# TRANSITION OF THE DEGREE SEQUENCE IN THE RANDOM GRAPH MODEL OF COOPER, FRIEZE AND VERA

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Abstract. We study the transition of the expected degree sequence from power law to exponential decay in the random graph process introduced by Cooper, Frieze and Vera. We prove that there is a threshold on the probabilities of introduction of vertices (with probability $\pi_1$), edges ($\pi_2$) and deletion of edges ($\pi_3$) or vertices ($\pi_4$). If $\pi_1 + 2\pi_2 - 2\pi_3 - \pi_4 > 0$, the expected fraction of vertices of degree $k$ follows a power law whereas for $\pi_1 + 2\pi_2 - 2\pi_3 - \pi_4 < 0$, it decays exponentially fast. This work extends the previous results of Wu et al. and Deijfen and Lindholm to the whole model defined by Cooper et al.

## 1. Introduction

The discovery of the scale free property in many real world networks brought attention from various fields of science. It has been shown by empirical observations that real-life networks such as internet [5], protein interaction, food webs [7] among many examples typically exhibit a degree sequence following a power law. That is, the proportion of vertices with degree $k$ decreases as $P(k) \sim Ck^{-\gamma}$ where $C$ and $\gamma$ are constants. In the attempt to develop a model reflecting the same properties as those of real networks, Barabási and Albert introduced the preferential attachment model [1]. In this model, at each time step, a new vertex is introduced together with an edge where the target vertex is chosen with probability proportional to the degree. Their simulations indicated that the proportion of vertices with degree $k$ followed a power law $P(k) \to Ck^{-3}$. A rigorous proof is given in [2].

The preferential attachment model follows a power law with exponent 3 and therefore lacks to explain the scale free networks with other exponents. Moreover, it can seem unnatural that vertices and edges are not subject to any deletion. In the model of Cooper, Frieze and Vera (CFV in short), at each time, it is possible to either, introduce a new vertex with preferential attachment (with a probability $\pi_1$) or a new edge with preferential attachment (with probability $\pi_2$) to exhibit new acquaintances between existing subjects, or it is possible to choose uniformly and delete an existing edge (with a probability $\pi_3$) or vertex (with probability $\pi_4 = 1 - \pi_1 - \pi_2 - \pi_3$). This model brings more flexibility and thus has a wide range of possible types of degree sequence depending on the parameters of introduction and deletion of edges and vertices. Cooper, Frieze and Vera showed that the exponent $\gamma$ could take any value in $(2, +\infty)$.

A simpler submodel with no deletion of vertices has been studied in details by Wu et al. (see [8]). They show a phase transition of the expected degree sequence from power law to exponential decay as the rate of deletion of edges increases. This work determines the critical amount of edge deletion a graph can sustain before losing its scale free property, $\pi_3^{cr} = \frac{1}{3} + \frac{\pi_2}{3}$.

Independently, Deijfen and Lindholm studied in [4] a slightly different model where the added edge (with probability $\pi_2$) links preferentialy for only one end of the edge, the other being chosen uniformly. Deijfen and Lindholm showed a phase transition at $\pi_3^{cr} = \frac{1}{3}$ while in the model of CFV, Wu et al. showed a phase transition at $\pi_3^{cr} = \frac{1}{3} + \frac{\pi_2}{3}$. The difference shows that the preferential attachment of the added edges strengthen the scale free property and the graph can therefore sustain more deletion. It would be interesting for comparison to study for example the critical probability if the extra edges are added with both ends chosen uniformly. The results are given in the discussion thereafter.

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We extend the previous results to all the parameters of the model of CFV by considering $\pi_4$, the probability of deletion of vertices with a threshold for $\pi_1 + 2\pi_2 - 2\pi_3 - \pi_4 = 0$. It is easily seen that when the probability of edge deletion is $\pi_3 = 0$ (no edge removal), then the graph always has a power law degree distribution. However, for any given $\pi_3 > 0$, there exists a critical probability of deletion of vertices $\pi_4^{cr} = \pi_1 + 2\pi_2 - 2\pi_3$ such that if we increase the probability of deletion above $\pi_4^{cr}$ then the degree distribution decreases exponentially fast. In addition, we extend the results to the model of Deijfen and Lindholm (DL) [4] and to a model with extra edges added uniformly.

