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ROBUSTNESS OF PREFERENTIAL ATTACHMENT UNDER DELETION OF EDGES

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□ *The effect of deletion of old edges in the preferential attachment model introduced by Barabási and Albert*[1] *is studied. We consider a model where every edge is deleted after a time* Δ. *The resulting graph has only* Δ *edges and with a high probability* $(1 + c)\Delta$ *nodes for some positive* c. *However, its structure doesn't resemble the structure of the former model even for large* Δ. *In particular, we prove that the expected degrees of the resulting graph are uniformly bounded by a constant that does not depend on* Δ. *We discuss applications of our model for the evolution of networks where competition occurs.*

Keywords Preferential attachment; Random graphs.

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1. INTRODUCTION

Recently, many new random graph models have been introduced, inspired by certain features observed in large-scale real-world networks such as the World Wide Web, interactions between proteins, genetic networks or social and business networks (see, e.g., Refs.[2,6] for a survey or Ref.[10]). The main observation is that in many real-world networks, the fraction $\mathbb{P}(d)$ of vertices with degree d is proportional to $d^{-\gamma}$ where γ is a constant independent of the size of the network. The study of these so-called scale-free random graphs is highly motivated by the model of preferential attachment introduced in Barabási and Albert[1]. Barabási and Albert showed by approximate arguments supported by simulation that this model leads to a graph with degree distribution proportional to d^{-3}. Rigorous proof of the power law distribution of degrees was given in Ref.[3]. In this model, at each time step t a new vertex v_t is introduced with m edges

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linking v_t to the previous vertices with probabilities proportional to their degrees or attractiveness. Note that the same principle has been previously introduced in the little-cited paper by Szymański[15]. This concept models real networks where the attractive components are more likely to receive new connections (as frequently visited sites are likely to get links from new sites). This process was rigorously redefined in Ref.[5].

Start with G^1, the graph with one vertex and one loop. Given G^{t-1} we form G^t by adding a vertex t together with a single edge $(t, g(t))$, where $g(t)$ is a random vertex chosen as follows. Let $D_i(t)$ denote the degree of vertex i at time t. The degree is the number of edges incident to the vertex.

$$\mathbf{P}(g(t) = i \mid D_i(t-1) = d_i) = \begin{cases} \dfrac{d_i}{2t-1}, & \text{if } 1 \leq i \leq t-1, \\ \dfrac{1}{2t-1}, & \text{if } i = t. \end{cases}$$

Thus, closed loops are allowed to occur when new vertices are nucleated, although in such cases the new sites are disconnected from the existing graph. Apart from short loops resulting from the connection of a site to itself, all connected components of the graph are essentially tree-like. However, if we introduce several connections (m) from a single vertex then the graph features will differ and cycles may occur, but the essential property of the graph will remain; i.e., the degree distribution follows a power law[3].

In the preferential attachment model, vertices and edges are added at each time step but never deleted. However, an evolving graph model incorporating deletion may model the evolution of the web graph more accurately. In this view, Cooper, Frieze and Vera[7] introduced random deletion of vertices and edges to the process to model networks such as P2P. Their model yields $\gamma \in]2; \infty[$ according to the parameters of accumulation and deletion.

The first paper on the deletion of vertices in the preferential attachment model is Bollobás et al.[4]. The authors raised the following question: “How robust and vulnerable are scale free graphs?” The robustness of a graph is defined in Ref.[4] as the stability of its properties under a *random* deletion of vertices. The vulnerability is defined as the stability of the properties of a graph but under a *malicious attack*, i.e., when a specific part of the graph is deleted. It was proved in Ref.[4] that a scale-free random graph is more robust than a classical random graph but is also more vulnerable. Due to the definition of the preferential attachment process, vertices with a high degree attract more edges and therefore become even more attractive. This phenomenom leads to a graph which accumulates most of the edges on the first vertices. This observation incites the authors of Ref.[4] to delete the first (or the oldest) vertices in order to change the structure of the graph.

