Algebra I, harj. 2, k2009

1. Olk. $f: A \to B$, $A_1, A_2 \subset A$ & $B_1, B_2 \subset B$.
Tällöin

i) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$ &

ii) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

Tod. i) $b \in f(A_1 \cup A_2)$

$\iff \exists a \in A_1 \cup A_2$ s.e. $f(a) = b$

$\iff \exists a \in A_1$ s.e. $f(a) = b$ t. $\exists a \in A_2$ s.e. $f(a) = b$

$\iff b \in f(A_1)$ t. $b \in f(A_2)$

$\iff b \in f(A_1) \cup f(A_2)$

ii) $a \in f^{-1}(B_1 \cup B_2)$

$\iff \exists b \in B_1 \cup B_2$ s.e. $f(a) = b$

$\iff \exists b \in B_1$ s.e. $f(a) = b$ t.
$\exists b \in B_2$ s.e. $f(a) = b$

$\iff a \in f^{-1}(B_1)$ t. $a \in f^{-1}(B_2)$

$\iff a \in f^{-1}(B_1) \cup f^{-1}(B_2)$ $\square$

Välttämättä ei päde $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$
(Huom. Aina kuitenkin $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$)

(Vastа)esim. Olk. $A = B = \{1, 2\}$, $f(1) = f(2) = 1$,
$A_1 = \{1\}$, $A_2 = \{2\}$

$\implies f(A_1 \cap A_2) = f(\emptyset) = \emptyset$, mutta
$f(A_1) \cap f(A_2) = \{1\} \cap \{1\} = \{1\} \neq \emptyset$ $\square$

$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$ pätee:

Tod. $x \in f^{-1}(B_1 \cap B_2) \iff f(x) \in B_1 \cap B_2$

$\iff f(x) \in B_1$ & $f(x) \in B_2 \iff x \in f^{-1}(B_1)$ & $x \in f^{-1}(B_2)$

$\iff x \in f^{-1}(B_1) \cap f^{-1}(B_2)$ $\square$

2. Olk. A tason ympyröiden & B tason pisteiden joukko & $f: A \to B$ kuvaus, jolla $f(x)$ on x:n keskipiste.

Merk. kutakin ympyrää $x \in A$
$x = B(a_x, r_x)$, missä $a_x \in \mathbb{R}^2$ on ympyrän keskipiste & $r_x > 0$ on ympyrän säde.

a) f ei ole injektio : Olk.

$x_1 = B(\bar{0}, 1)$ & $x_2 = B(\bar{0}, 2)$ (0-keskiset, 1- & 2-säteiset ympyrät). Tällöin $x_1 \neq x_2$ mutta $f(x_1) = \bar{0} = f(x_2)$. □

b) f on surjektio:

Olk. $y \in B$ $(= \mathbb{R}^2)$. Val. esim.

$x = B(y, 1) \in A$ (y-keskinen yksikköympyrä).

$\Rightarrow f(x) = y$ □

3. Olk. $f, g: \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} 0 & , x \in \mathbb{Q} \\ 1 & , x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}, \quad f(x) = (2 - g(x))x + g(x), x \in \mathbb{R}.$$

V. f on bijektio.

Tod. 1° Ol. $\Rightarrow$

$$f(x) = \begin{cases} (2-0)x + 0 = 2x, & x \in \mathbb{Q} \\ (2-1)x + 1 = x+1, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Selvästi yllä $x \in \mathbb{Q}$ joss $2x \in \mathbb{Q}$ & $x+1 \in \mathbb{R} \setminus \mathbb{Q}$ joss $x \in \mathbb{R} \setminus \mathbb{Q}$.

Siis $f(x) \in \mathbb{Q}$ joss $x \in \mathbb{Q}$

2° f on injektio: $f(x_1) = f(x_2)$

$$\Leftrightarrow \begin{cases} 2x_1 = 2x_2 \ \& \ x_1, x_2 \in \mathbb{Q} \ \text{t.} \\ x_1 + 1 = x_2 + 1 \ \& \ x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = x_2 \ \& \ x_1, x_2 \in \mathbb{Q} \ \text{t.} \\ x_1 = x_2 \ \& \ x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \Leftrightarrow x_1 = x_2.$$ Siis f on injektio.