## 2. Model

We consider a graph process $(G(t))_{t\geq 1}$ consisting of graphs $\big(V(t), E(t)\big)$. Let $v_t = |V(t)|$ and $e_t = |E(t)|$.

To initialize the process, we start with $G(1)$ consisting of an isolated vertex with a loop attached to it. The graph is constructed recursively and at time $t+1$ the possible steps are the following:

- With probability $\pi_1$ a new vertex with an edge attached to it is introduced. The other extremity of this edge is then attached to an existing vertex with a probability proportional to its degree.
$$\mathbb{P}\{w = v_{t+1}\} = \frac{d(w)}{2e_t}.$$
- With probability $\pi_2$ a new edge is added between two existing vertices where both ends are chosen with a probability proportional to their degrees.
- With probability $\pi_3$ an edge chosen uniformly at random is deleted.
- With probability $\pi_4 = 1 - \pi_1 - \pi_2 - \pi_3$, a vertex chosen uniformly at random is deleted.

If $v_t = 0$ or $e_t = 0$, we introduce a new vertex with a loop attached to it. Notice that a vertex of degree 0 can not be attached anymore to the graph by an edge but it remains as long as it has not been deleted. This is the model of CFV but where we allow multiple edges. We introduce/delete a single edge/vertex at a time for clarity. The results hold if we introduce/delete $m$ edges/vertices at a time. In the following, we use the notations of Deijfen and Lindholm for clarity again.

## 3. Preliminary results

Before we proceed to the computation of the expected degree sequence, some concentration results on the number of vertices $v_t$ and edges $e_t$ at time $t$ are needed.

3.1. **Number of vertices.** The number of vertices in the graph can be described as a simple random walk with reflecting barrier at 0. If $X_0(t) \geq 1$ then $X_0(t+1) = X_0(t) + w(t+1) = \sum_{i=1}^{t+1} w(i)$ with

$$w(i) = \begin{cases} \phantom{-}1, & \text{with probability } \pi_1, \\ \phantom{-}0, & \text{with probability } \pi_2 + \pi_3, \\ -1, & \text{with probability } \pi_4 = 1 - (\pi_1 + \pi_2 + \pi_3), \end{cases} \tag{3.1}$$

and if $X_0(t) = 0$ then
$$X_0(t+1) = 1.$$
If $\pi_4 \geq \pi_1$, it holds that
$$\mathbb{P}\{v_t = 0 \ i.o.\} = 1.$$
where $i.o.$ stands for infinitely often.

The case when the rate of deletion is smaller than the rate of introduction of vertices $\pi_4 < \pi_1$ corresponds to a random walk with reflecting barrier where all the states are transient. It can be approximated by a simple random walk $X(t) = \sum_{i=1}^{t} w(i)$ where the random variables $w(i)$ are independent and having the same transition probability as in (3.1) (see [6]). Let
$$\nu = \pi_1 - \pi_4,$$

then by the Hoeffding's inequality

$$\mathbb{P}(|X(t)-\nu t|)=\mathbb{P}(|\sum_{i=1}^{t} w(i)-\nu t|\geq u)\leq 2exp(-\frac{u^2}{2t}). \tag{3.2}$$

We obtain that

$$|v_t-\nu t|\leq ct^{1/2}\log t \quad \mathbf{qs},$$

for any constant $c$ where **qs** stands for $\mathbb{P}\{|v_t-\nu t|\leq ct^{1/2}\log t\ \}\leq t^{-a}$ for any $a>0$.

Let $\omega(t)$ with $\lim_{t\to\infty}\omega(t)=\infty$, then

$$\mathbb{P}\left\{\exists\ s\geq \omega(t)\log t, v_s=0\right\}=o\left(t^{-k}\right) \text{ for any } k.$$

Therefore $v_s>0$ **qs** for $s\geq\omega(t)\log(t)$ and we derive that $|X_0(t)-X(t)|\leq\omega(t)\log(t)$ **qs**. With help of inequality (3.2), we find that

$$|v_t-\nu t|\leq \omega(t)\log t \quad \mathbf{qs}. \tag{3.3}$$

We now turn to the number of edges in the graph at time $t$.