The introduction of deletion can allow the generalization of a model. Slater et al. introduced death of vertices with a rate either constant or proportional to their degree on a continuous time model of stochastically evolving networks[14].

The growth-deletion model introduced in Ref.[8] incorporates deletion of both vertices and edges, which are chosen uniformly at random. In another model of duplication-deletion describing genes, vertices are again deleted uniformly at random[9].

In this paper, we introduce another growth-deletion model where the choice of deletion is not made uniformly at random, conversely to Refs.[8,9]. Note that need for such models is discussed in Ref.[6]. Instead of considering deletion of the vertices, we will consider the effect of deletion of the first edges. More precisely, we delete any edge at time Δ after its introduction in the graph. Thus, each edge is given a lifetime of Δ. The choice of this attack is natural if one thinks about social, biological or in particular neural networks where the connections (impulses) are temporal in their nature (see, e.g., Ref.[11] for a relevant description and citations).

One typical property of the preferential attachment model is the accumulation of most edges on the first vertices. The deletion introduced in our model thwarts this effect. We prove that for any fixed Δ, the degree of any vertex will reach zero with time. Furthermore, when the degree of a vertex becomes zero it remains at zero forever (Theorem 3.1).

Our mechanism of deletion of edges is similar to the one introduced in Ref.[16]. It was shown in Ref.[16] that if old connections are abandoned, it is still possible to keep some macro-properties of the original graph (uniformly grown in the case of Ref.[16]) almost unchanged.

Here, we observe totally different behaviour. The main result of our study shows that the expected degree of any vertex is uniformly bounded in Δ. To be more precise, no matter how large one chooses Δ, the expectation of the degree in the resulting graph is bounded by a constant $me^{\frac{1}{2}}$ (Corollary 3.1). On the other hand, if $\Delta = \infty$ we recover the model of preferential attachment where the expected degree is unbounded. This yields a sharp transition at $\Delta = \infty$ proving that the preferential attachment model is not robust against the aging of edges. All results are still valid if instead of considering the lifetime of any edge to be constant, we allow the deletion to happen randomly on an interval of fixed length centered on Δ. Here, unlike in the previous study [16], abandoning old connections leads to a loss of original properties. But as a result of the introduced dynamics our graph becomes more homogeneous and in turn less vulnerable than the original preferential attachment model. Examples of possible applications of our model are given in the conclusion.

2. MODEL

We denote by $G^t = (V^t, L^t)$ the graph at time $t = \{1, 2, \dots\}$ where $V^t = \{1, \dots, t\}$ is the set of vertices and L^t is the set of edges. Set G^1 to be the graph with one vertex and one loop. Fix $\Delta > 0$ to be the life-time of any edge in the graph. This means that an edge introduced at time t is no longer present in the graph at time $s \geq t + \Delta$. Then as long as $t \leq \Delta$, the graph evolves as in the preferential attachment model[5]. Namely, given G^{t-1} we form G^t by adding a vertex t together with a single edge $(t, g(t))$, where $g(t)$ is a random vertex chosen as follows.

Let $D_i^{\Delta}(t)$ denote the degree of vertex i at time t. The degree is the number of edges incident to the vertex.

$$\mathbf{P}(g(t) = i \,|\, D_i^{\Delta}(t-1) = d_i) = \begin{cases} \dfrac{d_i}{2t-1}, & \text{if } 1 \leq i \leq t-1, \\ \dfrac{1}{2t-1}, & \text{if } i = t. \end{cases}$$

Hence, $L^t = L^{t-1} \cup \{(t, g(t))\}$.

The graph first accumulates vertices and edges (the growth phase) until it reaches its maximum capacity of Δ edges. Then the oldest edge is deleted to make way for the new one.

When $t > \Delta$, we define

$$L^t = L^{t-1} \cup \{(t, g(t))\} \backslash \{(t-\Delta, g(t-\Delta))\},$$

i.e.,

$$L^t = \{(s, g(s)), t - \Delta < s \leq t\}.$$

Clearly, if $\Delta = +\infty$, we recover the model of preferential attachment[1]. From this point on, we will denote D_i^{∞} the degree of the vertex i in this case.

