Canonical forms of 2×2 matrices

Given a matrix $\underset{2\times 2}{A}$, determine its eigenvalues (there are two of them). There are three possibilities:

1. Distinct real $\lambda_1 \neq \lambda_2$
2. Complex $\mu = a+ib$, $(b\neq 0)$
   $\bar{\mu} = a-ib$
3. Double real root $\lambda \in \mathbb{R}$

Case 1: determine eigenvectors $\vec{v}_1 \neq 0$ and $\vec{v}_2 \neq 0$ satisfying

$$A\vec{v}_1 = \lambda_1 \vec{v}_1,$$
$$A\vec{v}_2 = \lambda_2 \vec{v}_2.$$

Construct $P = [\vec{v}_1 \ \vec{v}_2]$, where vertical vectors $\vec{v}_1$ and $\vec{v}_2$ appear as columns.

Then $P^{-1}AP = D$ with

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Case 2: determine complex eigenvector $\vec{w} \in \mathbb{C}^2$ with $\vec{w} \neq 0$. Here $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ with $w_1 \in \mathbb{C}$ and $w_2 \in \mathbb{C}$. That is, solve the system

$$(A - \mu I)\vec{w} = 0.$$

Denote $\vec{w} = \vec{u} + i\vec{v}$; then $\vec{u} \in \mathbb{R}^2$ and $\vec{v} \in \mathbb{R}^2$ are linearly independent.

Set $P = [\vec{u} \quad \vec{v}]$ and $D = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Then $P^{-1}AP = D$.

Case 3: if the geometric multiplicity of λ is 2, then proceed as in 1.

However, if the geometric multiplicity is 1, then we can find only one linearly independent eigenvector $\vec{v} \neq 0$ satisfying

$$A\vec{v} = \lambda \vec{v}.$$

Solve equation

$$(A - \lambda I)\vec{w} = \vec{v}$$

for a vector $\vec{w} \in \mathbb{R}^2$ that is nonzero and

linearly independent of $\vec{v}$.
Then set $P = [\vec{v} \;\; \vec{w}]$
and $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Now

$$P^{-1}AP = J,$$

and the matrix J is called the Jordan canonical form of A.