Merging Percolation on Z^d and Classical Random Graphs: Phase Transition*

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ABSTRACT: We study a random graph model which is a superposition of bond percolation on Z^d with parameter p, and a classical random graph $G(n, c/n)$. We show that this model, being a homogeneous random graph, has a natural relation to the so-called "rank 1 case" of inhomogeneous random graphs. This allows us to use the newly developed theory of inhomogeneous random graphs to describe the phase diagram on the set of parameters $c \geq 0$ and $0 \leq p < p_c$, where $p_c = p_c(d)$ is the critical probability for the bond percolation on Z^d. The phase transition is of second order as in the classical random graph. We find the scaled size of the largest connected component in the supercritical regime. We also provide a sharp upper bound for the largest connected component in the subcritical regime. The latter is a new result for inhomogeneous random graphs with unbounded kernels. © 2009 Wiley Periodicals, Inc. Random Struct. Alg., 36, 185–217, 2010

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1. INTRODUCTION

1.1. Model

We shall begin by constructing for each integer $N \geq 1$ a random graph $G_N(p, c)$ on the set of vertices $B(N) := \{-N, \dots, N\}^d$ in Z^d, $d \geq 1$. Consider the two sets of edges

$$\mathcal{S}_N = \{\{u, v\} : u, v \in B(N), \|u - v\| = 1\}$$

and

$$\mathcal{L}_N = \{\{u, v\} : u, v \in B(N), \|u - v\| \geq 1\}.$$

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We assume that each edge from $\mathcal{S}_N$ is present in graph $G_N(p,c)$ with probability p independent of other edges, and also each edge from $\mathcal{L}_N$ is present in graph $G_N(p,c)$ with probability $c/|B(N)|$ independent of other edges. (Here, for any set A we denote $|A|$ the number of the elements in A.)

Call the edges from set $\mathcal{S}_N$ short-range edges and the edges from set $\mathcal{L}_N$ long-range edges. By this definition, if $\|u-v\| = 1$ then with probability p there is a short-range edge between u and v, whereas with probability $c/|B(N)|$ there is a long-range edge between any two vertices in graph $G_N(p,c)$. Clearly, there can be at most two edges between any two vertices in the graph $G_N(p,c)$.

The random graph $G_N(p,c)$ is a superposition of the bond percolation model restricted to $B(N)$ (see, e.g., [3]), where each pair of neighbors in $B(N)$ is connected with probability p, and the random graph model $G_{n,c/n}$ (see, e.g., [5]) on $n = |B(N)|$ vertices, where each vertex is connected to any other vertex with probability c/n. All of the edges in both models are independent.

Our model is a simplification of the most common graphs that have been designed to study natural phenomena such as biological neural networks [12]. Our random graph $G_N(p,c)$ is different to the so-called "small-world" models, intensively studied starting with [13]. In "small-world" models, the edges of the grid may be kept or removed, and only a finite number (often at most $2d$) of the long-range edges may emerge from each vertex, and the probability of those is a fixed number.

We are interested in the connectivity of the introduced graph $G_N(p,c)$ as $N \to \infty$. We say that two vertices are connected if there is a path of edges, independent of the type, between them. Clearly, if $c = 0$, we have a purely bond percolation model on Z^d, where any edge from the grid is kept (i.e., "is open" in the terminology of percolation theory) with a probability p, or, alternatively, removed with a probability $1-p$. Let us recall some basic facts from percolation theory which we need here.

Let C denote an open cluster containing the origin of Z^d in the bond percolation model. Clearly, the distribution of $|C|$ depends on the parameter p. It is known (see, e.g., [3]) that for any $d \geq 1$ there is $p_c = p_c(d)$ such that

$$\mathbf{P}\{|C| = \infty\} \begin{cases} = 0, & \text{if } p < p_c, \\ > 0, & \text{if } p > p_c, \end{cases}$$

where $0 < p_c < 1$, unless $d = 1$, in which case $p_c = 1$. We shall assume here that $0 < p < p_c$, and thus the connected components formed by the short-range edges are finite with probability one. Recall also that for all $0 < p < p_c$ the limit

$$\zeta(p) = \lim_{n\to\infty} \left(-\frac{1}{n} \log \mathbf{P}\{|C| = n\} \right) \tag{1.1}$$

exists and satisfies $\zeta(p) > 0$ (Theorem (6.78) from [3]).

1.2. Results

Let $C_1(G)$ denote the size (i.e., the number of vertices) of the largest connected component in a graph G.

Theorem 1.1. *Assume, $d \geq 1$ and $0 \leq p < p_c(d)$. Let*

$$c^{cr}(p) = \frac{1}{\mathbf{E}|C|}. \tag{1.2}$$

i. *For $c < c^{cr}(p)$ define*

$$y = \begin{cases} \text{the root of} \quad \mathbf{E}c|C|e^{c|C|y} = 1, & \text{if} \quad \mathbf{E}c|C|e^{|C|\zeta(p)} \geq 1, \\ \zeta(p)/c, & \text{otherwise,} \end{cases}$$

and set

$$\alpha(p,c) := (c + cy - \mathbf{E}ce^{c|C|y})^{-1}. \tag{1.3}$$

If $c < c^{cr}(p)$, then for any $\alpha > \alpha(p,c)$

$$\mathbf{P}\{C_1(G_N(p,c)) > \alpha \log |B(N)|\} \to 0. \tag{1.4}$$

as $N \to \infty$.

ii. *If $c \geq c^{cr}(p)$, then*

$$\frac{C_1(G_N(p,c))}{|B(N)|} \xrightarrow{P} \beta \tag{1.5}$$

as $N \to \infty$, with $\beta = \beta(p,c)$ defined as the maximal solution to

$$\beta = 1 - \mathbf{E}\{e^{-c\beta|C|}\}. \tag{1.6}$$

Remark 1.1. Only when $d = 1$ do we know the exact distribution of $|C|$ (see (1.7) later), in which case $\mathbf{E}|C|e^{|C|\zeta(p)} = \infty$. This obviously yields for $d = 1$ that the constant y in Theorem 1.1 is defined simply as the root of $\mathbf{E}\, c|C|e^{c|C|y} = 1$ for any $c < c^{cr}(p)$.

In view of (1.1) it is obvious that $\chi(p) := \mathbf{E}|C| < \infty$ for all $0 \leq p < p_c$. It is also known (see Theorem (6.108) and (6.52) in [3]) that $\chi(p)$ is an analytic function of p on $[0, p_c)$ and $\chi(p) \to \infty$ as $p \uparrow p_c$. This implies that $c^{cr}(p)$ is a continuous, strictly decreasing function on $[0, p_c)$ with $c^{cr}(0) = 1$ and $c^{cr}(p) \to 0$ as $p \uparrow p_c$. Hence, c^{cr} has an inverse, i.e., for any $0 < c < 1$ there is a unique $0 < p^{cr}(c) < p_c = p_c(d)$ such that $c^{cr}(p^{cr}(c)) = c$. This leads to the following duality of Theorem 1.1.

Corollary 1.1. *For any $0 < c < 1$ there is a unique $0 < p^{cr}(c) < p_c$ such that for any $p^{cr}(c) < p < p_c$ graph $G_N(p,c)$ has a giant component with size $\Theta(|B(N)|)$* **whp** *(i.e., with probability tending to one as $N \to \infty$), whereas for any $p < p^{cr}(c)$ the size of the largest connected component in $G_N(p,c)$ is $O(\log |B(N)|)$* **whp**. ■

Hence, Theorem 1.1 tells us something about the "distances" between the components of a random graph if it is considered on the vertices of $\mathbf{Z}^d$.

It is worth mentioning that the symmetry between c^{cr} and p^{cr} is most spectacular in dimension one, when $p_c(1) = 1$. Notice, that if $d = 1$ this case is exactly solvable, and we know the distribution of $|C|$:

$$\mathbf{P}\{|C| = k\} = (1-p)^2 kp^{k-1}, \quad k \geq 1. \tag{1.7}$$

Hence, if $d = 1$ we compute for all $0 \le p < 1 = p_c(1)$

$$c^{cr}(p) = \frac{1-p}{1+p}, \tag{1.8}$$

which yields

$$p^{cr}(c) = \frac{1-c}{1+c},$$

for all $0 \le c < 1$. (For more details on $d = 1$ case we refer to [11].)

Remark 1.2. For any fixed c function $\beta(p, c)$ is continuous at $p = 0$. If $p = 0$, i.e., when our graph is merely a classical $G_{n,c/n}$ random graph, then $|C| \equiv 1$ and (1.6) becomes a well-known relation.

Furthermore, for any fixed $c < 1$, if $p = 0$, it is not difficult to derive from (1.3) that

$$\alpha(0, c) = \frac{1}{c - 1 + |\log c|}. \tag{1.9}$$

But $\log n/(c - 1 + |\log c|)$ is known (see Theorem 7a in [2]) to be the principal term in the asymptotics (in probability) of the largest connected component of $G_{n,c/n}$ when $c < 1$. This leads to the following conjecture.

Conjecture 1.1. *For all* $\alpha_1 < \alpha(p, c)$ *when* $0 < p < p_c$ *and* $c < c^{cr}(p)$

$$\mathbf{P}\{C_1(G_N(p, c)) < \alpha_1 \log |B(N)|\} \to 0 \quad \text{as } N \to \infty. \tag{1.10}$$

It is easy to check that if $c \le c^{cr}(p)$ then Eq. (1.6) does not have a strictly positive solution, whereas $\beta = 0$ is always a solution to (1.6). Indeed, let $c \le c^{cr}(p)$ and consider the right-hand side of (1.6) as a function of $\beta \in [0, 1]$. It is an increasing concave function, whose right derivative at $\beta = 0$ is $c\mathbf{E}|C|$. Hence, if $c \le c^{cr}(p) = 1/\mathbf{E}|C|$ the only solution to (1.6) is $\beta = 0$. We conjecture that

$$\frac{\partial}{\partial c}\beta(p, c) \mid_{c\downarrow c^{cr}(p)} = 2\frac{(\mathbf{E}|C|)^3}{\mathbf{E}(|C|^2)}. \tag{1.11}$$

This would confirm that the phase transition remains to be of second order for any $p < p_c$, as it is for $p = 0$, i.e., in the case of classical random graphs.

One heuristic argument for the equality in (1.11) is the following. Compute a second derivative $\beta'' = \frac{\partial^2}{\partial c^2}\beta(p, c)$, which by (1.6) is

$$\beta'' = \mathbf{E}[(2\beta' + c\beta'') - (\beta + c\beta')^2|C|]|C|e^{-c\beta|C|},$$

and pass to the limit $c \downarrow c^{cr}(p) = (\mathbf{E}|C|)^{-1}$ on both sides, taking into account that $\beta(p, c) \mid_{c\downarrow c^{cr}(p)} = 0$ for all $p < p_c(d)$:

$$\begin{aligned}\beta'' \mid_{c\downarrow c^{cr}(p)} &= 2\beta' \mid_{c\downarrow c^{cr}(p)} \mathbf{E}|C| + \beta'' \mid_{c\downarrow c^{cr}(p)} c^{cr}(p)\mathbf{E}|C| - (\beta' \mid_{c\downarrow c^{cr}(p)})^2 (c^{cr}(p))^2 \mathbf{E}(|C|^2) \\ &= \beta'' \mid_{c\downarrow c^{cr}(p)} + \beta' \mid_{c\downarrow c^{cr}(p)} (\mathbf{E}|C|)(2 - \beta' \mid_{c\downarrow c^{cr}(p)} (c^{cr}(p))^3 \mathbf{E}(|C|^2)).\end{aligned}$$

Then (1.11) may follow from here by the fact that β is the maximal solution to (1.6).

For the proofs of similar to (1.11) statements one can consult [1], Section 16.4.