For all $t > \Delta$ we have $|L^t| = \Delta$ since every time one edge is introduced one edge is deleted. This implies that for all $t > \Delta$,

$$\sum_{i=1}^{t} D_i^{\Delta}(t) = 2\Delta,$$

and the transition probabilities are

$$\mathbf{P}\big(g(t) = i \,|\, D_i^{\Delta}(t-1) = d_i\big) = \begin{cases} \dfrac{d_i}{2\Delta+1}, & \text{if } i < t, \\ \dfrac{1}{2\Delta+1}, & \text{if } i = t. \end{cases}$$

We can consider an extension of the previous model where each new vertex is introduced together with m edges (i.e, it has out-degree m). Each new edge attaches to previous vertices independently to the other edges introduced at the same time. Thus, multiple edges are allowed and for each edge, the probability law is

$$\mathbf{P}\big(g(t) = i \mid D_i^{\Delta}(t-1) = d_i\big) = \begin{cases} \dfrac{d_i}{(2\Delta+1)m}, & \text{if } i < t, \\ \dfrac{1}{(2\Delta+1)m}, & \text{if } i = t. \end{cases}$$

3. RESULTS

When $\Delta = \infty$ (model of preferential attachment[3]) the vertices with the highest degree are first. The deletion of edges in our model leads to the opposite situation. Namely, with probability one, every vertex introduced in the graph will have degree 0 after some time as it is stated in Theorem 3.1.

Theorem 3.1. *For any out-degree $m \geq 1$ and any $i \geq 1$*

$$\lim_{t\to\infty} \mathbf{P}\{D_i^{\Delta}(t) = 0\} = 1.$$

Recall that when $\Delta = \infty$, the expected degree has the following formula for $m = 1$[3]

$$\mathbf{E}(D_i^{\infty}(t)) = \prod_{j=i}^{t} \frac{2j}{2j-1} \sim \sqrt{t/i}, \tag{3.1}$$

i.e., the expectation is unbounded in t.

In our case, the expected degree of a vertex is no longer a monotone increasing function of time. In Theorem 3.2 below, we give the time when the expected degree of a fixed vertex reaches its maximum.

Theorem 3.2. *Let $m \geq 1$. For any $i \geq 1$*

$$\max_{t\geq 1} \mathbf{E}\big(D_i^{\Delta}(t)\big) = \mathbf{E}\big(D_i^{\Delta}(i+\Delta-1)\big), \tag{3.2}$$

and for any $t \geq \Delta$

$$\max_{1\leq i\leq t} \mathbf{E}\big(D_i^{\Delta}(t)\big) = \mathbf{E}\big(D_{t-\Delta+1}^{\Delta}(t)\big). \tag{3.3}$$

The next corollary shows that the expectations of the degrees in our graph are uniformly bounded, unlike in the corresponding model without deletion[3].

Corollary 3.1. *Let $m \geq 1$. For any $t \geq 2\Delta$*

$$\max_{1\leq i\leq t} \mathbf{E}\big(D_i^{\Delta}(t)\big) = m\bigg(1 + \frac{1}{2\Delta+1}\bigg)^{\Delta}, \tag{3.4}$$

and

$$\lim_{\Delta\to\infty} \lim_{t\to\infty} \max_{1\leq i\leq t} \mathbf{E}\big(D_i^{\Delta}(t)\big) = me^{\frac{1}{2}}. \tag{3.5}$$

One should remark that in the model of preferential attachment,

$$\lim_{t\to\infty} \lim_{\Delta\to\infty} \max_{1\leq i\leq t} \mathbf{E}(D_i^{\Delta}(t)) = \lim_{t\to\infty} \max_{1\leq i\leq t} \mathbf{E}(D_i^{\infty}(t)) = \infty. \tag{3.6}$$

Relations (3.1), (3.5), and (3.6) show that the transition of the expected degree between the model with ephemeral edges and the model with infinite-lived connection is not continuous.

