An explicit relation between monodromy of the Knizhnik-Zamolodchikov equations and braiding of quantum $\mathfrak{sl}_2$

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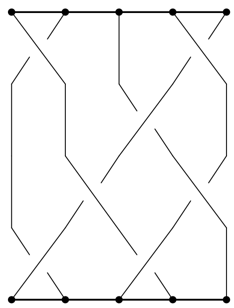

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Contents

Preface

The purpose of this thesis is to construct an *explicit equivalence* of representations of the braid group B_n, arising on the one hand from the monodromy of the Knizhnik-Zamolodchikov equations (KZ) and on the other hand from an R-matrix of the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. The equivalence of these two representations is a most remarkable and suprising result, relating two apparently different fields of mathematics. The general result, concerning any semisimple Lie algebra, which connects the braid group representation defined by the monodromy of KZ and the braid group representation induced by the quantum group of the associated Lie algebra, carries the name Drinfeld-Kohno theorem. The theorem is proved by using topological algebra [Kas95]. We will not consider the proof in this thesis, but rather give an explicit relation between these two representations in the case of the (semi)simple Lie algebra $\mathfrak{sl}_2$. Moreover, this method applies also to other systems of linear partial differential equations whose solutions are of similar form as solutions of KZ.

The above equivalence was first stated by Toshitake Kohno in 1986 [Koh87]. He gave a description of the monodromy representation of the braid group arising from the KZ-equations in terms of quantum groups. In 1990 Vladimir Drinfeld [Dri90] established the relation between the monodromy of KZ and the braid group representation defined by the universal R-matrix of the associated quantum group in a more general framework. Drinfeld accepted the prestigious Fields medal in 1990.

The *Knizhnik-Zamolodchikov equations* were introduced by Vadim G. Knizhnik and Alexander B. Zamolodchikov in the early eighties [KZ84] as the differential equations satisfied by certain correlation functions in conformal field theory. More precisely, the equations arose originally from the Wess-Zumino-(Novikov)-Witten-model, which is a two dimensional nonlinear sigma-model describing the propagation of strings on a group manifold, that is, a Lie group. Since then the KZ-equations have found applications in several areas of mathematics, among others the representation theory of affine Lie algebras, quantum groups, braid groups, and also in topology of hyperplane complements and the theory of knots and three-folds.

In this thesis we solve these equations for $\mathfrak{sl}_2$ and study the monodromy of the solutions. The theory of solutions of the KZ-equations generalises the classical theory of the hypergeometric equation (HGE), and in low dimensions the monodromy of the solutions is similar to the monodromy of HGE. Hence we first present some of the classical theory of HGE. We also introduce the so called “contour deformation“ method, which is a procedure to compute monodromy of solutions of differential equations written in integral form. With help of this method we compute also the monodromy of KZ explicitly.

On the other hand, we introduce Hopf algebras and quantum groups, and explain how R-matrices of quantum groups define representations of the braid group B_n. Moreover, we construct an R-matrix for the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, and show that it defines a representation of B_n equivalent to the monodromy of KZ for the Lie algebra $\mathfrak{sl}_2$.

The *braid group* B_n is a finitely generated group closely related to the symmetric group S_n. It is a group intuitively easy to picture, consisting of isotopy classes of braids with n strands. The group operation in B_n is defined simply by placing two braids on top of each other. The relation with the symmetric group is expressed by a surjective group homomorphism. Indeed, every braid $b \in B_n$ defines a unique permutation $\tau(b) \in S_n$ of the set of its starting and end points. The map $b \mapsto \tau(b)$ defines a surjective homomorphism from B_n to S_n.

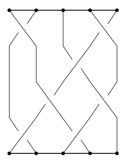

Figure 1: A braid with five strands

The KZ-equations define a connection on a trivial vector bundle E over the complex manifold

$$Y_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n\} \setminus \bigcup_{i<j} \{z_i = z_j\}.$$

The solutions of KZ are to be considered as horizontal sections of this bundle. The connection defined by KZ is flat, which implies that monodromy can be defined as parallel transport along smooth paths on Y_n, and also that in the vicinity of any point $(z_1, \ldots, z_n) \in Y_n$ there exists a basis of local solutions for KZ. The fundamental group of Y_n is isomorphic to a subgroup P_n of the braid group B_n, called the *pure braid group*. It consists of braids which preserve the order of their starting and end points; in other words it is the kernel of the homomorphism $b \mapsto \tau(b)$ from B_n to S_n. So the monodromy of KZ defines a representation of P_n acting on the fibers of the vector bundle E. By incorporating an action of S_n on E we will be able to obtain a representation of the whole braid group B_n.

In the first chapter we recall some complex analysis needed for studying the monodromy of linear differential equations. The most important tool for us from complex analysis is *analytic continuation* of holomorphic functions. We show that analytic continuation of solutions of linear ODE's defines a linear representation of the fundamental group of the domain where the equation and its solutions are analytically defined, acting on the space of local solutions. This is called the *monodromy* of the ODE. The reader should have some experience in the fundamentals of complex analysis, covering basic analytic functions and the most common examples of multivalued functions, such as the complex power function. Knowledge of the theory of linear ODE's will also be useful.