3.2. **Number of edges.** At each time step, we introduce an edge with probability $\pi_1+\pi_2$, delete one with probability $\pi_3$ and delete a vertex with a probability $\pi_4$, therefore the number of edges changes at each step. The difficulty with the number of edges is that the deletion of a vertex implies the deletion of the edges adjacent to that vertex. Therefore the deletion of a vertex with a large degree may have a more dramatic impact on the number of edges than the deletion of a single edge. However, the proportion of vertices of so-called large degree is not so large, the degree sequence being at most polynomial (see [3]). It is shown in [3], p. 473 that

$$\mathbb{P}\left(|e_t-\eta t|\geq t^{1-\frac{\rho}{8}}\right)=O\left(t^{-\frac{\rho}{4}}\left(\log t\right)^{\xi+5}\right) \tag{3.4}$$

with $\eta=\frac{\pi_1+\pi_2-\pi_3}{\pi_1+\pi_4}\nu$ and $\nu=\pi_1-\pi_4$.

It is natural for clarity to express $\eta$ in the form:

$$\eta=\pi_1+\pi_2-\pi_3+\pi_4 f(\pi_1,\pi_2,\pi_3,\pi_4)=\pi_1+\pi_2-\pi_3-2\pi_4\frac{\pi_1+\pi_2-\pi_3}{\pi_1+\pi_4}.$$

## 4. The degree sequence

4.1. **Recursion formula for the expected asymptotic degree distribution.** From [3], we know that for any vertex $d(t)<t^{1-\rho/2}(\log t)^5$ **qs** so we determine the degree sequence for $k<t^{1-\rho/2}(\log t)^5$ and then take the limit as $t\to\infty$. Let $N_k(t)$ denote the number of vertices with degree $k$ in $G(t)$. Given $G(t)$, the expected number of vertices of degree $k$ at next step is given by

$$\begin{aligned}\mathbb{E}\left(N_k(t+1)|G(t)\right)=&N_k(t)+\pi_1\Big((k-1)\frac{N_{k-1}(t)}{2e_t}-k\frac{N_k(t)}{2e_t}|e_t>0\Big)\\&+\pi_2\Big(2(k-1)\frac{N_{k-1}(t)}{2e_t}-2k\frac{N_k(t)}{2e_t}|e_t>0\Big)\\&+\pi_3\Big(2(k+1)\frac{N_{k+1}(t)}{2e_t}-2k\frac{N_k(t)}{2e_t}|e_t>0\Big)\\&+\pi_4\Big((k+1)(\frac{N_{k+1}(t)}{v_t}-\frac{N_k(t)}{v_t})|v_t>0)\Big)\\&+\mathbb{P}(v_t=0)+\mathbb{P}(e_t=0).\end{aligned}$$

Taking the expectation on both sides, one finds

$$
\begin{aligned}
\mathbb{E}\left(N_k(t+1)\right) =& \mathbb{E}\left(N_k(t)\right) + \pi_1 \mathbb{E}\left((k-1)\frac{N_{k-1}(t)}{2e_t} - k\frac{N_k(t)}{2e_t} \big| e_t > 0\right) \\
&+ \pi_2 \mathbb{E}\left(2(k-1)\frac{N_{k-1}(t)}{2e_t} - 2k\frac{N_k(t)}{2e_t} \big| e_t > 0\right) \\
&+ \pi_3 \mathbb{E}\left(2(k+1)\frac{N_{k+1}(t)}{2e_t} - 2k\frac{N_k(t)}{2e_t} \big| e_t > 0\right) \\
&+ \pi_4 \Big((k+1)\big(\frac{N_{k+1}(t)}{v_t} - \frac{N_k(t)}{v_t}\big) \big| v_t > 0)\Big) \\
&+ \mathbb{P}(v_t = 0) + \mathbb{P}(e_t = 0).
\end{aligned} \tag{4.5}
$$