1.3. Methods

Although our model (it can be considered on a torus, in the limit the result is the same) is a perfectly homogeneous random graph, in the sense that the degree distribution is the same for any vertex, we study it via inhomogeneous random graphs, making use of the recently developed theory from [1]. The idea is the following. First, we consider the subgraph induced by the short-range edges, i.e., the edges which connect two neighboring nodes with probability p. It is composed of connected clusters (which may consist just of one single vertex) in $B(N)$. Define a macrovertex to be a connected component of this subgraph. We say that a macrovertex is of type k, if k is the number of vertices in it. Note also that the distribution of a size of a cluster is given (approximately) by the distribution of $|C|$. Conditionally on the set of macrovertices, we consider a graph on these macrovertices induced by the long-range connections. Two macrovertices are said to be connected if there is at least one (long-range type) edge between two vertices belonging to different macrovertices. Thus, the probability of an edge between two macrovertices v_i and v_j of types x and y, correspondingly, is

$$\tilde{p}_{xy}(N) := 1 - \left(1 - \frac{c}{|B(N)|}\right)^{xy} \approx \frac{c}{|B(N)|}xy \tag{1.12}$$

for large N. Later, we show that this graph on macrovertices fits the conditions of a general inhomogeneous graph model defined in [1], with a kernel proportional to xy (which is unbounded for $x, y \geq 1$). Then, formula (1.2) for the critical parameters follows in an almost straightforward fashion by [1].

The size of a component in our original model is the sum of the types of the macrovertices of the correspondent component in the inhomogeneous model. We still use essentially the results from [1] to derive (1.6). Notice, however, that (1.6) is no longer an equation for a survival probability as in [1].

The result in the subcritical phase (part i of Theorem 1.1) is new, it does not follow from the theory in [1]. The essential feature of the derived inhomogeneous model is that it has an unbounded kernel function. Up until now, the size of the largest connected component in an inhomogeneous random graph below the phase transition has been studied only for uniformly bounded kernels, for which it was proved to be $O(\log n)$ **whp** (see Theorem 3.12, [1]). Our method, which is based on the moment generating functions, not only allows us to treat unbounded kernels but also yields a sharp bound for the size of the component. We discuss this in more detail in the end of Section 2.4.

Notice also that to analyze the introduced model, we derive here some results on the joint distribution of the sizes of clusters in the percolation model (see Lemma 2.1 later), which may be of interest on its own.

The principle of treating some local structures in a graph as new vertices (macrovertices), and then considering a graph induced by the original model on these vertices appears to be rather general. For example, in [4] a different graph model was also put into a framework of inhomogeneous graphs theory by certain restructuring. This method should be useful for analysis of a broad class of complex structures, whenever one can identify local and global connections. Some examples of such models one can find in [6].

Finally, we comment that our result should help to study a model for the propagation of the neuronal activity introduced in [12]. Here, we show that a giant component in the graph can emerge from two sources, neither of which can be neglected, but each of which may be in the subcritical phase, i.e., even when both $p < p_c$ and $c < 1$. In particular, for any

$0 < c < 1$ we can find $p < p_c$, which allows with positive probability the propagation of impulses through the large part of the network due to the local activity.

2. MODEL $G_N(p, c)$ AS AN INHOMOGENEOUS RANDOM GRAPH

2.1. Bond Percolation Model

Consider the subgraph on $B(N)$ induced by the short-range edges only, which is a purely bond percolation model. By the construction this subgraph, call it $G_N^{(s)}(p)$, is composed of a random number of clusters (of connected vertices) of random sizes. We call the size of a cluster the number of its vertices (it may be just one). We recall here more results from percolation theory, which we shall use later on.

Let K_N denote the number of the connected components (clusters) in $G_N^{(s)}(p)$, and let

$$\mathbf{X} = \{X_1, X_2, \ldots, X_{K_N}\} \tag{2.1}$$

denote the collection of all connected clusters X_i in $G_N^{(s)}(p)$. We shall also use X_i to denote the set of vertices in the $i-$th cluster. By this definition $\sum_{i=1}^{K_N} |X_i| = |B(N)|$.

Theorem ([3], (4.2) Theorem, p. 77)**.**

$$\frac{K_N}{|B(N)|} \to \kappa(p) := \mathbf{E}\frac{1}{|C|} \tag{2.2}$$

a.s. and in L^1 as $N \to \infty$. ∎

Note (see, e.g., [3]) that $\kappa(p)$ is strictly positive and finite for all $0 < p < p_c$. Next, we cite the large deviations property of K_N from [14].

Theorem ([14], Theorem 2)**.** *Given $\kappa(p) > \varepsilon > 0$, there exists $\sigma_j(\varepsilon, p) > 0$ for $j = 1, 2$ such that*

$$\lim_{N\to\infty} \frac{-1}{|B(N)|} \log \mathbf{P}\left(\frac{K_N}{|B(N)|} \geq \kappa(p) + \varepsilon\right) = \sigma_1(\varepsilon, p)$$

and

$$\lim_{N\to\infty} \frac{-1}{|B(N)|} \log \mathbf{P}\left(\frac{K_N}{|B(N)|} \leq \kappa(p) - \varepsilon\right) = \sigma_2(\varepsilon, p).$$

This theorem immediately implies the following.

Corollary 2.1. *Define for any $0 < \delta < \kappa(p) = \mathbf{E}(|C|^{-1})$ an event*

$$\mathcal{A}_{\delta,N} = \left\{\left|\frac{K_N}{|B(N)|} - \mathbf{E}(|C|^{-1})\right| \leq \delta\right\}. \tag{2.3}$$

There exists a positive constant $\sigma = \sigma(\delta, p)$ such that for all large N

$$\mathbf{P}(\mathcal{A}_{\delta,N}) \geq 1 - e^{-\sigma|B(N)|}. \tag{2.4}$$

Next, we define for any $k \geq 1$ and $x \geq 0$ an indicator function:

$$I_k(x) = \begin{cases} 1, & \text{if } x = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.1. *For any fixed $k \geq 1$*

$$\frac{1}{K_N} \sum_{i=1}^{K_N} I_k(|X_i|) \to \frac{1}{\kappa(p)} \frac{\mathbf{P}\{|C| = k\}}{k} =: \mu(k) \tag{2.5}$$

a.s. and in L^1 as $N \to \infty$.

Proof. Let $C_N(z)$, $z \in B(N)$, denote the connected cluster in $B(N)$, which contains vertex z, and let $C(z)$ denote the open cluster in $\mathbf{Z}^d$, which contains vertex z. We write

$$\frac{1}{K_N} \sum_{i=1}^{K_N} I_k(|X_i|) = \frac{|B(N)|}{K_N} \frac{1}{k} \frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C_N(z)|). \tag{2.6}$$

Observe that

$$\begin{aligned} \sum_{z \in B(N)} I_k(|C_N(z)|) &= \sum_{z \in B(N)} I_k(|C(z)|) + \sum_{z \in B(N)} (I_k(|C_N(z)|) - I_k(|C(z)|)) \\ &= \sum_{z \in B(N)} I_k(|C(z)|) + \sum_{z \leftrightarrow \partial B(N)} (I_k(|C_N(z)|) - I_k(|C(z)|)), \end{aligned} \tag{2.7}$$

where the last summation is over all vertices z of $B(N)$ which are connected to the surface

$$\partial B(N) = \left\{ x \in B(N) : \max_{1 \leq i \leq d} |x_i| = N \right\},$$

and hence the last sum in (2.7) contains at most $k|\partial B(N)|$ nonzero terms. Now, we can rewrite (2.6) as follows

$$\frac{1}{K_N} \sum_{i=1}^{K_N} I_k(|X_i|) = \frac{|B(N)|}{K_N} \frac{1}{k} \frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C(z)|) + \frac{|B(N)|}{K_N} \frac{\Delta_N}{|B(N)|}, \tag{2.8}$$

where

$$|\Delta_N| \leq |\partial B(N)|.$$

Taking into account (2.2), we conclude

$$\frac{|B(N)|}{K_N} \frac{\Delta_N}{|B(N)|} \to 0 \tag{2.9}$$

in L^1 and *a.s.* By the ergodic theorem

$$\frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C(z)|) \to \mathbf{P}\{|C| = k\} \tag{2.10}$$

a.s. as $N \to \infty$, and in L^1 as well, since

$$0 \le \frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C(z)|) \le 1.$$

Hence, statement (2.5) follows by (2.8), (2.9), and (2.10) combined with (2.2). ∎

Finally, we mention here one more helpful result.

Proposition 2.2. *Let $p < p_c(d)$. Then for all $L > 0$ and all $N > 1$*

$$\mathbf{P}\left\{ \max_{1 \le i \le K_N} |X_i| > L \right\} \le |B(N)|\, \mathbf{P}\{|C| > L\}, \tag{2.11}$$

and in particular

$$\mathbf{P}\left\{ \max_{1 \le i \le K_N} |X_i| > \frac{2}{\zeta(p)} \log |B(N)| \right\} \to 0 \tag{2.12}$$

as $N \to \infty$.

Proof. First, we observe that

$$\mathbf{P}\left\{ \max_{1 \le i \le K_N} |X_i| > L \right\} \le \mathbf{P}\left\{ \max_{z \in B(N)} |C(z)| > L \right\} \le |B(N)|\, \mathbf{P}\{|C| > L\},$$

where $C(z)$ is the open cluster containing z. This proves (2.11).

By (1.1) for any $0 < \alpha < \zeta(p)$ there is constant $b > 0$ such that

$$\mathbf{P}\{|C| \ge L\} \le be^{-\alpha L}$$

for all $L \ge 1$, which together with (2.11) implies (2.12). ∎

2.2. Random Graph on Macrovertices

Given a collection of clusters $\mathbf{X} = \{X_1, \dots, X_{K_N}\}$ defined in (2.1), we introduce another graph $\tilde{G}_N(\mathbf{X}, p, c)$ as follows. The vertices of graph $\tilde{G}_N(\mathbf{X}, p, c)$ are the *sets* $X_1, \dots, X_{K_N}$. We write

$$v_i = X_i, \quad 1 \le i \le K_N, \tag{2.13}$$

and call a vertex v_i of $\tilde{G}_N(\mathbf{X}, p, c)$ a macrovertex. Each vertex v_i is said to be of type $|X_i|$, which means that the type of a macrovertex v_i is simply its cardinality. The set of types of macrovertices is $\{1, 2, \dots\}$.

The edges between the vertices of $\tilde{G}_N(\mathbf{X}, p, c)$ are presented independently with probabilities induced by the original graph $G_N(p, c)$. We say that two macrovertices are connected if there is at least one (long-range type) edge between two vertices belonging to different macrovertices. Then, the probability of an edge between any two vertices v_i and v_j of types x and y, correspondingly, is $\tilde{p}_{xy}(N)$ introduced in (1.12). Clearly, this construction provides a one-to-one correspondence between the connected components in the graphs $\tilde{G}_N(\mathbf{X}, p, c)$

and $G_N(p,c)$: the number of the connected components is the same for both graphs, as well as the number of the involved vertices from $B(N)$ in two corresponding components. In fact, considering conditionally on $\mathbf{X}$ the graph $\tilde{G}_N(\mathbf{X},p,c)$ we neglect only those long-range edges from $G_N(p,c)$, which connect vertices within each v_i, i.e., the vertices which are already connected through the short-range edges.

Now the aim is to place our model into the framework of inhomogeneous random graphs from [1]. First, we recall some basic definitions from [1].

Let S be a separable metric space and μ be a Borel probability measure on S. A generalized vertex space is a triple $\mathcal{V} = (S, \mu, (\mathbf{x}_n)_{n\geq 1})$, where $\mathbf{x}_n = (x_1, \dots, x_{v_n})$ for each n is a random sequence of random length v_n of points of S such that for any μ-continuity set $A \subseteq S$

$$\frac{\#\{i : x_i \in A\}}{v_n} \xrightarrow{P} \mu(A) \tag{2.14}$$

as $n \to \infty$. Given the sequence $x_1, \dots, x_{v_n}$, we let $G^{\mathcal{V}}(n, \kappa_n)$ be the random graph on vertices $\{1, \dots, v_n\}$, such that any two vertices i and j are connected by an edge independently of the others and with probability

$$p_{x_i x_j}(n) = \min\left\{\frac{\kappa_n(x_i, x_j)}{v_n}, 1\right\}, \tag{2.15}$$

where the kernel κ_n is a symmetric non-negative measurable function on $S \times S$.