Remark 3.1. Using the bound (3.5) in Corollary 3.1 and the fact that for $i > \Delta$, $t > i + \Delta$, we have $\mathbf{E}(D_i(t + k\Delta)) \leq (\frac{1}{2})^k \mathbf{E}(D_i(t))$, one can prove that the variance is also bounded independently of Δ.

Remark 3.2. It is clear that our model admits generalizations. Allowing some freedom on the deletion of edges would fit reality even more. If, for example, the deletion happens randomly on an interval of a fixed length δ centered on Δ, then our results are still valid.

4. PROOFS

4.1. Proof of Theorem 3.1

Without loss of generality, here we consider the case $m = 1$.

Let us fix a vertex i and consider $D_i^{\Delta}(t)$, $t \geq i$. We introduce a Markov chain $\{\mathbf{X}(t)\}_{t\geq i}$ with

$$\mathbf{X}(t) = (X_0(t), X_1(t), \dots, X_{\Delta}(t)) \in \{0, 1\}^{\Delta+1}.$$

$X_0(t)$ is deterministic

$$X_0(t) = \begin{cases} 1 & \text{if } i \leq t < i + \Delta, \\ 0 & \text{if } t \geq i + \Delta, \end{cases}$$

and

$$X_k(t) = \begin{cases} 0 & \text{if } (i, t-k+1) \notin L^t, \\ 1 & \text{if } (i, t-k+1) \in L^t. \end{cases}$$

$X_k(t), 1 \leq k \leq \Delta$ is the number of edges introduced at time $t-k+1$ incident with i. According to the definition of our model at each time $t \geq i$ a new vertex is added together with an edge $(t, g(t))$. If $g(t) = i$ then $X_1(t) = 1$ and $X_1(t) = 0$ otherwise.

At time $t+1$, the value $X_1(t)$ is translated to the next coordinate: $X_2(t+1) = X_1(t)$ and $X_{k+1}(t+1) = X_k(t),\ 1 \leq k \leq \Delta - 1$. Then

$$\sum_{k=0}^{\Delta} X_k(t) = D_i^{\Delta}(t), \quad \text{for all } t \geq i.$$

Set

$$\mathbf{X}(i) = \begin{cases} (1,0,0,\ldots,0) & \text{with probability } 1 - \dfrac{1}{2\Delta+1}, \\ (1,1,0,\ldots,0) & \text{with probability } \dfrac{1}{2\Delta+1}. \end{cases}$$

The endpoint $g(t)$ of the edge $(t, g(t))$ introduced at time t is chosen according to the transition probability

$$\begin{aligned} \mathbf{P}\big(g(t) = i \,|\, D_i^{\Delta}(t-1) = d_i\big) &= \frac{d_i}{\min\{2\Delta+1, 2t-1\}} \\ \mathbf{P} = \mathbf{P}\big(X_1(t) = 1 \,|\, \mathbf{X}(t-1) = \overline{\mathbf{X}}\big) &= \frac{\sum_{k=0}^{\Delta} \overline{X}_k}{\min\{2\Delta+1, 2t-1\}}. \end{aligned} \tag{4.1}$$

To prove the statement of Theorem 3.1, we need to show that the vector $(0,\ldots,0)$ is absorbing and reachable from every state. Indeed, according to the transition probability (4.1) we have

$$\mathbf{P}(\mathbf{X}(t+1) = (0,\ldots,0) \,|\, \mathbf{X}(t) = (0,\ldots,0)) = 1,$$

i.e., $(0,\ldots,0)$ is an absorbing state.

Now we show that $(0,\ldots,0)$ is reachable from any state. For any $\overline{\mathbf{X}} \in \{0,1\}^{\Delta+1}$, there exists with a positive probability a sequence of states from $\overline{\mathbf{X}}$ to $(0,\ldots,0)$.