Using the concentration results on $v_t$ (3.3) and $e_t$ (3.4) in the equation (4.5) we get

$$
\begin{aligned}
\mathbb{E}\left(N_k(t+1)\right) - \mathbb{E}\left(N_k(t)\right) =& \frac{\mathbb{E}\, N_{k+1}(t)}{t}(k+1)\Big(\frac{\pi_3}{\eta} + \frac{\pi_4}{\nu}\Big) \\
&- \frac{\mathbb{E}\, N_k}{t} k \Big(\Big(\frac{\pi_1}{2} + \pi_2 + \pi_3\Big)\frac{1}{\eta} + \frac{\pi_4}{\nu}\Big) - \frac{\mathbb{E}\, N_k}{t}\Big(1 + \frac{\pi_4}{\nu}\Big) \\
&+ \frac{\mathbb{E}\, N_{k-1}(t)}{t}(k-1)\Big(\frac{\pi_1}{2\nu}\Big) \\
&+ O\big(t^{-\pi_1/8(\pi_1+\pi_2)}\big).
\end{aligned} \tag{4.6}
$$

Following the notations of Deijfen and Lindholm, we write $\frac{\mathbb{E}\, N_k}{t} = p_k$. Thus for large $t$ we have, $\mathbb{E}\left(N_k(t)\right) \sim p_k t$ (see [3] Lemma 5.1) and derive $\mathbb{E}\left(N_k(t+1)\right) - \mathbb{E}\left(N_k(t)\right) \sim p_k(t+1) - p_k t = p_k$. The equation (4.6) can be approximated by

$$
p_{k+2}(k+2)\alpha_2 + p_{k+1}\big((k+1)\alpha_1 + \beta_1\big) + p_k k \alpha_0 = 0, \tag{4.7}
$$

where

$$
\begin{aligned}
\alpha_2 &= \frac{\pi_3}{\eta} + \frac{\pi_4}{\nu}, \\
\alpha_1 &= -\Big(\frac{\pi_1}{2} + \pi_2 + \pi_3\Big)\frac{1}{\eta} - \frac{\pi_4}{\nu}, \qquad \beta_1 = -\Big(1 + \frac{\pi_4}{\nu}\Big), \\
\alpha_0 &= \Big(\frac{\pi_1}{2} + \pi_2\Big)\frac{1}{\eta}.
\end{aligned}
$$

Notice that $\alpha_1 = -\alpha_0 - \alpha_2$.

4.2. **Resolution of the recursive equation (4.7).** Following the steps of [3], [4] or [8], equation (4.7) can be solved by using the method of Laplace. Set

$$
p_k = \int_a^b t^{k-1} h(t) dt. \tag{4.8}
$$

By integration by parts, we get

$$
k p_k = \Big[t^k h(t)\Big]_a^b - \int_a^b t^k h'(t) dt. \tag{4.9}
$$

Furthermore, we define

$$
\begin{aligned}
\Psi_\alpha(t) &= \alpha_2 t^2 + \alpha_1 t + \alpha_0 = \alpha_2 (1-t)\big(\frac{\alpha_0}{\alpha_2} - t\big), \\
\Psi_\beta(t) &= \beta_1 t.
\end{aligned}
$$

By substituting (4.8) and (4.9) in (4.7), one gets

$$
\Big[t^k \Psi_\alpha(t) h(t)\Big]_a^b + \int_a^b t^k \Big(h(t)\frac{\Psi_\beta(t)}{t} - \Psi_\alpha(t) h'(t)\Big) dt = 0. \tag{4.10}
$$

The equation (4.10) is satisfied if $a$, $b$ and $h(t)$ are chosen such that

$$\Big[t^k\Psi_\alpha(t)h(t)\Big]_a^b = 0 \tag{4.11}$$

and

$$\frac{h'(t)}{h(t)} = \frac{\Psi_\beta(t)}{t\Psi_\alpha(t)}. \tag{4.12}$$

Take $a = 0$ and $b$ as a root of $\Psi_\alpha(t)$ so that condition (4.11) is fulfilled. Notice that $\Psi_\alpha(t)$ has two roots 1 and $\frac{\alpha_0}{\alpha_2}$ so depending if $\frac{\alpha_0}{\alpha_2} > 1$ or $\frac{\alpha_0}{\alpha_2} < 1$, we choose the smallest root for the integration in (4.10). Condition (4.12) gives

$$\frac{h'(t)}{h(t)} = \frac{\beta_1}{\alpha_2}\frac{1}{(1-t)(\frac{\alpha_0}{\alpha_2}-t)}. \tag{4.13}$$