Definition A. *A kernel κ is graphical on $\mathcal{V}$ if the following conditions hold:*

(i) *κ is continuous a.s. on $S \times S$;*
(ii) *$\kappa \in L^1(S \times S, \mu \times \mu)$;*
(iii)

$$\frac{1}{n}\mathbf{E}e(G^{\mathcal{V}}(n,\kappa)) \to \frac{1}{2}\int_S\int_S \kappa(x,y)d\mu(x)d\mu(y),$$

where $e(G)$ denotes the number of edges in a graph G.

Definition B. *A sequence (κ_n) of kernels on $S \times S$ is* graphical *on $\mathcal{V}$* with limit *κ if for a.e. $(y,z) \in S \times S$*

$$y_n \to y \text{ and } z_n \to z \text{ imply that } \kappa_n(y_n, z_n) \to \kappa(y,z), \tag{2.16}$$

κ satisfies conditions (i) and (ii) from Definition A, and also

$$\frac{1}{n}\mathbf{E}e\left(G^{\mathcal{V}}(n,\kappa_n)\right) \to \frac{1}{2}\int_S\int_S \kappa(x,y)d\mu(x)d\mu(y). \tag{2.17}$$

Now consider the graph $\tilde{G}_N(\mathbf{X},p,c)$. Let us rewrite probabilities $\tilde{p}_{xy}(N)$ for the edges in this model taking into account the size of the graph:

$$\tilde{p}_{xy}(N) = 1 - \left(1 - \frac{c}{|B(N)|}\right)^{xy} =: \frac{K_N(x,y)}{K_N}. \tag{2.18}$$

By (2.2) and (2.18) for every $x, y \in \{1, 2, \ldots\}$ we have: if $x(N) \to x$ and $y(N) \to y$ (which in our case simply means that $x(N) = x$ and $y(N) = y$ for all large N) then

$$\mathcal{K}_N(x(N), y(N)) = \frac{K_N}{|B_N|}(|B_N|\tilde{p}_{x(N)y(N)}(N)) \stackrel{a.s.}{\to} \kappa(p)\, cxy =: \mathcal{K}(x, y) \tag{2.19}$$

as $N \to \infty$. This confirms condition (2.16) for our model. Also, by Proposition 2.1 for all $k \in \{1, 2, \ldots\}$

$$\frac{\#\{i : |X_i| = k\}}{K_N} \stackrel{P}{\to} \mu(k) := \frac{1}{\kappa(p)} \frac{\mathbf{P}\{|C| = k\}}{k}, \tag{2.20}$$

as $N \to \infty$ which gives us condition (2.14) for our model.

Note that because of (1.1) the function $\mu(k)$ decays exponentially. This implies that

$$\mathcal{K} \in L^1(S \times S, \mu \times \mu). \tag{2.21}$$

Finally, one can verify with the help of (2.2) and Proposition 2.1 that for any $t(N)$ such that $t(N)/|B(N)| \to \mathbf{E}\{|C|^{-1}\}$

$$\frac{1}{t(N)}\mathbf{E}\{e(\tilde{G}_N(\mathbf{X}, p, c))|K_N = t(N)\} \to \frac{1}{2}\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\mathcal{K}(x, y)\mu(x)\mu(y), \tag{2.22}$$

and the convergence in (2.17) follows.

Hence, we conclude that our model $\tilde{G}_N(\mathbf{X}, p, c)$ is the particular case of the general inhomogeneous random graph model, which we denote $G^{\mathcal{V}}(n, \mathcal{K}_n)$ with the vertex space

$$\mathcal{V} = (S, \mu, (v_1, \ldots, v_{t(n)})_{n \geq 1}),$$

where (here and in the rest of this article)

$$S = \{1, 2, \ldots\},$$

μ is a probability on S defined by (2.20), and the sequence of kernels $\mathcal{K}_n$ is graphical on $\mathcal{V}$ with limit

$$\mathcal{K}(x, y) = \kappa(p)\, cxy.$$

Notice here that kernels of this type, i.e., when $\mathcal{K}(x, y) = \psi(x)\psi(y)$, fall into the "rank 1 case" (see [1], Chapter 16.4).

2.3. A Branching Process Related to $\tilde{G}_N(X, p, c)$

Here, we follow closely the approach in [1]. We shall use a well-known technique of branching processes to reveal the connected component in the graph $\tilde{G}_N(\mathbf{X}, p, c)$. Recall first the usual exploration algorithm of finding a connected component. Conditionally on the set of macrovertices, take any vertex v_i to be the root. Reveal all the vertices $\{v^1_{i_1}, v^1_{i_2}, \ldots, v^1_{i_n}\}$ connected to this vertex v_i in the graph $\tilde{G}_N(\mathbf{X}, p, c)$, call them the offspring of v_i, and then mark v_i as "saturated." Then for each nonsaturated but already revealed vertex, we find all the vertices connected to them but which have not been used previously. We continue this process until we end up with a tree of saturated macrovertices whose types are in the set $S = \{1, 2, \ldots\}$.

Denote $\tau_N(v)$ the set of the macrovertices in the tree constructed according to the exploration algorithm with the root at a vertex v. It is plausible to think (and in our case it is correct, as will be seen later) that $\tau_N(v)$ is well approximated by the following multitype Galton-Watson process. Let $x \in S$ and let $B_x^{c,p} = \{B_x^{c,p}(n), n \geq 0\}$, where $B_x^{c,p}(n)$ for each $n \geq 0$ denotes a set of particles, each of which is assigned some type, a value from the set S. We assume that initially $B_x^{c,p}(0)$ consists of a single particle of type x, and then at any step, a particle of type $x' \in S$ is replaced in the next generation by a set of particles where the number of particles of type y has a Poisson distribution

$$\mathrm{Po}(\mathcal{K}(x', y)\mu(y)).$$

We shall study now the properties of the process $B_x^{c,p}$. Let $\rho(x)$ denote the probability that a particle of type x produces by this process an infinite population. First, we state a general result on $\rho(x)$, which was proved in [1]. Define

$$T_{\mathcal{K}} f(x) = \int_S \mathcal{K}(x, y) f(y) d\mu(y),$$

and

$$\|T_{\mathcal{K}}\| = \sup\{\|T_{\mathcal{K}} f\|_2 : f \geq 0, \|f\|_2 \leq 1\}.$$

Theorem A ([1], Theorem 6.1). *Suppose that $\mathcal{K}$ is the kernel on (S, μ), that $\mathcal{K} \in L^1$, and*

$$\int_S \mathcal{K}(x, y) d\mu(y) < \infty$$

for every $x \in S$. Then $\rho(x)$ is the maximal solution to

$$f(x) = 1 - \exp\{-T_{\mathcal{K}} f(x)\}. \tag{2.23}$$

Furthermore:

(i) *If $\|T_{\mathcal{K}}\| \leq 1$ then $\rho(x) = 0$ for every x, and* (2.23) *has only the zero solution.*
(ii) *If $1 < \|T_{\mathcal{K}}\| \leq \infty$ then $\rho(x) > 0$ on a set of a positive measure. If, in addition, $\mathcal{K}$ is irreducible then $\rho(x) > 0$ for a.e. x, and $\rho(x)$ is the only nonzero solution of* (2.23).

(Notice that $f = 0$ is always a solution to (2.23) independent of value $\|T_{\mathcal{K}}\|$.)

Let us verify the conditions of this theorem for our model. Recall that a kernel $\mathcal{K}$ on a ground space (S, μ) is called irreducible if

$$A \subseteq S \text{ and } \mathcal{K} = 0 \text{ a.e. on } A \times (S \setminus A) \text{ implies } \mu(A) \text{ or } \mu(S \setminus A) = 0.$$

This condition trivially holds in our case since $\mathcal{K}(x, y) > 0$ for all $(x, y) \in S \times S$. Also, it is straightforward to derive

$$\sum_{y=1}^{\infty} \mathcal{K}(x, y)\mu(y) = \sum_{y=1}^{\infty} \kappa(p) cxy \frac{1}{\kappa(p)} \frac{\mathbf{P}\{|C| = y\}}{y} = cx < \infty$$

for any x. This together with (2.21) confirms that the conditions of Theorem A hold for our model. Hence, in our case function $\rho(x)$, $x \in S$, is the maximum solution to

$$\rho(x) = 1 - e^{-\sum_{y=1}^{\infty} \mathcal{K}(x,y)\mu(y)\rho(y)}. \tag{2.24}$$

We remark that in the rank 1 case of the kernel, i.e., when $\mathcal{K}(x, y) = \psi(x)\psi(y)$, we have

$$\|T_{\mathcal{K}}\| = \left(\int_S \int_S \mathcal{K}^2(x, y)d\mu(x)d\mu(y)\right)^{1/2} = \int_S \psi^2(x)d\mu(x). \tag{2.25}$$

(For further details refer [1].) Therefore, in the case of our model, when $\mathcal{K}(x, y) = c\kappa(p)xy = \psi(x)\psi(y)$ with $\psi(x) = \sqrt{c\kappa(p)}\,x$, and μ is a probability on a countable space S, we find by (2.25) that

$$\|T_{\mathcal{K}}\| = c\kappa(p)\sum_{y=1}^{\infty} y^2\mu(y) = c\,\kappa(p)\sum_{y=1}^{\infty} y^2 \frac{1}{\kappa(p)} \frac{\mathbf{P}\{|C| = y\}}{y} = c\,\mathbf{E}|C|.$$

Hence, by the cited above Theorem A we have for our model $\rho(x) > 0$ for all $x \in S$ if and only if

$$\|T_{\mathcal{K}}\| = c\mathbf{E}|C| > 1; \tag{2.26}$$

otherwise, $\rho(x) = 0$ for all $x \in S$.

Let us state another general result from [1], which we need here. Let $\rho(x)$ be the maximum solution to (2.23), and define

$$\rho = \int_S \rho(x)d\mu(x).$$

Theorem B ([1], Theorem 3.1). *Let $(\mathcal{K}_n)$ be a graphical sequence of kernels on a generalized vertex space $\mathcal{V}$ with limit $\mathcal{K}$.*

(i) *If $\|T_{\mathcal{K}}\| \le 1$, then $C_1(G^{\mathcal{V}}(n, \mathcal{K}_n)) = o_P(n)$, while if $\|T_{\mathcal{K}}\| > 1$, then $C_1(G^{\mathcal{V}}(n,\mathcal{K}_n)) = \Theta(n)$ whp.*

(ii) *For any $\varepsilon > 0$, whp we have*

$$\frac{1}{n}C_1(G^{\mathcal{V}}(n, \mathcal{K}_n)) \le \rho + \varepsilon.$$

(iii) *If $\mathcal{K}$ is quasi-irreducible, then*

$$\frac{1}{n}C_1(G^{\mathcal{V}}(n, \mathcal{K}_n)) \xrightarrow{P} \rho.$$

In all cases $\rho > 0$ if and only if $\|T_{\mathcal{K}}\| > 1$.

Let us apply this general result to our model.

Corollary 2.2. (i) *For all $0 \le p < p_c$ and $c \ge 0$*

$$\frac{C_1(\tilde{G}_N(\mathbf{X}, p, c))}{K_N} \xrightarrow{P} \rho := \sum_{x=1}^{\infty} \rho(x)\mu(x), \tag{2.27}$$

where $\rho(x)$ is the maximum solution to (2.24).

