hölder continuous with exponent $\beta > 1$. Show that f is a constant, equal to $f(0)$.

Solution 1: Let us assume, on the contrary, that $\exists x_0$ such that $f(x_0) \neq f(0)$. We may assume $x_0 > 0$. Fix $n \geq 1$, $n \in \mathbb{N}$ and denote

$$a_k = \frac{x_0}{n}k, \quad k=0,\ldots,n; \quad a_n = x_0.$$

Now

$$\begin{aligned} 0 < |f(x_0) - f(0)| &\le \left|\sum_{k=0}^{n-1}(f(a_{k+1}) - f(a_k))\right| \\ &\le \sum_{k=0}^{n-1} |f(a_{k+1}) - f(a_k)| \\ &\le nc|a_{k+1} - a_k|^\beta \\ &= nc\left(\frac{x_0}{n}\right)^\beta \\ &= n^{1-\beta}cx_0^\beta \end{aligned}$$

and since $1-\beta < 0$ we have $n^{1-\beta}cx_0^\beta \to 0$ as $n \to \infty$.
Solution 2: Let us fix $x \in \mathbb{R}$. By the definition

$$\frac{|f(x) - f(y)|}{|x-y|} \le c|x-y|^{\beta-1},$$

where $\beta - 1 > 0$, and therefore $|f(x) - f(y)|/|x-y| \to 0$ as $y \to x$ implying $f'(x) = 0$. Since this is true for all $x \in \mathbb{R}$ the function f is a constant.

4. Let $\sigma\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $\mathbb{R}_+ = (0,\infty)$, be defined by $\rho(x,y) = |\log(x/y)|$. Show that ρ is a metric.
(a) Suppose that $u\colon (\mathbb{R}_+,\rho) \to (\mathbb{R}_+,\rho)$ is uniformly continuous. Show that $u(x) \le Ax^B$ for some constants A and B for all $x \ge 1$.
(b) Suppose that $u\colon (\mathbb{R}_+,d) \to (\mathbb{R}_+,\rho)$ is uniformly continuous, where d is the euclidean metric. Find an inequality of the same type as in (a) [but some other function in place of Ax^B].
Solution: We will show that ρ is a metric.
a) Clearly $\rho(x,y) = \rho(y,x) \ge 0$.
b) Triangle inequality:

$$\rho(x,y) = \left|\log\frac{x}{y}\right| = \left|\log\frac{x}{z} + \log\frac{z}{y}\right| \le \left|\log\frac{x}{z}\right| + \left|\log\frac{z}{y}\right| = \rho(x,z) + \rho(z,y).$$

c) $\rho(x,y) = 0$ iff $x/y = 1$ iff $x = y$.
(a) Let us choose $p \in \mathbb{N}$ such that $2^p \le x < 2^{p+1}$ implying $p \le \log x/\log 2$. Let us denote $a_k = 2^k$. Now for all $x \in [a_k, a_{k+1}]$

$$\rho(a_k, x) \le \log 2$$

and by the uniform continuity

$$\rho(u(a_k), u(x)) \le \omega(\log 2).$$

Therefore

$$\begin{aligned}\rho(u(x),u(1)) &= |\log(u(x)/u(1))| \\ &= \left|\log\left(\frac{u(2)}{u(1)}\frac{u(4)}{u(2)}\cdots\frac{u(x)}{u(2^p)}\right)\right| \\ &\le p\omega(\log 2) \\ &\le \frac{\log x}{\log 2}\omega(\log 2).\end{aligned}$$

If $|\log(u(x)/u(1))| = \log(u(x)/u(1))$ then $u(x)/u(1) \le x^{\omega(\log 2)/\log 2}$ and the assertion follows. If $|\log(u(x)/u(1))| = \log(u(1)/u(x))$ then we can make similar estimations by using the fact that $p+1 > \log x/\log 2$.
(b) Let us choose $n \in \mathbb{N}$ be such that $n \le x < n+1$. Similarly as in (a) we obtain that $u(x) \le Ae^{xB}$.

5. Let $D = H^2 = \{(x,y) \in \mathbb{R}^2 \colon y > 0\}$. The modulus $M(D;-1,0,s,\infty)$ of a quadrilateral $(D;-1,0,s,\infty)$, $s > 0$, is usually denoted $\tau(s)/2$ [we will define the function $\tau(s)$ later]. Fix $\alpha \in (0,\pi)$ and denote $D_\alpha = \{z \in \mathbb{C} \colon 0 < \arg z < \alpha\}$, $r > s > 0$, $t > u > 0$. Using the above notation, find $M(D_\alpha; re^{i\alpha}, se^{i\alpha}, u, t)$.
Solution: Let us first map D_α on to half-plane D by $z \mapsto z^{\pi/\alpha}$. Now

$$\begin{aligned} r &\mapsto r^{\pi/\alpha} = r', \\ s &\mapsto s^{\pi/\alpha} = s', \\ t &\mapsto t^{\pi/\alpha} = t', \\ u &\mapsto u^{\pi/\alpha} = u', \end{aligned}$$

$r', s', t', u' \in \mathbb{R}$ and $r' < s' < t' < u'$. Let us then find a mapping such that $r' \mapsto -1$, $s' \mapsto 0$, $t' \mapsto v$ and $u' \mapsto \infty$. This can be done as in exercise 2 or obtained from the lecture notes

$$v = \frac{(u'-r')(t'-s')}{(u'-t')(r'-s')} - 1 = \frac{(u'-s')(t'-r')}{(u'-t')(r'-s')}.$$

Finally

$$M(D_\alpha; re^{i\alpha}, se^{i\alpha}, u, t) = \tau(v)/2 = \frac{1}{2}\tau\left(\frac{(u^{\pi/\alpha}-s^{\pi/\alpha})(t^{\pi/\alpha}-r^{\pi/\alpha})}{(u^{\pi/\alpha}-t^{\pi/\alpha})(r^{\pi/\alpha}-s^{\pi/\alpha})}\right).$$

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