The equation (4.13) can be integrated and is solved by

$$h(t) = (1-t)^{\gamma_1}\Big(\frac{\alpha_0}{\alpha_2}-t\Big)^{\gamma_2}, \tag{4.14}$$

where $\gamma_1 = \frac{\beta_1}{\alpha_2-\alpha_0}$ and $\gamma_2 = -\gamma_1 = \frac{\beta_1}{\alpha_0-\alpha_2}$.

In the particular case $\alpha_0 = \alpha_2$, we have

$$\frac{h'(t)}{h(t)} = \frac{\beta_1}{\alpha_2}\frac{1}{(1-t)^2},$$

which implies

$$h(t) = \exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big). \tag{4.15}$$

with $\beta_1 = -\left(1+\frac{\pi_4}{\nu}\right)$ and $\alpha_2 = \frac{\pi_3}{\eta}+\frac{\pi_4}{\nu}$ thus $\frac{\beta_1}{\alpha_2} < 0$.

- The power law regime: This stage has been studied in [3].

  If $\frac{\alpha_0}{\alpha_2} > 1$, then $\gamma_1 = -\frac{\beta_1}{\alpha_0-\alpha_2} > 0$ and $\gamma_2 < 0$, using (4.14), we have

$$\begin{aligned} p_k &= \int_a^b t^{k-1}h(t)dt = \int_0^1 t^{k-1}h(t)dt \\ &= \int_0^1 t^{k-1}(1-t)^{\gamma_1}\Big(\frac{\alpha_0}{\alpha_2}-t\Big)^{-\gamma_1}dt \\ &= \int_0^1 t^{k-1}\Big(\frac{1-t}{\frac{\alpha_0}{\alpha_2}-t}\Big)^{\gamma_1}dt = \int_0^1 t^{k-1}\frac{\alpha_2}{\alpha_0}\Big(\frac{1-t}{1-\frac{\alpha_2}{\alpha_0}t}\Big)^{\gamma_1}dt. \end{aligned}$$

  By [3] Lemma 6.1, we have that

$$\begin{aligned} p_k &= C_1k^{-(1-\gamma_1)}\big(1+O(k^{-1})\big) \\ &= C_1k^{-(1-\frac{\beta_1}{\alpha_0-\alpha_2})}\big(1+O(k^{-1})\big). \end{aligned}$$

  The sequence of the expected degrees follows a power law with exponent $1-\frac{\beta_1}{\alpha_0-\alpha_2} = 2+\frac{\pi_1+\pi_4}{\pi_1+2\pi_2-2\pi_3-\pi_4} \geq 2$.

  Notice that the exponent increases to infinity as $\pi_4$ increases to $\pi_4^{cr}$.

- The exponential decay phase:

  If $\frac{\alpha_0}{\alpha_2} < 1$, then $\gamma_1 = -\frac{\beta_1}{\alpha_0-\alpha_2} < 0$ and $\gamma_2 > 0$

$$\begin{aligned} p_k &= \int_a^b t^{k-1}h(t)dt = \int_0^{\frac{\alpha_0}{\alpha_2}} t^{k-1}\Big(\frac{\frac{\alpha_0}{\alpha_2}-t}{1-t}\Big)^{-\gamma_1}dt \\ &= \Big(\frac{\alpha_0}{\alpha_2}\Big)^{k-\gamma_1}\int_0^1 u^{k-1}\Big(\frac{1-u}{1-\frac{\alpha_0}{\alpha_2}u}\Big)^{-\gamma_1}dt, \end{aligned}$$

  where $u = \frac{\alpha_0}{\alpha_2}t$.