(*ii*) *$\rho > 0$ if and only if $c\mathbf{E}|C| > 1$, i.e., if and only if $c > c^{cr}(p)$; otherwise, $\rho = 0$.*

Proof. As we previously showed, conditionally on K_N so that $K_N/|B(N)| \to \mathbf{E}(|C|^{-1})$, the sequence $\mathcal{K}_n(x,y)$ is graphical on $\mathcal{V}$. Hence, together with the fact that our kernel $\mathcal{K}$ is positive on S (hence, quasi-irreducible as well) the conditions of the cited Theorem B (*iii*) are satisfied for our model $\tilde{G}_N(\mathbf{X},p,c)$. Therefore, Theorem B (*iii*) yields statement (*i*) of the corollary.

Statement (*ii*) of the corollary follows by (2.26) and Theorem A. ∎

Remark 2.1. Corollary 2.2 together with (2.2) implies

$$\frac{C_1(\tilde{G}_N(\mathbf{X},p,c))}{|B(N)|} \xrightarrow{P} \mathbf{E}(|C|^{-1})\ \rho, \tag{2.28}$$

where $\rho > 0$ if and only if $c > c^{cr}(p)$.

Notice, however, that here $C_1(\tilde{G}_N(\mathbf{X},p,c))$ is the number of macrovertices in the largest connected component of $\tilde{G}_N(\mathbf{X},p,c)$, whereas our aim is to find the size of the largest connected component in the original graph $G_N(p,c)$.

Finally, we quote a general result from [1] on the second largest component, which we denote by $C_2(G^{\mathcal{V}}(n,\mathcal{K}_n))$.

Theorem C ([1], Theorem 12.6)**.** *Let* $(\mathcal{K}_n)$ *be a graphical sequence of kernels on a generalized vertex space* $\mathcal{V}$ *with irreducible limit* $\mathcal{K}$. *If* $\|T_{\mathcal{K}}\| > 1$, *and* $\inf_{n,x,y} \mathcal{K}_n(x,y) > 0$ *then*

$$C_2\left(G^{\mathcal{V}}(n,\mathcal{K}_n)\right) = O(\log n)$$

whp.

This theorem together with the convergence in (2.2) implies for our case the following result.

Corollary 2.3. *For all* $0 \le p < p_c$ *and* $c > c^{cr}$ *one has*

$$C_2(\tilde{G}_N(\mathbf{X},p,c)) = O(\log |B(N)|)$$

whp.

2.4. On the Distribution of Types of Vertices in $\tilde{G}_N(X,p,c)$

Given a collection of clusters $\mathbf{X}$ (see (2.1)), we define for all $1 \le k \le |B(N)|$

$$\mathcal{N}_k = \mathcal{N}_k(\mathbf{X}) = \sum_{i=1}^{K_N} I_k(|X_i|).$$

In words, $\mathcal{N}_k$ is the number of (macro)vertices of type k in the set of vertices of graph $\tilde{G}_N(\mathbf{X},p,c)$.

First, we establish that $\mathcal{N}_k$ for all k and large N satisfies Talagrand's inequality [8], which we cite here from the book [5].

Theorem (Talagrand's inequality [5, p. 40])**.** *Suppose that* $Z_1,\ldots,Z_n$ *are independent random variables taking their values in some sets* $\Lambda_1,\ldots,\Lambda_n$, *respectively. Suppose further* *that* $W = f(Z_1, \dots, Z_n)$*, where* $f : \Lambda_1 \times \cdots \times \Lambda_n \to \mathbb{R}$ *is a function such that, for some constants* c_k*,* $k = 1, \dots, n$*, and some function* Ψ*, the following two conditions hold:*

1) *If* $z, z' \in \Lambda = \prod_1^n \Lambda_i$ *differ only in the i-th coordinate, then* $|f(z) - f(z')| \le c_i$*.*
2) *If* $z \in \Lambda$ *and* $r \in \mathbb{R}$ *with* $f(z) \ge r$*, then there exists a set* $J \subseteq \{1, \dots, n\}$ *with* $\sum_{i \in J} c_i^2 \le \Psi(r)$*, such that for all* $y \in \Lambda$ *with* $y_i = z_i$ *when* $i \in J$*, we have* $f(y) \ge r$*.*
   Then, for every $r \in \mathbb{R}$ *and* $t \ge 0$*,*

$$\mathbf{P}(W \le r - t)\mathbf{P}(W \ge r) \le e^{-t^2/4\Psi(r)}. \tag{2.29}$$

Corollary 2.4. *For every* $0 < a \le r$ *and all* $k \ge 1$

$$\mathbf{P}\{\mathcal{N}_k \le a\}\mathbf{P}\{\mathcal{N}_k \ge r\} \le \exp\left\{-\frac{(r-a)^2}{32dkr}\right\}. \tag{2.30}$$

Proof. We shall show that the function $\mathcal{N}_k$ satisfies the conditions of the cited theorem on Talagrand's inequality. Let $\{e_1, \dots, e_n\}$ be the set of all edges from the lattice $\mathbf{Z}^d$, which have both end points in $B(N)$. Define

$$Z_i = \begin{cases} 1, & \text{if } e_i \text{ is open in } G_N^s(p), \\ 0, & \text{if } e_i \text{ is closed in } G_N^s(p). \end{cases}$$

According to the definition of our model, $Z_i \in Be(p)$, $i = 1, \dots, n$, are independent random variables, and

$$\mathcal{N}_k = \mathcal{N}_k(Z_1, \dots, Z_n)$$

since the number of the components of size k (open k-clusters) in $G_N^s(p)$ is defined completely by $Z_1, \dots, Z_n$. Furthermore, it is clear that removing or adding just one edge in $G_N^s(p)$ may increase or decrease by at most 2 the number of k-clusters. Hence, the first condition of Talagrand's inequality is satisfied with $c_i = 2$ for all $1 \le i \le n$: if configurations $z, z' \in \{0, 1\}^n$ differ only in the ith coordinate, then

$$|\mathcal{N}_k(z) - \mathcal{N}_k(z')| \le 2.$$

Next, we check that the second condition is fulfilled as well, and we shall determine the function Ψ. Assume, $z \in \{0, 1\}^n$ corresponds to such a configuration of the edges in $B(N)$ that $\mathcal{N}_k(z) \ge r$, for some $r \in \{1, 2, \dots\}$, i.e., there are at least r clusters of size k. Let $\{e_j, j \in J\} \subset \{e_1, \dots, e_n\}$ be a set of edges, which have at least one common vertex with a set of exactly r (arbitrarily chosen out of $\mathcal{N}_k(z)$) clusters of size k. Clearly, $|J| \le 2dkr$, and for any $z' \in \{0, 1\}^n$ with $z'_j = z_j$ if $j \in J$, we have

$$\mathcal{N}_k(z') \ge r.$$

Hence, the second condition of Talagrand's inequality is satisfied with $\Psi(r) = 8dkr$ for positive integers r, since

$$\sum_{i \in J} c_i^2 = 4|J| \le 8dkr. \tag{2.31}$$

The case when r is not an integer is treated as explained in Example 2.33, page 41, in [5]. Then, inequality (2.30) follows by (2.29). ∎

Now we are able to prove a useful result on the distribution of the entire vector $\mathcal{N}/K_N = (\mathcal{N}_1/K_N, \ldots, \mathcal{N}_{|B(N)|}/K_N)$. Recall that for each fixed k convergence of $\mathcal{N}_k/K_N$ as $N \to \infty$ was proved in Proposition 2.1.

Lemma 2.1. *Assume, $p < p_c$. Set*

$$\tilde{\mu}(k) = \sum_{n=k}^{\infty} \mathbf{P}\{|C| = n\} = \mathbf{P}\{|C| \geq k\}.$$

Then for any fixed $\nu > 2$ and $\varepsilon > 0$

$$\mathbf{P}\{|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon\, k^{\nu}\, \tilde{\mu}(k) \quad \text{for some } 1 \leq k \leq |B(N)|\} \to 0 \tag{2.32}$$

as $N \to \infty$.

Remark 2.2. Because of (1.1) $k^{\nu}\, \tilde{\mu}(k) \to 0$ as $k \to \infty$. Therefore, statement (2.32) does not follow simply by Proposition 2.1.

Proof of Lemma 2.1. Let us fix $\nu > 2$ and $\varepsilon > 0$ arbitrarily. Notice that for any $1 \leq L_0 < L < |B(N)|$, we have

$$\begin{aligned}
&\mathbf{P}\big\{|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon\, k^{\nu}\, \tilde{\mu}(k) \quad \text{for some } 1 \leq k \leq |B(N)|\big\} \\
&\quad \leq \mathbf{P}\big\{|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon\, k^{\nu}\, \tilde{\mu}(k) \quad \text{for some } 1 \leq k \leq L_0\big\} \\
&\qquad + \mathbf{P}\big\{|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon\, k^{\nu}\, \tilde{\mu}(k) \quad \text{for some } L_0 < k \leq L\big\} \\
&\qquad + \mathbf{P}\big\{|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon\, k^{\nu}\, \tilde{\mu}(k) \quad \text{for some } k > L\big\} \\
&\quad =: \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3.
\end{aligned} \tag{2.33}$$

We shall choose later on an appropriate constant L_0 and an increasing function $L = L(N)$ so that we will be able to bound from above by $o(1)$ (as $N \to \infty$) each of three summands on the right-hand side in (2.33). Notice here that the first and the last terms are the easiest to estimate, while the main job concerns $\mathbf{P}_2$. We shall make it in two steps. ∎

Step I: Preliminary bounds of $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$. Consider the first term on the right-hand side in (2.33). Observe that for any fixed constant $L_0 \geq 1$, we have by Proposition 2.1

$$\mathbf{P}_1 = \mathbf{P}\{|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon\, k^{\nu}\, \tilde{\mu}(k) \quad \text{for some } 1 \leq k \leq L_0\} = o(1) \tag{2.34}$$

as $N \to \infty$.

To bound $\mathbf{P}_3$ we first choose a constant L_0 so that

$$\varepsilon L_0^{\nu} \geq \frac{1}{\mathbf{E}(|C|^{-1})}. \tag{2.35}$$

Then for all $k > L_0$

$$\varepsilon\, k^{\nu}\, \tilde{\mu}(k) \geq \frac{\tilde{\mu}(k)}{\mathbf{E}(|C|^{-1})} > \mu(k), \tag{2.36}$$

and hence, inequality

$$|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon\, k^\nu\, \tilde{\mu}(k)$$

may hold only if $\mathcal{N}_k/K_N > 0$. This implies that for any $L > L_0$

$$\mathbf{P}_3 = \mathbf{P}\{|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon\, k^\nu\, \tilde{\mu}(k) \quad \text{for some } k > L\}$$
$$\leq \mathbf{P}\left\{\max_{1\leq i\leq K_N} |X_i| > L\right\}. \tag{2.37}$$

Making use of Proposition 2.2, we get from here

$$\mathbf{P}_3 \leq |B(N)|\,\mathbf{P}\{|C| > L\} \leq |B(N)|\,\tilde{\mu}(L). \tag{2.38}$$

Consider now $\mathbf{P}_2$. Taking into account (2.36), we can write

$$\mathbf{P}_2 = \mathbf{P}\left\{\frac{\mathcal{N}_k}{K_N} - \mu(k) > \varepsilon\, k^\nu\, \tilde{\mu}(k) \quad \text{for some } L_0 < k \leq L\right\}. \tag{2.39}$$

Using the events $\mathcal{A}_{\delta,N}$ from Corollary 2.1, we define

$$P(k) := \mathbf{P}\left\{\left(\frac{\mathcal{N}_k}{K_N} - \mu(k) > \varepsilon\, k^\nu\, \tilde{\mu}(k)\right) \cap \mathcal{A}_{\delta,N}\right\}. \tag{2.40}$$

Then by Corollary 2.1 and (2.39), we have

$$\mathbf{P}_2 \leq \sum_{k=L_0+1}^{L} P(k) + o(1), \tag{2.41}$$

as $N \to \infty$.

Claim 1. *For some $b > 0$*

$$P(k) \leq 4\exp\{-b|B(N)|k^{\nu-1}\,\tilde{\mu}(k)\} \tag{2.42}$$

uniformly in $k > L_0$ and all large N.