3° f on surjektio: Olk. $y \in \mathbb{R}$

i) Ol. $y \in \mathbb{Q}$: $y = f(x) \overset{1°}{\Leftrightarrow} y = 2x \Leftrightarrow x = y/2$

Siis $\exists\, x = y/2$ s.e. $y = f(x)$.

ii) Ol. $y \in \mathbb{R} \setminus \mathbb{Q}$. $y = f(x) \overset{1°}{\Leftrightarrow} y = x + 1 \Leftrightarrow x = y - 1$

Siis $\exists\, x = y - 1$ s.e. $y = f(x)$

i) – ii) $\Rightarrow \forall y \in \mathbb{R}\ \exists x \in \mathbb{R}$ s.e. $y = f(x)$

$\Rightarrow f$ on surj.

2° – 3° $\Rightarrow f$ on bij. $\square$

4. V. $f : A \to B$ on surjektio, jos & vain jos $f(f^{-1}(B_0)) = B_0 \quad \forall B_0 \subset B$.

Tod. 1° Ol. $f(f^{-1}(B_0)) = B_0 \ \forall B_0 \subset B$.

Olk. $y \in B$ (mieliv.). Ol. $\Rightarrow \{y\} = f(f^{-1}(\{y\}))$

$\Rightarrow \exists x \in f^{-1}(\{y\}) (\subset A)$ s.e. $f(x) = y \Rightarrow f(A) = B$ l.

f on surjektio.

2° Ol., että f on surj.. Luentojen (s. 9) lemman mukaan $f(f^{-1}(B_0)) \subset B_0 \ \forall B_0 \subset B$, joten riittää tod., että $f(f^{-1}(B_0)) \supset B_0$.

Olk. $y \in B_0$. Koska f on surj., $\exists x \in A$ s.e. $y = f(x) \Rightarrow x \in f^{-1}(B_0) \Rightarrow y = f(x) \in f(f^{-1}(B_0))$

$\Rightarrow B_0 \subset f(f^{-1}(B_0))$ □

5. $2n = \sum_{k=1}^{2n} (-1)^k (2k+1) \quad \forall\, n \in \mathbb{N}_+ \ (= \mathbb{N} \setminus \{0\})$.

Tod. 1° $n=1$: $\sum_{k=1}^{2\cdot 1} (-1)^k(2k+1) = (-1)^1(2\cdot 1+1) + (-1)^2(2\cdot 2+1)$

$= -3 + 5 = 2 = 2\cdot 1$

2° Ol. $n \in \mathbb{N}_+$

Ol. $\sum_{k=1}^{2n} (-1)^k (2k+1) = 2n$

3° $\sum_{k=1}^{2(n+1)} (-1)^k (2k+1)$

$= \sum_{k=1}^{2n} (-1)^k (2k+1) + (-1)^{2n+1}(2(2n+1)+1) + (-1)^{2n+2}(2(2n+2)+1)$

$\overset{2°}{=} 2n + (-1)(4n+3) + (4n+5) = 2n + 2 = 2(n+1)$ □

6. Olk. $f: \mathbb{N}_+ \to \mathbb{N}_+$, $f(1) = 1$, $f(n+1) = 2f(n)$, $n \in \mathbb{N}_+$.

V1. $f(n) = 2^{n-1}$, kun $n \in \mathbb{N}_+$.

Tod. 1° $f(1) = 1 = 2^0 = 2^{1-1}$. 2° Olk. $n \in \mathbb{N}_+$ sellainen, että $f(n) = 2^{n-1}$.

3° $f(n+1) = 2f(n) \overset{2°}{=} 2\cdot 2^{n-1} = 2^{(n+1)-1}$ □

Olk. $S(n) = f(1) + \dots + f(n) \left(\overset{V1}{=} 1 + 2 + 2^2 + \dots + 2^{n-1}\right.$,
geom. sarja$\left.\right)$

V2. $S(n) \left(= \frac{1-2^n}{1-2}\right) = 2^n - 1 \quad \forall\, n \in \mathbb{N}_+$.

Tod. 1° $S(1) = f(1) = 1 = 2^1 - 1$.

2° Ol. $S(n) = 2^n - 1$ jollain TIETYLLÄ $n \in \mathbb{N}_+$.

3° $S(n+1) = \sum_{k=1}^{n+1} f(k) = \sum_{k=1}^{n} f(k) + f(n+1) = S(n) + f(n+1)$

$= 2^n - 1 + 2^n = 2^{n+1} - 1$ □