The integral can be estimated by [3] Lemma 6.1, and

$$p_k = \Big(\frac{\alpha_0}{\alpha_2}\Big)^{k-\gamma_1} C_1 k^{-(1-\gamma_1)}\big(1+O(k^{-1})\big).$$

The sequence of the expected degrees decreases exponentially at rate $\frac{\alpha_0}{\alpha_2} = 1 - \frac{\pi_1\left(\frac{\pi_1}{2}+\pi_2-\pi_3\right)}{\pi_1\pi_3+\pi_1\pi_4+\pi_2\pi_4} > 1 - \frac{1}{2}\frac{\pi_1\pi_4}{\pi_1\pi_4+\pi_1\pi_3+\pi_2\pi_4}$.

- <u>The critical regime:</u>
  If $\frac{\alpha_0}{\alpha_2} = 1$, then by (4.15)

$$p_k = \int_0^1 t^{k-1} \exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)dt. \tag{4.16}$$

We have

$$\lim_{k\to+\infty} \frac{\ln p_k}{k} = 0 = \lim_{k\to+\infty} \frac{\ln k}{\ln p_k}. \tag{4.17}$$

This relation means that we have an intermediate state at criticality. The right hand side of equation (4.17) implies that the degree sequence decreases faster than any polynomial i.e. $p_k = o(k^{-a})$ for any $a > 0$. The left hand side means that the degree sequence decreases slower than any exponential function i.e. $p_k \gg \alpha^k$ for any $\alpha \in (0,1)$. For any $a > 0$ and $\alpha \in (0,1)$,

$$k^{-\gamma} \gg p_k \gg \alpha^k.$$

We prove here the left hand side of equation (4.17). We have for $\delta \geq 0$

$$0 \geq \frac{1}{k}\ln p_k = \frac{1}{k}\int_0^1 t^{k-1}\exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)dt \geq \frac{1}{k}\ln\left(\int_0^{1-\delta} t^{k-1}\exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)dt\right)$$

The fact that $\exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)$ is a decreasing function of $t$ yields $\exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big) \geq \exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{\delta}\Big)$ on $(0, 1-\delta)$ and thus

$$\begin{aligned}\frac{1}{k}\ln p_k &\geq \frac{1}{k}\ln\left(\int_0^{1-\delta} t^{k-1}\exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{\delta}\Big)dt\right)\\ &\geq \frac{1}{k}\frac{\beta_1}{\alpha_2} + \frac{1}{k}\ln\frac{1}{\delta} + \frac{1}{k}\ln\frac{1}{k} + \ln(1-\delta)\end{aligned}$$

Take $\delta = \delta(k) = \frac{1}{k}$ to ensure that $\lim_{k\to\infty}\ln(1-\delta) = 0$ and $\lim_{k\to\infty}\frac{1}{k}\ln k = 0$. That implies the right hand side of (4.17), that is

$$\lim_{k\to\infty}\frac{1}{k}\ln p_k = 0 \tag{4.18}$$

Instead of proving the right hand side of (4.17), we show that $\lim_{k\to\infty} k^a p_k = 0$ for any $a$. That means that $p_k$ decreases to 0 faster than any polynomial.

We split the integral in equation (4.16) in two parts. Let $\epsilon \in (0,1)$ then

$$\begin{aligned}p_k = \int_0^1 t^{k-1}\exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)dt &= \int_0^{1-\epsilon} t^{k-1}\exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)dt + \int_{1-\epsilon}^1 t^{k-1}\exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)dt\\ &\leq \int_0^{1-\epsilon} t^{k-1}dt + \int_{1-\epsilon}^1 \exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)dt\end{aligned}$$

That gives

$$\lim_{k\to\infty} k^a p^k \leq \lim_{k\to\infty} k^a \int_0^{1-\epsilon} t^{k-1}dt + \lim_{k\to\infty} k^a \int_{1-\epsilon}^1 \exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{1-t}\Big)dt$$