Assume that $\mathbf{X}(t) = \overline{\mathbf{X}} \in \{0,1\}^{\Delta+1}$ and at each time $s, t \leq s \leq t+\Delta$ enforce $X_1(s) = 0$. After Δ steps, it reaches $(0,\ldots,0)$.

$$\begin{aligned} &\mathbf{P}(\mathbf{X}(t+\Delta) = (0,\ldots,0) \,|\, \mathbf{X}(t) = \overline{\mathbf{X}}) \\ &\qquad \geq \frac{\prod_{i=-1}^{\Delta-1}(\Delta+i)}{(2\Delta+1)^{\Delta}} = \frac{2\Delta!}{(\Delta-1)!(2\Delta-1)(2\Delta+1)^{\Delta}} \geq \frac{1}{2^{\Delta}}. \end{aligned}$$

This implies that the limit distribution is concentrated on the point $(0, \dots, 0)$,

$$\lim_{t\to\infty} \mathbf{P}(X(t) = (0, \dots, 0)) = 1.$$

Hence $\lim_{t\to\infty} \mathbf{P}\{D_i^\Delta(t) = 0\} = 1$ as stated in Theorem 3.1.

4.2. Recursive Formula for the Expectation

From now on, we fix $1 \le m$ and $0 < \Delta < \infty$ arbitrarily and write $D_i^\Delta(t) = D_i(t)$.

In order to prove Theorem 3.2 and Corollary 3.1, we first need the following proposition.

Proposition 4.2.1. *For all $i > \Delta > 0$, we have the following recursive formulas:*

$$\mathbf{E}(D_i(t)) = \begin{cases} 0, & t < i, \\ m + \dfrac{m}{2\Delta+1}, & t = i, \\ \left(1 + \dfrac{1}{2\Delta+1}\right)\mathbf{E}(D_i(t-1)), & i+1 \le t \le i+\Delta-1, \\ \left(1 + \dfrac{1}{2\Delta+1}\right)\mathbf{E}(D_i(t-1)) & \\ \quad - \left(m + \dfrac{m}{2\Delta+1}\right), & t = i+\Delta, \\ \left(1 + \dfrac{1}{2\Delta+1}\right)\mathbf{E}(D_i(t-1)) & \\ \quad - \dfrac{1}{2\Delta+1}\mathbf{E}(D_i(t-\Delta-1)), & t > i+\Delta. \end{cases}$$

Proof of Proposition 4.2.1. Let $t \ge i > \Delta$. Given $D_i(s) = d_i(s), t-\Delta \le s < t$ we have

$$D_i(i) = m + mI_{\{g(t)=i\}},$$

and

$$D_i(t) = D_i(t-1) + mI_{\{g(t)=i\}} - \begin{cases} 0, & \text{if } i < t < i+\Delta, \\ 1 + mI_{\{g(t-\Delta)=i\}}, & \text{if } t = i+\Delta, \\ mI_{\{g(t-\Delta)=i\}}, & \text{if } t > i+\Delta. \end{cases}$$

We prove here only the last case when $t > i + \Delta$. The rest can be found in the same way.

$$\begin{aligned}
&\mathbf{E}(D_i(t) \mid D_i(t-1), D_i(t-\Delta-1)) \\
&\quad = D_i(t-1) + \mathbf{E}(I_{\{g(t)=i\}} \mid D_i(t-1)) - \mathbf{E}(I_{\{g(t-\Delta)=i\}} \mid D_i(t-\Delta-1)) \\
&\quad = \left(1 + \frac{1}{2\Delta+1}\right) D_i(t-1) - \frac{D_i(t-\Delta-1)}{2\Delta+1}.
\end{aligned}$$

Taking the expectation of both sides, we derive

$$\mathbf{E}(D_i(t)) = \left(1 + \frac{1}{2\Delta+1}\right)\mathbf{E}(D_i(t-1)) - \frac{1}{2\Delta+1}\mathbf{E}(D_i(t-\Delta-1)),$$

as required.