Proof of Claim 1. Because of the definition of $\mathcal{A}_{\delta,N}$ in Corollary 2.1, we have

$$P(k) \leq \mathbf{P}\{\mathcal{N}_k > (\kappa(p) - \delta)|B(N)|(\varepsilon\, k^\nu\, \tilde{\mu}(k) + \mu(k))\}. \tag{2.43}$$

Now set

$$k_N = (\kappa(p) - \delta)|B(N)|,$$

and consider Talagrand's inequality (2.30) with

$$r = k_N(\varepsilon\, k^\nu\, \tilde{\mu}(k) + \mu(k)),$$
$$a = k_N\left(\frac{\varepsilon}{2}\, k^\nu\, \tilde{\mu}(k) + \mu(k)\right). \tag{2.44}$$

By Markov's inequality

$$\mathbf{P}\{\mathcal{N}_k \le a\} \ge 1 - \frac{\mathbf{E}\mathcal{N}_k}{a}. \tag{2.45}$$

Now write

$$\mathcal{N}_k = \sum_{i=1}^{K_N} I_k(|X_i|) = \frac{1}{k} \sum_{z\in B(N)} I_k(|C_N(z)|), \tag{2.46}$$

where $C_N(z)$ denotes an open cluster in $B(N)$, which contains the vertex z. Further let $C(z)$ denote an open cluster in $\mathbf{Z}^d$, which contains the vertex z. Then by (2.7) we have that

$$\sum_{z\in B(N)} I_k(|C_N(z)|) \le \sum_{z\in B(N)} I_k(|C(z)|) + \sum_{z\leftrightarrow \partial B(N)} I_k(|C_N(z)|), \tag{2.47}$$

where the last sum contains at most $k|\partial B(N)|$ nonzero terms. Note also that for any $z \leftrightarrow \partial B(N)$

$$\mathbf{P}\{|C_N(z)| = k\} \le \mathbf{P}\{|C(z)| \ge k\} = \mathbf{P}\{|C| \ge k\}.$$

Therefore, we derive from (2.46) and (2.47)

$$\begin{aligned}\mathbf{E}\mathcal{N}_k &\le \frac{1}{k}|B(N)|\mathbf{P}\{|C| = k\} + |\partial B(N)|\mathbf{P}\{|C| \ge k\} \\ &= \kappa(p)\mu(k)|B(N)| + |\partial B(N)|\tilde{\mu}(k).\end{aligned} \tag{2.48}$$

This yields that for any $0 < \delta \le \kappa(p)/18$, all $k > L_0$ (in which case $\varepsilon k^\nu\,\tilde{\mu}(k) \ge \mu(k)$) and all large N

$$\frac{\mathbf{E}\mathcal{N}_k}{a} = \frac{\mathbf{E}\mathcal{N}_k}{(\kappa(p) - \delta)|B(N)|\left(\frac{\varepsilon}{2}\, k^\nu\,\tilde{\mu}(k) + \mu(k)\right)} \le 3/4,$$

which together with (2.45) implies

$$\mathbf{P}\{\mathcal{N}_k \le a\} \ge \frac{1}{4}. \tag{2.49}$$

Using (2.49) in Talagrand's inequality (2.30) with r and a defined in (2.44), we derive for all $k > L_0$

$$\begin{aligned}\mathbf{P}\{\mathcal{N}_k \ge r\} &\le (\mathbf{P}\{\mathcal{N}_k \le a\})^{-1} \exp\left\{-\frac{(r-a)^2}{32dkr}\right\} \\ &\le 4\exp\left\{-\frac{\left(\frac{\varepsilon}{2}k_N\, k^\nu\,\tilde{\mu}(k)\right)^2}{32dk(k_N(\varepsilon\, k^\nu\,\tilde{\mu}(k) + \mu(k)))}\right\} \\ &\le 4\exp\left\{-\frac{\varepsilon k_N\, k^\nu\,\tilde{\mu}(k)}{2^8 dk}\right\} = 4\exp\left\{-\frac{\varepsilon(\kappa(p)-\delta)}{2^8 d}|B(N)|k^{\nu-1}\,\tilde{\mu}(k)\right\}.\end{aligned} \tag{2.50}$$

Substituting (2.50) into (2.43) we get the bound (2.42), where

$$b := \frac{\varepsilon(\kappa(p)-\delta)}{2^8 d}.$$

This finishes the proof of the claim. ∎

Step II: Final bound of $\mathbf{P_1} + \mathbf{P_2} + \mathbf{P_3}$. Combining all the preliminary bounds (2.34), (2.38), and (2.41) together with (2.42), we obtain

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 \leq o(1) + 4 \sum_{k=L_0}^{L} \exp\left\{-b|B(N)|k^{\nu-1}\,\tilde{\mu}(k)\right\} + |B(N)|\,\tilde{\mu}(L), \tag{2.51}$$

as $N \to \infty$ for any $L \geq L_0$.

From now on choose

$$L = L(N) = \min\left\{k \geq 1 : k^{\alpha}\tilde{\mu}(k) < \frac{1}{|B(N)|}\right\}, \tag{2.52}$$

where

$$\alpha = \frac{\nu - 2}{2},$$

which is positive by the assumption of Lemma 2.1.

Claim 2. *For any $\Delta > 0$ one can choose a finite constant L_0 so that*

$$\sum_{k=L_0}^{L(N)} \exp\{-b|B(N)|k^{\nu-1}\,\tilde{\mu}(k)\} < \Delta, \tag{2.53}$$

for all large N, and

$$|B(N)|\,\tilde{\mu}(L(N)) \to 0, \tag{2.54}$$

as $N \to \infty$.

Before we proceed with the proof of our claim, we note that this claim together with (2.51) implies that

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = o(1)$$

as $N \to \infty$, which by (2.33) yields the statement of Lemma 2.1.

Proof of Claim 2. By the result (1.1) on the exponential tail of the distribution of the open cluster, we have $k^{\alpha}\tilde{\mu}(k) \to 0$, as $k \to \infty$ for any fixed α, but $k^{\alpha}\tilde{\mu}(k) > 0$ for all $k \geq 1$. This yields that $L(N) \to \infty$ as $N \to \infty$, which in turn implies that there exists

$$\lim_{N\to\infty} \tilde{\mu}(L(N))|B(N)| \leq \lim_{N\to\infty} L(N)^{-\alpha} = 0,$$

and (2.54) follows.

To prove (2.53) we note first that by the definition (2.52) of $L(N)$

$$(L(N) - 1)^{\alpha}\tilde{\mu}(L(N) - 1) \geq \frac{1}{|B(N)|}. \tag{2.55}$$

Recall that by Lemma 6.102 from [3] (p. 139), for all $n, m \geq 0$

$$\frac{1}{m+n}\mathbf{P}(|C| = n + m) \geq p(1-p)^{-2}\frac{1}{m}\mathbf{P}(|C| = m)\frac{1}{n}\mathbf{P}(|C| = n). \tag{2.56}$$

When $m = 1$ inequality (2.56) implies that

$$\mathbf{P}(|C| = n + 1) \geq p(1 - p)^{2(d-1)}\mathbf{P}(|C| = n), \tag{2.57}$$

for all $n \geq 0$. This clearly yields

$$\tilde{\mu}(n + 1) \geq p(1 - p)^{2(d-1)}\tilde{\mu}(n)$$

for all $n \geq 0$, and in particular

$$\tilde{\mu}(L(N)) \geq p(1 - p)^{2(d-1)}\tilde{\mu}(L(N) - 1). \tag{2.58}$$

Notice that $\gamma := p(1 - p)^{2(d-1)} \leq p < 1$ for all $d \geq 1$. Combining (2.55) with (2.58), we immediately get

$$L(N)^{\alpha}\tilde{\mu}(L(N)) \geq \frac{\gamma}{|B(N)|}, \tag{2.59}$$

and also by definition (2.52) for all $k < L(N)$

$$k^{\alpha}\tilde{\mu}(k) \geq \frac{1}{|B(N)|} \geq \frac{\gamma}{|B(N)|}. \tag{2.60}$$

Making use of (2.59) and (2.60), we derive

$$\sum_{k=L_0}^{L(N)} \exp\left\{-b|B(N)|k^{\nu-1}\,\tilde{\mu}(k)\right\} \leq \sum_{k=L_0}^{L(N)} \exp\left\{-b\gamma k^{\nu-1-\alpha}\right\} \leq a_1 \exp\left\{-b\gamma L_0^{\nu-2-\alpha}\right\}, \tag{2.61}$$

where a_1 is some positive constant independent of L_0. Hence, for any $\Delta > 0$ we can fix L_0 so large that (2.61) implies (2.53), and in the same time L_0 satisfies (2.35) and $L_0 < L(N)$. This completes the proof of Claim 2 and, therefore, finishes the proof of Lemma 2.1. ∎

Corollary 2.5. *For any fixed $\nu > 2$, $\varepsilon > 0$ and $\delta > 0$ define an event*

$$\mathcal{B}_N = \mathcal{A}_{\delta,N} \cap \left(\max_{1\leq i\leq K_N} |X_i| \leq \frac{2}{\zeta(p)} \log |B(N)|\right) \cap \left(\cap_{k=1}^{|B(N)|} \left\{\left|\frac{\mathcal{N}_k}{K_N} - \mu(k)\right| \leq \varepsilon k^{\nu}\tilde{\mu}(k)\right\}\right). \tag{2.62}$$

Then

$$\mathbf{P}\{\mathcal{B}_N\} = 1 - o(1) \tag{2.63}$$

as $N \to \infty$.

Proof. Statement (2.63) follows immediately by Corollary 2.1, Proposition 2.2, and Lemma 2.1. ∎

3. PROOF OF THEOREM 1.1 IN THE SUBCRITICAL CASE $C < C^{CR}(P)$

Let us fix $0 \le p < p_c$ and then $c < c^{cr}(p)$ arbitrarily. First, given $\mathbf{X} = (X_1, \dots, X_{K_N})$ consider the graph $\tilde{G}_N(\mathbf{X}, p, c)$ introduced in Section 2.2. Recall that we denote by X_i the vertices (of types $|X_i|$) of this graph. Let $\tilde{L}$ denote a connected component in $\tilde{G}_N(\mathbf{X}, p, c)$. Observe that given $\mathbf{X} = (X_1, \dots, X_{K_N})$, the size of a connected component in $G_N(p, c)$ equals the sum of the types of the vertices in some component of $\tilde{G}_N(\mathbf{X}, p, c)$. Therefore, we have

$$C_1(G_N(p,c))|_{\mathbf{X}} = \max_{\tilde{L}} \sum_{X_i \in \tilde{L}} |X_i|,$$

where the maximum is taken over all components of $\tilde{G}_N(\mathbf{X}, p, c)$. Using the exploration algorithm defined in Section 2.3, we can rewrite the last formula as follows:

$$C_1(G_N(p,c))|_{\mathbf{X}} = \max_{\tilde{L}} \sum_{X_i \in \tilde{L}} |X_i| = \max_{1 \le i \le K_N} \sum_{X_j \in \tau_N(X_i)} |X_j|, \tag{3.1}$$

where $\tau_N(X_i)$ denotes the set of macrovertices in a component revealed by the exploration algorithm starting with vertex X_i. Set

$$S_N(X_i) = \sum_{X_j \in \tau_N(X_i)} |X_j|$$

to be the number of vertices from $B(N)$, which compose the macrovertices of $\tau_N(X_i)$. Then by (3.1) we have for any positive w

$$\mathbf{P}\{C_1(G_N(p,c)) > w\} = \mathbf{P}\left\{ \max_{1 \le i \le K_N} S_N(X_i) > w \right\}. \tag{3.2}$$

Our aim is to show that if $w = \alpha \log |B(N)|$ this probability tends to zero as $N \to \infty$, and then the statement (1.4) of Theorem 1.1 follows.