We take $u = \frac{1}{1-t}$ in the second integral to derive

$$\begin{aligned}
\lim_{k\to\infty} k^a p^k &\leq \lim_{k\to\infty} k^a \left[\frac{1}{k} t^k\right]_0^{1-\epsilon} + \lim_{k\to\infty} k^a \int_{\frac{1}{\epsilon}}^{\infty} \exp\Big(\frac{\beta_1}{\alpha_2} u\Big) \frac{1}{u^2} du \\
&\leq \lim_{k\to\infty} k^a \frac{1}{k}(1-\epsilon)^k + \lim_{k\to\infty} k^a \int_{\frac{1}{\epsilon}}^{\infty} \exp\Big(\frac{\beta_1}{\alpha_2} u\Big) du \qquad \text{where } \frac{1}{\epsilon} \geq 1 \\
&\leq \lim_{k\to\infty} k^a \frac{1}{k}(1-\epsilon)^k + \lim_{k\to\infty} k^a \left(-\frac{\beta_1}{\alpha_2} \exp\Big(\frac{\beta_1}{\alpha_2}\frac{1}{\epsilon}\Big)\right) du \qquad (4.19)
\end{aligned}$$

Choose $\epsilon = \left(\frac{1}{k}\right)^{\lambda}$ where $0 < \lambda < 1$, say $\epsilon = \frac{1}{\sqrt{k}}$, to derive that both limits on the right hand side of equation (4.19) converge to 0 for any $a$. Thus we find that for any $a$,

$$\lim_{k\to\infty} k^a p_k = 0 \qquad (4.20)$$

Together, equations (4.18) and (4.20) imply that at criticality, we have an intermediate stage.

To fully prove our statement, we are left to estimate $\frac{\alpha_0}{\alpha_2}$. We recall that $\alpha_0 = \left(\frac{\pi_1}{2} + \pi_2\right)\frac{1}{\eta}$, $\alpha_2 = \frac{\pi_3}{\eta} + \frac{\pi_4}{\nu}$ where $\nu = \pi_1 - \pi_4$ and $\eta = \frac{\pi_1+\pi_2-\pi_3}{\pi_1+\pi_4}\nu$. This gives the condition

$$\frac{\alpha_0}{\alpha_2} < 1 \Leftrightarrow \pi_1 + 2\pi_2 - 2\pi_3 - \pi_4 < 0. \qquad (4.21)$$

## 5. Summary and discussion

We have proved that in the case of the model of CFV, there exist critical probabilities of deletion of edges and vertices under which the graph keeps its scale free property, but if the rate of deletion increases above then the expected fraction of vertices of degree $k$ decreases exponentially fast at rate $\frac{\alpha_0}{\alpha_2}$ with an intermediate state at criticality. The critical threshold is given by equation (4.21).

A similar behaviour happens with the model of Deijfen and Lindholm where the process of activation differs on the introduction of edges which happens with probability $\pi_2$. In the model of Deijfen and Lindholm, only one extremity of the edge is chosen with preferential attachment, the other being chosen uniformly among all the vertices. We can also consider the model where both extremities of the added edges are chosen uniformly.

The same steps as the one detailed in the previous section hold in the two variations presented above. The number of vertices is clearly the same in all models. We need to determine the number of edges at time $t$. Lemma 1 from [3] holds for these models also since for the vertices with large degree the probability of receiving a link from an added edge is larger than the mean probability $\frac{1}{v_t}$. Therefore, the maximal degree in both models is dominated by the maximal degree in the model of [3]. It follows that Lemma 4.1 in [3] holds and the number of edges $e_t$ is well defined and the same in all models. We can solve the recursive formula as previously and we can derive the critical probabilities

$$\begin{aligned}
\pi_{4,DL}^{cr} &= \pi_1 + \pi_2\left(1 - \frac{\pi_4}{\pi_1}\right) - 2\pi_3 && \text{for the model introduced in [4] and} \\
\pi_{4,U}^{cr} &= \pi_1 - 2\pi_2\frac{\pi_4}{\pi_1} - 2\pi_3 && \text{where the index }_U\text{ stands for uniform.}
\end{aligned}$$

Notice that $\pi_{4,DL}^{cr} = \pi_4^{cr} - \frac{\pi_4}{\pi_1}\pi_2$ and $\pi_{4,U}^{cr} = \pi_{4,DL}^{cr} - \pi_2\left(1 + \frac{\pi_4}{\pi_1}\right)$ which indicates the natural relation $\pi_{4,U}^{cr} \leq \pi_{4,DL}^{cr} \leq \pi_4^{cr}$.

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