4.3. Maximum of Expectation

Proof of Theorem 3.2 *and Corollary* 3.1 Consider $\mathbf{E}(D_i(i+n))$. We may refer to $n \geq 0$ as the age of the vertex i.

Then if $n \leq \Delta - 1$ we have

$$\mathbf{E}(D_i(i+n)) = m\left(1 + \frac{1}{2\Delta+1}\right)^{n+1}.$$

The right-hand side is obviously increasing in n and reaches its maximum value for $n = \Delta - 1$.

To prove that $m\left(1 + \frac{1}{2\Delta+1}\right)^{\Delta}$ is the maximum of the expected degree we need the following lemma.

Lemma 4.3.1. *Set* $k \geq 1$. *If* $\boldsymbol{E}(D_i(i+n+1)) \leq \boldsymbol{E}(D_i(i+n))$ *for* $k \leq n \leq k + \Delta$ *then* $\boldsymbol{E}(D_i(i+n+1)) \leq \boldsymbol{E}(D_i(i+n))$ *for* $n \geq k$.

Thanks to this lemma, we will only need to prove that the sequence $\mathbf{E}(D_i(i+n))$ is decreasing for $\Delta - 1 \leq n \leq 2\Delta - 1$ to prove that it's decreasing for $n \geq \Delta - 1$.

Proof of Lemma 4.3.1. For $k \geq \Delta + 1$, the expectation follows the recursive formula

$$\mathbf{E}(D_i(i+k+1)) = \mathbf{E}(D_i(i+k)) + \frac{1}{2\Delta+1}(\mathbf{E}(D_i(i+k)) - \mathbf{E}(D_i(i+k-\Delta))).$$

We deduce that $\mathbf{E}(D_i(i+k)) \leq \mathbf{E}(D_i(i+k-\Delta))$ implies $\mathbf{E}(D_i(i+k)) \leq \mathbf{E}(D_i(i+k))$.

Finally, if

$$\mathbf{E}(D_i(i+k-\Delta)) \geq \mathbf{E}(D_i(i+k-\Delta+1)) \geq \cdots \geq \mathbf{E}(D_i(i+k)),$$

then

$$\mathbf{E}(D_i(i+n)) \geq \mathbf{E}(D_i(i+n+1)) \quad \text{for any } n \geq k.$$

Let $\Delta \leq n \leq 2\Delta - 1$. We are going to show that the expectation of the degree is monotone decreasing. One can find by induction that

$$\begin{aligned}\mathbf{E}(D_i(i+n)) &= m\left(1+\frac{1}{2\Delta+1}\right)^{n+1}\left(1-\frac{1}{2\Delta+1}\left(1+\frac{1}{2\Delta+1}\right)^{-\Delta}(n+\Delta+2)\right)\\ &= me^{n+1\ln\left(1+\frac{1}{2\Delta+1}\right)}(1-C(n+\Delta+2)),\end{aligned}$$

with $C = \frac{1}{2\Delta+1}\left(1+\frac{1}{2\Delta+1}\right)^{-\Delta}$.

Let $f(x)$, $\Delta \leq x \leq 2\Delta - 1$ be the extension to $\mathbb{R}$ of $\mathbf{E}(D_i(i+n))$, $\Delta \leq n \leq 2\Delta - 1$.

We differentiate this function and prove that for every x, it has the same sign.

$$f'(x) = m\left(1+\frac{1}{2\Delta+1}\right)^{x+1}\left(\ln\left(1+\frac{1}{2\Delta+1}\right)(1-C(x+\Delta+2))-C\right).$$

It vanishes for $x_0 = \frac{1-C(\Delta+2)}{C} - \frac{1}{\ln\left(1+\frac{1}{2\Delta+1}\right)}$ but $x_0 < \Delta$ for any $\Delta > 0$ which contradicts $\Delta \leq x \leq 2\Delta - 1$.