Observe that the distribution of $S_N(X_i)$ depends only on $|X_i|$, i.e.,

$$S_N(X_i)|_{|X_i|=k} =_d S_N(X_j)|_{|X_j|=k}$$

for all $k \ge 1$. Therefore, we shall use notation

$$S_N(k) :=_d S_N(X_i)|_{|X_i|=k}.$$

By (3.2) and Corollary 2.5, we have for all $w > 0$

$$\begin{aligned} \mathbf{P}\{C_1(G_N(p,c)) > w\} &= \mathbf{P}\left\{ \max_{1 \le i \le K_N} S_N(X_i) > w \mid \mathcal{B}_N \right\} + o(1) \\ &\le |B(N)|(\delta + \mathbf{E}(|C|^{-1})) \sum_{k=1}^{|B(N)|} (\mu(k) + \varepsilon k^{\nu} \tilde{\mu}(k)) \mathbf{P}\{S_N(k) > w \mid \mathcal{B}_N\} + o(1) \end{aligned} \tag{3.3}$$

as $N \to \infty$.

3.1. Approximation by Branching Processes

We shall approximate $S_N(k)$ using the branching process introduced in Section 2.3. Let $S^{c,p}(y)$ denote the sum of types including the one of the initial particle, in the total progeny of the branching process $B_y^{c,p}$ starting with initial particle of type y.

Proposition 3.1. *For any $c' > c$ and $p' > p$ such that*

$$c < c' < c^{cr}(p') < c^{cr}(p),$$

one has

$$\mathbf{P}\{S_N(k) > w \mid \mathcal{B}_N\} \le \mathbf{P}\{S^{c',p'}(k) > w\} \tag{3.4}$$

for all large N.

Proof. Observe that at each step of the exploration algorithm, the number of the type y offspring of a particle of type x has a binomial distribution $\mathrm{Bin}(N'_y, \tilde{p}_{xy}(N))$, where N'_y is the number of remaining vertices of type y. In particular, $N'_y \le \mathcal{N}_y$.

We shall explore the well-known fact that a binomial $\mathrm{Bin}(n,p)$ distribution is stochastically dominated by a Poisson distribution $\mathrm{Po}(-n\log(1-p))$; denote this by

$$\mathrm{Bin}(n,p) \prec \mathrm{Po}(-n\log(1-p)).$$

Hence, we have

$$\mathrm{Bin}(N'_y, \tilde{p}_{xy}(N)) \prec \mathrm{Po}\big(-N'_y\log(1-\tilde{p}_{xy}(N))\big). \tag{3.5}$$

Consider the parameter of the last distribution. ∎

Claim. *Conditionally on $\mathcal{B}_N$ for any $x, y \in \{X_1, \ldots, X_{K_N}\}$ we have*

$$-N'_y\log(1-\tilde{p}_{xy}(N)) \le \mu_{p'}(y)\mathcal{K}_{c',p'}(x,y) \tag{3.6}$$

for all large N, where c', p' satisfy the conditions of Proposition 3.1.

Before we prove (3.6), let us derive the statement of Proposition 3.1 from here. Bound (3.6) (and properties of the Poisson distribution) implies that

$$\mathrm{Po}\big(-N'_y\log(1-\tilde{p}_{xy}(N))\big) \prec \mathrm{Po}(\mu_{p'}(y)\mathcal{K}_{c',p'}(x,y)),$$

which together with (3.5) yields

$$\mathrm{Bin}(N'_y, \tilde{p}_{xy}(N)) \prec \mathrm{Po}(\mu_{p'}(y)\mathcal{K}_{c',p'}(x,y)).$$

Hence, $\mathrm{Bin}(N'_y, \tilde{p}_{xy}(N))$, which is the distribution of the offspring in the exploration algorithm, is stochastically dominated by $\mathrm{Po}(\mu_{p'}(y)\mathcal{K}_{c',p'}(x,y))$, which is the distribution of offspring in the branching process $B_y^{c,p}$. Therefore, $S_N(k)$ is also stochastically dominated by $S^{c',p'}(k)$ for any $k \ge 1$, and the statement of Proposition 3.1 follows.

Proof of the Claim. Notice that for all $1 \le x, y \le \frac{2}{\zeta(p)} \log |B(N)|$

$$\tilde{p}_{xy}(N) = 1 - \left(1 - \frac{c}{|B(N)|}\right)^{xy} = \frac{c}{|B(N)|}xy(1 + o(1)). \tag{3.7}$$

Therefore, for any fixed positive ε_1, we can choose small ε and δ in (2.62) so that conditionally on $\mathcal{B}_N$ we have

$$-N'_y \log(1 - \tilde{p}_{xy}(N)) \le -\mathcal{N}_y \log(1 - \tilde{p}_{xy}(N)) \le (\mu(y) + y^{\nu}\varepsilon_1\tilde{\mu}(y))\mathcal{K}(x, y)(1 + o(1)) \tag{3.8}$$

for all large N.

Let us write further

$$\mu(y) = \mu_p(y) = \frac{1}{\kappa(p)}\frac{\mathbf{P}_p\{|C| = y\}}{y}, \quad \tilde{\mu}(y) = \tilde{\mu}_p(y), \quad \mathcal{K}(x, y) = \mathcal{K}_{c,p}(x, y) = c\kappa(p)xy,$$

emphasizing dependence on p and c. Recall that along with the result (1.1), it is also proved in [3] that for all $0 < p < p_c$

$$\zeta(p) = \lim_{n\to\infty}\left(-\frac{1}{n}\log \mathbf{P}\{|C| \ge n\}\right). \tag{3.9}$$

Then (1.1) and (3.9) immediately imply the existence and equality of the following limits for all $0 < p < p_c$

$$\zeta(p) = \lim_{n\to\infty}\left(-\frac{1}{n}\log \mu(n)\right) = \lim_{n\to\infty}\left(-\frac{1}{n}\log \tilde{\mu}(n)\right), \tag{3.10}$$

i.e., that both $\mu(n)$ and $\tilde{\mu}(n)$ decay exponentially fast, and moreover with the same exponent in the limit. This allows us to find for any $p < p' < p_c$ positive constants ε_2 and $\varepsilon_1 = \varepsilon_1(\varepsilon_2, p')$ such that

$$\mu_p(y) + y^{\nu}\varepsilon_1\tilde{\mu}_p(y) \le (1 + \varepsilon_2)\mu_{p'}(y), \tag{3.11}$$

and moreover $\varepsilon_2 \downarrow 0$ as $p' \downarrow p$. Setting now

$$c' = \left(1 + \frac{3}{2}\varepsilon_2\right)\frac{\kappa(p)}{\kappa(p')}c$$

we derive from (3.8) with the help of (3.11) that conditionally on $\mathcal{B}_N$ with an appropriate choice of constants

$$-N'_y \log(1 - \tilde{p}_{xy}(N)) \le \left(1 + \frac{3}{2}\varepsilon_2\right)\mu_{p'}(y)\mathcal{K}_{c,p}(x, y) = \mu_{p'}(y)\mathcal{K}_{c',p'}(x, y) \tag{3.12}$$

for all large N. Recall that above we fixed p and $c < c^{cr}(p)$, where $c^{cr}(p)$ is strictly decreasing and continuous in p. Furthermore, the function $\kappa(p)$ is analytic on $[0, p_c)$. Hence, we can choose $p' > p$ and $c' = (1 + \frac{3}{2}\varepsilon_2)\frac{\kappa(p)}{\kappa(p')}c$ so that

$$c < c' < c^{cr}(p') < c^{cr}(p), \tag{3.13}$$

and moreover c' and p' can be chosen arbitrarily close to c and p, respectively. Hence, bound (3.6) follows, and this finishes the proof of the Proposition. ∎

3.2. Proof of (1.4)

Using Proposition 3.1 and bound (3.11) in (3.3), we derive for all c', p', which satisfy conditions of Proposition 3.1 that

$$\mathbf{P}\{C_1(G_N(p,c)) > w\} \le b|B(N)| \sum_{k=1}^{|B(N)|} k\mu_{p'}(k)\mathbf{P}\{S^{c',p'}(k) > w\} + o(1) \tag{3.14}$$

as $N \to \infty$, where b is some positive constant. Now, using Markov's inequality for a non-negative random variable X

$$\mathbf{P}\{X > w\} \le z^{-w}\mathbf{E}z^X,$$

we derive from (3.14) with $w = a\log|B(N)|$ that for any $z \ge 1$

$$\mathbf{P}\{C_1(G_N(p,c)) > a\log|B(N)|\} \le b|B(N)|z^{-a\log|B(N)|}A_z(c',p') + o(1), \tag{3.15}$$

where

$$A_z(c,p) = \sum_{k=1}^{\infty} k\mu_p(k)\mathbf{E}z^{S^{c,p}(k)}. \tag{3.16}$$

It is clear that if $A_z(c',p') < \infty$ for some $z \ge 1$, then for any $a > (\log z)^{-1}$ the right-hand part of (3.15) tends to zero as $N \to \infty$. Therefore, we search for all $z \ge 1$ for which the series $A_z(c,p)$ converge.

Proposition 3.2. *Let* $0 \le p < p_c$ *and* $c < c^{cr}(p)$. *Then,* $A_z(c,p) < \infty$ *for all*

$$1 < z \le z_0(c,p) := \exp\left\{c\left(1 + y_0 - \mathbf{E}\,e^{c|C|y_0}\right)\right\},$$

where

$$y_0 = \begin{cases} \text{the root of } \mathbf{E}\,c|C|e^{|C|x} = 1, & \text{if } \mathbf{E}\,c|C|e^{|C|\zeta(p)} \ge 1, \\ \zeta(p)/c, & \text{otherwise.} \end{cases}$$

Before we proceed with the proof of this proposition, let us derive the statement (1.4) of Theorem 1.1 from (3.15) and Proposition 3.2. By Proposition 3.2 for any $1 < z \le z_0(c',p')$ we have $A_z(c',p') < \infty$ in (3.15). Therefore, if

$$a > (\log z)^{-1} \ge (\log z_0(c',p'))^{-1} > 0, \tag{3.17}$$

then the right-hand part of (3.15) tends to zero as $N \to \infty$. Note that function $z_0(c,p)$ is continuous at any (c,p), such that $0 \le p < p_c$ and $c < c^{cr}(p)$. Hence, for any

$$a > (\log z_0(c,p))^{-1} = \left(c + cy_0 - c\mathbf{E}\,e^{c|C|y_0}\right)^{-1}$$

we can choose c' and p' (satisfying the conditions of Proposition 3.1) so that also

$$a > (\log z_0(c',p'))^{-1} > 0.$$

With these c' and p', and with $z = z_0(c',p')$ condition (3.17) is satisfied, and therefore the right-hand part of (3.15) is $o(1)$. This proves (1.4).

Proof of Proposition 3.2. We shall study first function $g_z(k) = \mathbf{E}z^{S^{c,p}(k)}$. (Refer, e.g., a book [7] on the theory of multitype branching process.) Recall that $S^{c,p}(k)$ denotes the sum of types in the total progeny of the branching process $B_k^{c,p}$ starting with initial particle of type k. To simplify further notations, we shall omit here indices c, p. Let $S_n(k) = S_n^{c,p}(k)$ denote for all $n \geq 0$ the sum of types of the particles in the first n generations. In particular, $S_0(k) = k$. It is clear that

$$S_n(k) \uparrow S^{c,p}(k)$$

as $n \to \infty$. By the definition of branching process $B_k^{c,p}$, we have

$$S_1(k) =_d k + \sum_{x=1}^{\infty} x N_{kx}, \tag{3.18}$$

where N_{kx} is the number of the offspring of type x. By the definition of the process $B_k^{c,p}$ random variables $N_{kx} \in Po(\mathcal{K}_{c,p}(k,x)\mu_p(x))$ are independent for different x, k. It also follows that

$$S_{n+1}(k) =_d k + \sum_{x=1}^{\infty} \sum_{i=1}^{N_{kx}} S_n^i(x) \tag{3.19}$$

for all $n \geq 0$, where $S_n^i(x)$, $i = 1, \ldots,$ are independent copies of $S_n(x)$, independent for different x as well.

Let

$$g_{n,z}(k) = \mathbf{E}z^{S_n(k)}$$

for $n \geq 0$. The generating function for a Poisson random variable $Y \in Po(m)$ is

$$\phi_Y(z) = \mathbf{E}z^Y = e^{m(z-1)}.$$

Using this formula, it is straightforward to derive from (3.18) the generating function of a linear combination of independent Poisson random variables

$$g_{1,z}(k) = \mathbf{E}z^{S_1(k)} = z^k \prod_x \phi_{N_{kx}}(z^x) = z^k \exp\left\{\sum_{x=1}^{\infty} \mathcal{K}_{c,p}(k,x)\mu_p(x)(z^x - 1)\right\}. \tag{3.20}$$

Since

$$g_{0,z}(k) = z^k,$$

we can rewrite (3.20) as

$$g_{1,z}(k) = z^k \exp\left\{\sum_{x=1}^{\infty} \mathcal{K}_{c,p}(k,x)\mu_p(x)(g_{0,z}(x) - 1)\right\} =: \Phi_z[g_{0,z}](k).$$

In the same fashion, we derive from (3.19)

$$g_{n+1,z}(k) = \Phi_z[g_{n,z}](k) \tag{3.21}$$

for all $n \geq 0$. Observing that

$$\Phi_z[1](k) = z^k = g_{0,z}(k),$$

we get from (3.21) the following formula

$$g_{n,z}(k) = \Phi_z^{n+1}[1](k) \tag{3.22}$$

for all $n \geq 0$ and $k \geq 1$.

Lemma 3.1. *For any $z \geq 1$ function $g_z(k) = \mathbf{E} z^{S^{c,p}(k)}$ is the minimal solution $f \geq 1$ to the equation*

$$f = \Phi_z[f]. \tag{3.23}$$

Proof. Since $S_n(k) \uparrow S^{c,p}(k)$, by the Monotone Convergence Theorem we have for any $z \geq 1$

$$g_z(x) = \lim_{n\to\infty} g_{n,z}(x) = \lim_{n\to\infty} \Phi_z^n[1](x)$$

for all x, where the last equality is due to (3.22). Again using monotone convergence, we derive

$$\Phi_z[g_z](x) = \Phi_z\Big[\lim_{n\to\infty} \Phi_z^n[1]\Big](x) = \lim_{n\to\infty} \Phi_z^{n+1}[1](x) = g_z(x). \tag{3.24}$$

Hence, $g_z = \lim_{n\to\infty} \Phi_z^n[1]$ is a solution to (3.23), and by the definition $g_z(k) = \mathbf{E} z^{S^{c,p}(k)} \geq 1$ for all $z \geq 1$ and $x \geq 1$.

Let us show that g_z is the minimal solution $f \geq 1$ to (3.23). Assume that there is a solution $f \geq 1$ such that $1 \leq f(x) < g_z(x)$ at least for some $x \geq 1$. Then, because of the monotonicity of Φ_z, we have also

$$\Phi_z^k[1](x) \leq \Phi_z^k[f](x) = f(x) < g_z(x) = \lim_{n\to\infty} \Phi_z^n[1](x)$$

for all $k \geq 1$. Letting $k \to \infty$ in the last formula, we come to the contradiction with the strict inequality in the middle. Therefore, g_z is the minimal solution $f \geq 1$ to (3.23). ∎

By Lemma 3.1 function g_z is the minimal solution to

$$g_z(k) = z^k \exp\left\{\sum_{x=1}^{\infty} \mathcal{K}_{c,p}(k,x)\mu_p(x)(g_z(x) - 1)\right\}. \tag{3.25}$$

Recall that $\mathcal{K}_{c,p}(k,x) = \kappa(p)ckx$ and by the definition in (2.5)

$$\mu_p(k) = \frac{1}{\kappa(p)} \frac{\mathbf{P}\{|C| = k\}}{k}.$$

Hence,

$$\sum_{x=1}^{\infty} \mathcal{K}_{c,p}(k,x)\mu_p(x) = \sum_{x=1}^{\infty} \kappa(p)ckx \frac{1}{\kappa(p)} \frac{\mathbf{P}\{|C| = x\}}{x} = ck,$$

and we can rewrite (3.25) as

$$g_z(k) = z^k \exp\left\{ck\left(\sum_{x=1}^{\infty} x\mu_p(x)g_z(x) - 1\right)\right\}. \tag{3.26}$$

More precisely, g_z is the minimal solution to (3.26). This implies (multiply both sides of (3.26) by $k\mu_p(k)$ and sum over k) that

$$A_z = A_z(c,p) = \sum_{x=1}^{\infty} x\mu_p(x)g_z(x)$$

is the minimal solution to the equation

$$A_z = \sum_{k=1}^{\infty} k\mu_p(k)z^k \exp\{ck(\kappa(p)A_z - 1)\}, \tag{3.27}$$

or, equivalently, to the equation

$$A_z = (\kappa(p))^{-1}\mathbf{E}(z^{|C|}\, e^{c|C|(\kappa(p)A_z-1)}). \tag{3.28}$$

Hence, to establish Proposition 3.2, we are left to prove that Eq. (3.28) has a finite solution for any $z \le z_0$ and that $z_0 > 1$.

Notice that by the definition (3.16)

$$A_z \ge A_1 = \sum_{k=1}^{\infty} k\mu_p(k) = \sum_{k=1}^{\infty} k\frac{1}{\kappa(p)}\frac{\mathbf{P}\{|C| = k\}}{k} = \frac{1}{\kappa(p)}.$$

Therefore, Eq. (3.28) after the change $y = \kappa(p)A_z$ becomes

$$y = \mathbf{E}(z^{|C|}\, e^{c|C|(y-1)}) \tag{3.29}$$

for $y > 1$. Note that since the distribution of $|C|$ decays exponentially, at least for some $y > 1$ and $z > 1$ the function on the right in (3.29)

$$f(y,z) := \mathbf{E}(z^{|C|}\, e^{c|C|(y-1)})$$

is finite. Also, (wherever defined) it has all the derivatives of the second order, and $\frac{\partial^2}{\partial y^2}f(y,z) > 0$. Now compute

$$\frac{\partial}{\partial y}f(y,z)|_{y=1,z=1} = c\mathbf{E}|C| = \frac{c}{c^{cr}}. \tag{3.30}$$

Hence, if $c > c^{cr}$ there is no solution $y \ge 1$ to (3.29) for any $z > 1$. On the other hand, if $c < c^{cr}$ there exists $1 < z_0 < e^{\zeta(p)}$ such that for all $1 \le z \le z_0$ there is a finite solution $y \ge 1$ to (3.29).

Let us rewrite (3.29) as follows. Set

$$a = \frac{1}{c}\log z.$$

Then (3.29) is equivalent to

$$y = \mathbf{E}\, e^{c|C|(y-1+a)}, \tag{3.31}$$

which after the change $x = y - 1 + a$ becomes

$$x + 1 - a = \mathbf{E}\, e^{c|C|x}. \tag{3.32}$$

Here on the right-hand side, we have a convex function with positive second derivative (for all $x < \zeta(p)/c$). Hence, the function

$$\frac{\partial}{\partial x}\mathbf{E}\, e^{c|C|x} = \mathbf{E}\,(c|C|e^{c|C|x})$$

is strictly increasing in x when $x < \zeta(p)/c$ and continuous on the left at $x = \zeta(p)/c$. As $c < c^{cr} = 1/\mathbf{E}|C|$ we have

$$\frac{\partial}{\partial x}\mathbf{E}\, e^{c|C|x}|_{x=0} = \mathbf{E}\, c|C| < 1. \tag{3.33}$$

Therefore, if

$$\frac{\partial}{\partial x}\mathbf{E}\, e^{c|C|x}|_{x=\zeta(p)/c} = \mathbf{E}\, c|C|e^{|C|\zeta(p)} \geq 1,$$

then there exists unique $0 < x_0 \leq \zeta(p)/c$ such that

$$\frac{\partial}{\partial x}\mathbf{E}\, e^{c|C|x}|_{x=x_0} = \mathbf{E}\,(c|C|e^{c|C|x_0}) = 1. \tag{3.34}$$

Now define

$$y_0 = \begin{cases} x_0, & \text{if } \mathbf{E}\, c|C|e^{|C|\zeta(p)} \geq 1, \\ \zeta(p)/c, & \text{otherwise,} \end{cases}$$

and set

$$a_0 := 1 + y_0 - \mathbf{E}\, e^{c|C|y_0}. \tag{3.35}$$

Let us show that a_0 is strictly positive. Consider function $\mathbf{E}\, e^{c|C|x}$. It is convex on $x \geq 0$, and by (3.33) its first derivative is less than 1 for all $0 \leq x < y_0$. Hence, $\mathbf{E}\, e^{c|C|x} < 1 + x$ for all $0 \leq x \leq y_0$. This implies that

$$a_0 > 0. \tag{3.36}$$

Notice that by the construction function $1 + x - a_0$ is tangent to $\mathbf{E}\, e^{c|C|x}$ with the tangency point at $x = y_0$. Hence, for all $a \leq a_0$ Eq. (3.32) has at least one solution. This implies due to (3.31) and (3.36) that for all

$$1 < z \leq z_0 := e^{ca_0} = \exp\{c(1 + y_0 - \mathbf{E}\, e^{c|C|y_0})\}, \tag{3.37}$$

Eq. (3.29) has also at least one finite solution $y > 1$, which yields in turn that A_z is finite for all $1 < z \leq z_0$. This finishes the proof of Proposition 3.2 and completes the proof of statement (1.4) of Theorem 1.1. ∎

To conclude this section, we comment on the methods used here. It is shown in [9] that in the subcritical case of the classical random graph model $G_{n,c/n}$ (i.e., $p = 0$ in terms of our model) the same method of generating functions leads to a constant, which is exactly $\alpha(0, c)$ [see (1.9)]. The last constant is known to be the principal term for the asymptotics of the size of the largest component (scaled to $\log n$) in the subcritical case. This gives us hope that the constant $\alpha(p, c)$ is close to the optimal one also for $p > 0$.

Similar methods were used in [10] for some class of inhomogeneous random graphs, and in [1] for a general class of models. Note, however, some difference with the approach in [1]. It is assumed in [1], Section 12, that the generating function for the corresponding branching process with the initial state k (e.g., our function $g_z(k)$, $k \geq 1$) is bounded uniformly in k. As we prove here, this condition is not always necessary: we need only convergence of the series A_z, whereas $g_z(k)$ is unbounded in k in our case. Furthermore, our approach allows one to construct constant $\alpha(p, c)$ as a function of the parameters of the model.

4. PROOF OF THEOREM 1.1 IN THE SUPERCRITICAL CASE

Let $\mathcal{C}_k$ denote the set of vertices in the k-th largest component in the graph $G_N(p, c)$, and conditionally on $\mathbf{X}$ let $\tilde{\mathcal{C}}_k$ denote the set of macrovertices in the k-th largest component in the graph $\tilde{G}_N(\mathbf{X}, p, c)$ (ordered in any way if there are ties). Let also $C_k = |\mathcal{C}_k|$ and $\tilde{C}_k = |\tilde{\mathcal{C}}_k|$ denote correspondingly, their sizes.

According to our construction for any connected component $\tilde{L}$ in $\tilde{G}_N(\mathbf{X}, p, c)$, there is a unique component L in $G_N(p, c)$ such that $\tilde{L}$ and L are composed of the same vertices from $B(N)$, i.e., in the notations (2.13)

$$L = \cup_{X_i \in \tilde{L}} \cup_{z \in X_i} \{z\} =: V(\tilde{L}). \tag{4.38}$$

First, we shall prove that with a high probability the largest components in both graphs consist of the same vertices.

Lemma 4.1. *For any $0 \leq p < p_c$ if $c > c^{cr}(p)$ then*

$$\mathbf{P}\{\mathcal{C}_1 = V(\tilde{\mathcal{C}}_1)\} = 1 - o(1) \tag{4.39}$$

as $N \to \infty$.