So $f'(x) \neq 0 \quad \forall x \in [\Delta; 2\Delta - 1]$ and is continuous, which implies that all the derivatives have the same sign.

We now show that the derivative is negative for one precise n and with this result, we know that the expectation decreases for $\Delta \leq n \leq 2\Delta - 1$.

$$\begin{aligned}\operatorname{sgn}(f'(x)) &= \operatorname{sgn}(f'(x))_{|x=\Delta+1}\\ &= \operatorname{sgn}\left(\ln\left(1+\frac{1}{2\Delta+1}\right)\left(1+\frac{1}{2\Delta+1}\right)^{\Delta} - \frac{4}{2\Delta+1}\right) < 0 \quad \forall\Delta > 0.\end{aligned}$$

To use the Lemma 4.3.1, we need a Δ + 1st consecutive point for which it's decreasing.

$$\begin{aligned}&\mathbf{E}(D_i(i+\Delta-1)) - \mathbf{E}(D_i(i+\Delta))\\ &\quad = m\left(1+\frac{1}{2\Delta+1}\right)^{\Delta} - m\left(\left(1+\frac{1}{2\Delta+1}\right)^{\Delta+1} - \left(1+\frac{1}{2\Delta+1}\right)\right) > 0.\end{aligned}$$

One may remark that if we consider the in-degree then the inequality is reversed and the maximal in-degree is attained at time $i+\Delta$.

As we have $\Delta+1$ consecutive points for which the expectation is monotone decreasing, we can conclude that

$$\mathbf{E}(D_i(i+\Delta-1)) \geq \mathbf{E}(D_i(i+n)) \quad \forall n.$$

This proves Equation (3.2) of Theorem 3.2.

Even if for convenience the proof has been done for $i \geq \Delta$, it is also true for $i < \Delta$ as then edges are more intensively deleted.

The formula (3.3) of Theorem 3.2 and Corollary 3.1 follow immediately.

$$\begin{aligned}\max_{t\geq 1} \mathbf{E}(D_i(t)) &= \mathbf{E}(D_i(i+\Delta-1)), \quad \forall i,\\ &= m\left(1+\frac{1}{2\Delta+1}\right)^{\Delta}, \quad \forall i > 2\Delta.\end{aligned}$$

As a consequence:

$$\lim_{\Delta\to\infty} \lim_{t\to\infty} \max_{1\leq i\leq t} \mathbf{E}\big(D_i^{\Delta}(t)\big) = me^{\frac{1}{2}}.$$

5. CONCLUSION

We examined the impact of deletion of edges in the Preferential Attachment model. We showed that the transition of the expected degree between the model with ephemeral edges and the model with infinite-lived connection is not continuous. Our results (Theorem 3.1) show that the deletion of edges implies that with a high probability, the vertices are renewed too. In our model, the connections are not static and the system is more applicable to the real world.

The model is separated into two phases. The first is the growth phase when the graph follows the rules of the model of preferential attachment. In this phase, the graph accumulates vertices and edges until it reaches a maximum size Δ. Once reached the graph can't continue to grow. This is followed by the renewal phase where competition occurs. In this phase, the oldest edge is deleted so as to be replaced by a new one.

The model captures essential features of social networks: people with multiple connections are more likely to acquire new frienships but there is ultimately a limit on the number of friendships that one individual can maintain[12]. Our model fulfills conditions of an evolution model stated in Ref.[13] ("Constant change is a natural feature of evolution, on a sufficiently large time scale, there's nothing remotely stable about evolution.") and is in good agreement with observations such as the number of species, their lifetime and renewal. In particular, our model fits the description of neural networks.

We proved that the property of accumulating edges on the first vertices is removed by the concept of lifetime for edges. However, by keeping the principle of preferential attachment for the newly introduced edges our graph is still robust to random deletion as it has a part with high degree vertices but is also much less vulnerable than the preferential attachment model as the attractive vertices are numerous. Moreover, by changing the parameter m, which is the number of edges introduced into the graph with every vertex, we control the connectivity of the limiting graph.

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