Proof. By (4.38) there exists some component $\tilde{\mathcal{C}}_k$ with $k \geq 1$ such that $\mathcal{C}_1 = \tilde{\mathcal{C}}_k$. Hence,

$$\mathbf{P}\{\mathcal{C}_1 \neq V(\tilde{\mathcal{C}}_1)\} = \mathbf{P}\{\mathcal{C}_1 = V(\tilde{\mathcal{C}}_k) \text{ for some } k \geq 2\}.$$

We know already from Remark 2.1 that in the supercritical case $\tilde{C}_1 = \Theta(|B(N)|)$ **whp**, and therefore $C_1 = \Theta(|B(N)|)$ **whp**. Also, by Corollary 2.3, we have $\tilde{C}_2 = O(\log |B(N)|)$ **whp**. Hence, for some positive constants a and b

$$\begin{aligned}\mathbf{P}\{\mathcal{C}_1 \neq V(\tilde{\mathcal{C}}_1)\} &= \mathbf{P}\{\mathcal{C}_1 = V(\tilde{\mathcal{C}}_k) \text{ for some } k \geq 2\} \\ &\leq \mathbf{P}\left\{\left(\max_{k\geq 2} |V(\tilde{\mathcal{C}}_k)| > b|B(N)|\right) \cap \left(\max_{k\geq 2} \tilde{C}_k < a \log |B(N)|\right)\right\} + o(1).\end{aligned} \tag{4.40}$$

It follows from Proposition 2.2 (part (2.12)) that

$$\mathbf{P}\left\{\max_{1\le i\le K_N}|X_i|\ge\sqrt{|B(N)|}\right\}=o(1)$$

as $N\to\infty$. Now, we derive

$$\begin{aligned}
&\mathbf{P}\left\{\left(\max_{k\ge2}|V(\tilde{\mathcal{C}}_k)|>b|B(N)|\right)\cap\left(\max_{k\ge2}\tilde{C}_k<a\log|B(N)|\right)\right\}\\
&\le\mathbf{P}\left\{\left(\max_{k\ge2}|V(\tilde{\mathcal{C}}_k)|>b|B(N)|\right)\cap\left(\max_{k\ge2}\tilde{C}_k<a\log|B(N)|\right)\cap\left(\max_{1\le i\le K_N}|X_i|<\sqrt{|B(N)|}\right)\right\}\\
&\qquad+o(1)\ \le\mathbf{P}\left\{\sqrt{|B(N)|}\,a\log|B(N)|>b|B(N)|\right\}+o(1)=o(1).
\end{aligned}\tag{4.41}$$

Substituting this bound into (4.40) we immediately get (4.39). ∎

Now consider

$$C_1=C_1(G_N(p,c))=|\mathcal{C}_1|$$

conditionally on event $\mathcal{C}_1=V(\tilde{\mathcal{C}}_1)=\cup_{X_i\in\tilde{\mathcal{C}}_1}\cup_{z\in X_i}\{z\}$, which by Lemma 4.1 holds **whp**. We derive

$$\begin{aligned}
C_1|_{\mathcal{C}_1=V(\tilde{\mathcal{C}}_1)}&=\sum_{i=1}^{K_N}|X_i|\mathbf{1}\{X_i\in\tilde{\mathcal{C}}_1\}\\
&=\sum_{i=1}^{K_N}\sum_{k=1}^{|B(N)|}k\mathbf{1}\{|X_i|=k\}\mathbf{1}\{X_i\in\tilde{\mathcal{C}}_1\}\\
&=K_N\sum_{k=1}^{|B(N)|}k\frac{1}{K_N}\#\{X_i\in\tilde{\mathcal{C}}_1:|X_i|=k\}.
\end{aligned}\tag{4.42}$$

Let us denote

$$\nu_N(k):=\frac{1}{K_N}\#\{X_i\in\tilde{\mathcal{C}}_1:|X_i|=k\}.$$

Then, Lemma 4.1 together with (4.42) yields

$$\mathbf{P}\left\{C_1=K_N\sum_{k=1}^{|B(N)|}k\nu_N(k)\right\}=1-o(1)\tag{4.43}$$

as $N\to\infty$.

The rest of the proof of part (*ii*) of Theorem 1.1 will follow by the convergence of the sum

$$W_N:=\sum_{k=1}^{|B(N)|}k\nu_N(k),$$

which we shall establish later.

Proposition 4.1. (*i*) *For all* $p < p_c(d)$ *and* $c > c^{cr}(p)$

$$W_N \xrightarrow{P} \sum_{k=1}^{\infty} k\rho(k)\mu(k) =: \beta\,(\mathbf{E}(|C|^{-1}))^{-1} \tag{4.44}$$

as $N \to \infty$.

(*ii*) *Constant* β *defined in (4.44) is also the maximal solution to*

$$\beta = 1 - \mathbf{E}(e^{-c|C|\beta}).$$

Before we proceed with the proof, let us show that Proposition 4.1 and formula (4.43) imply part (*ii*) of Theorem 1.1. Indeed, for any positive ε

$$\mathbf{P}\left\{\left|\frac{C_1(G_N(p,c))}{|B(N)|} - \beta\right| > \varepsilon\right\} = \mathbf{P}\left\{\left|\frac{K_N}{|B(N)|}W_N - \beta\right| > \varepsilon\right\} + o(1) \to 0,$$

as $N \to \infty$, where the first equality is by (4.43), and the last convergence is by Proposition 4.1 and result (2.2). Hence, the statement (1.5) follows from here, whereas (1.6) is given directly by Proposition 4.1 part (*ii*). Therefore, we are left only with the proof of Proposition 4.1.

The essential part of the proof presented later is due to a result from [1], which we shall cite now.

Theorem (Theorem 9.10, [1]). *Let* $(\mathcal{K}_n)$ *be a graphical sequence of kernels on a vertex space* $\mathcal{V}$ *with quasi-irreducible limit* κ. *Then for every* μ*-continuity set A,*

$$\frac{1}{n}\#\left\{v_i \in C_1\left(G^{\mathcal{V}}(n,\mathcal{K}_n)\right) : \ v_i \in A\right\} \xrightarrow{P} \int_A \rho(x)d\mu(x)$$

($\rho(x)$ *in the last formula is defined by Theorem A).*

This theorem implies for our model that for each $k \geq 1$

$$\nu_N(k) := \frac{1}{K_N}\#\left\{X_i \in \tilde{\mathcal{C}}_1 : |X_i| = k\right\} \xrightarrow{P} \rho(k)\mu(k) \tag{4.45}$$

as $N \to \infty$, where $\rho(k)$ is the maximal solution to (2.24).

Proof of Proposition 4.1. First, we derive part (*ii*). Observe that according to (2.24) the constant β defined in (4.44) is the maximal solution to

$$\beta \equiv \mathbf{E}\left(|C|^{-1}\right)\sum_{k=1}^{\infty} k\rho(k)\mu(k) = \mathbf{E}(|C|^{-1})\sum_{k=1}^{\infty} k\left(1 - e^{-\sum_{y=1}^{\infty}\mathcal{K}(k,y)\mu(y)\rho(y)}\right)\mu(k)$$
$$= 1 - \mathbf{E}(e^{-c|C|\beta}).$$

This proves part (*ii*) of the proposition.

(i) For any $1 \leq R < |B(N)|$ write $W_N := W_N^R + w_N^R$, where

$$W_N^R := \sum_{k=1}^{R} k\nu_N(k), \quad w_N^R := \sum_{k=R+1}^{|B(N)|} k\nu_N(k).$$

By (4.45) we have for any fixed $R \geq 1$

$$W_N^R \xrightarrow{P} \sum_{k=1}^{R} k\rho(k)\mu(k) \tag{4.46}$$

as $N \to \infty$.

Note that for any $\varepsilon > 0$ we can choose R_0 so that for all $R \geq R_0$

$$\sum_{k=R+1}^{\infty} k\rho(k)\mu(k) < \varepsilon/3.$$

Then, we derive

$$\begin{aligned}
&\mathbf{P}\{|W_N - \sum_{k=1}^{\infty} k\rho(k)\mu(k)| > \varepsilon\} \\
&\quad = \mathbf{P}\left\{\left|\left(W_N^R - \sum_{k=1}^{R} k\rho(k)\mu(k)\right) + w_N^R - \sum_{k=R+1}^{\infty} k\rho(k)\mu(k)\right| > \varepsilon\right\} \\
&\quad \leq \mathbf{P}\left\{\left|W_N^R - \sum_{k=1}^{R} k\rho(k)\mu(k)\right| > \varepsilon/3\right\} + \mathbf{P}\left\{w_N^R > \varepsilon/3\right\} \\
&\quad \leq o(1) + \frac{3\mathbf{E}w_N^R}{\varepsilon}
\end{aligned} \tag{4.47}$$

as $N \to \infty$, where the last bound is due to (4.46) and the Markov's inequality.

Claim. *For some positive constants A_2 and a_2*

$$\mathbf{E}w_N^R = \sum_{k=R+1}^{|B(N)|} k\mathbf{E}\nu_N(k) \leq A_2 e^{-a_2 R}. \tag{4.48}$$

Proof of Claim. Note that for any $k \geq 1$

$$\nu_N(k) \leq \frac{1}{K_N} \sum_{i=1}^{K_N} I_k(|X_i|) = \frac{\mathcal{N}_k}{K_N} \leq 1. \tag{4.49}$$

Using the events $\mathcal{A}_{\delta,N}$ defined in Corollary 2.1, we obtain from (4.49) that for any fixed $0 < \delta < \mathbf{E}(|C|^{-1})/2$ and $k \geq 1$

$$\begin{aligned}
\mathbf{E}\nu_N(k) &\leq \mathbf{E}\left(\frac{\mathcal{N}_k}{K_N}\mathbf{1}\{\mathcal{A}_{\delta,N}\}\right) + \mathbf{E}\left(\frac{\mathcal{N}_k}{K_N}\mathbf{1}\{\overline{\mathcal{A}_{\delta,N}}\}\right) \\
&\leq \frac{\mathbf{E}\mathcal{N}_k}{(\mathbf{E}(|C|^{-1}) - \delta)|B(N)|} + \mathbf{P}\{\overline{\mathcal{A}_{\delta,N}}\}.
\end{aligned}$$

Substituting the bound from (2.48)

$$\mathbf{E}\mathcal{N}_k \leq \kappa(p)\mu(k)|B(N)| + |\partial B(N)|\tilde{\mu}(k)$$

into the last formula we obtain

$$\mathbf{E}\nu_N(k) \le \frac{\kappa(p)\mu(k)|B(N)| + |\partial B(N)|\tilde{\mu}(k)}{(\mathbf{E}(|C|^{-1}) - \delta)|B(N)|} + \mathbf{P}\{\overline{\mathcal{A}_{\delta,N}}\}.$$

Corollary 2.1 allows us to derive from here that

$$\mathbf{E}\nu_N(k) \le A_1(\mu(k) + \tilde{\mu}(k) + e^{-a_1|B(N)|}) \tag{4.50}$$

for some positive constants A_1 and a_1 independent of k and N. This together with the exponential decay of μ and $\tilde{\mu}$ yields (4.48). ▪

Using (4.48) we immediately derive from (4.47)

$$\mathbf{P}\left\{|W_N - \sum_{k=1}^{\infty} k\rho(k)\mu(k)| > \varepsilon\right\} \le o(1) + \frac{3A_2 e^{-a_2 R}}{\varepsilon} \tag{4.51}$$

as $N \to \infty$. Hence, for any given positive ε and ε_0 we can choose a finite sufficiently large number R that

$$\lim_{N\to\infty} \mathbf{P}\left\{|W_N - \sum_{k=1}^{\infty} k\rho(k)\mu(k)| > \varepsilon\right\} < \varepsilon_0. \tag{4.52}$$

This proves statement (4.44) and, therefore, finishes the proof of Proposition 4.1 and completes the proof of Theorem 1.1. ▪

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