In the second chapter we present the classical theory of the *hypergeometric equation*, including solutions and monodromy. We present two methods for solving the equation and computing its monodromy. The reason for this is that the monodromy of HGE is easy to compute by simply continuing analytically the power series solutions obtained from the Fröbenius method, whereas in computing the monodromy of the KZ-equations we use another method based on "contour deformation". For that we introduce integral expressions of solutions of HGE, presented by Leonhard Euler in 1748 in his book "Introduction to analysis of the infinite", and compute their monodromy by deforming and cutting up the paths of integration of each solution in a suitable way. Chapter two is quite self contained, and the reader need not to have previous experience on monodromy computations or the theory of HGE.

In the third chapter we introduce the language of connections and vector bundles on complex manifolds. We define *Pfaffian systems*, which are systems of linear partial differential equations defined analytically on a complex manifold M. We show that KZ is an example of a Pfaffian system defined on Y_n. We also explain how a Pfaffian system defines a connection on a trivial vector bundle over the manifold M, and show that if this connection is flat then the system has local fundamental solutions and, moreover, monodromy of the system is well defined. We define monodromy of this kind of Pfaffian systems as a linear representation of the fundamental group of M acting on fibers of the trivial vector bundle. For the reader it is useful to be familiar with smooth manifolds and their tangent and cotangent bundles. However, we recall the definitions of complex manifolds and holomorphic vector bundles, in analogy with the theory of smooth manifolds found e.g. in [Lee97] and [Lee03]. The reader might also want to take a look at [For91] for more details on complex structures. Knowledge of the theory of connections on vector bundles is not required, and we shall prove the used properties of connections in detail.

In the fourth chapter we recall briefly the structure of the Lie algebra $\mathfrak{sl}_2$, the notion of bialgebras and Hopf algebras, and definition of the so called *quantum* $\mathfrak{sl}_2$, denoted by $U_q(\mathfrak{sl}_2)$, which is a q-deformation of the universal enveloping algebra of $\mathfrak{sl}_2$. We also recall the necessary representation theory of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$, in the extent of irreducible highest weight representations and semisimplicity. The reader need not to have background from the theory of Lie algebras, representations or quantum groups, and we will at least state with references the results needed. However, knowledge of linear algebra is necessary, and we expect also that the reader is somewhat familiar with the concept of tensor products. For more details on Hopf algebras and quantum groups one may consult e.g. [Kas95] or [Kyt11], and on the theory of Lie algebras e.g. [Hum72] or [Kna04].

In the fifth chapter we first give the rigorous definition of the braid group B_n and show that it is the fundamental group of the configuration space $\mathcal{C}_n = Y_n/S_n$, which is the quotient space of the manifold Y_n equipped with the natural action of the symmetric group S_n. Furthermore, in the fifth chapter we solve the KZ-equations in the case of $\mathfrak{sl}_2$ and compute their monodromy using the *"contour deformation"* method. Our treatment of the monodromy of the KZ-equations for $\mathfrak{sl}_2$ can be generalised to any semisimple Lie algebra following the same lines. The reason why we have chosen to present the case of $\mathfrak{sl}_2$ is that for $\mathfrak{sl}_2$ the solutions of KZ are the simplest to obtain, although even for that case they include all generalised hypergeometric functions studied in the last century. We will state some facts concerning the solutions of KZ without proofs, referring to [EFK98]. In computing the monodromy we use suitable non-intersecting families of loops which enable us to compute the monodromy explicitly. For details relating these families to general solutions of KZ in integral form we refer to [FW90].

In the sixth chapter we introduce more of the theory of quantum groups and construct a representation of the braid group B_n from the *extended quantum group* $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. We show that this representation is actually equivalent to the monodromy representation of B_n arising from the monodromy of solutions of KZ. Moreover, we obtain an *explicit relation* between these two representations, in terms of matrix elements. In order to construct a braid group representation from $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ we introduce the Drinfeld double construction, which is a Hopf algebra that as a vector space is a tensor product of a Hopf algebra and a certain subspace of its algebraic dual. We also introduce the quantum Borel algebra $H_{q^2} \subset U_q(\mathfrak{sl}_2)[\sqrt{K}]$ and find that the Drinfeld double associated to this Hopf algebra produces as a quotient structure the quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. We will state most of the results concerning Hopf algebras and quantum groups without proofs, which can be found in [Kas95] and [Kyt11].

Finally, in the last chapter we present the *Drinfeld-Kohno theorem* for comparison to our results. It is formulated using the notion of topological modules and formal power series in an indeterminate h. Hence the proof of the Drinfeld-Kohno theorem is valid only for such h that are near one, whereas our concrete method has only the restriction that the deformation parameter q is not allowed to be a root of unity. Moreover, our method produces an *explicit relation* between the monodromy of KZ($\mathfrak{sl}_2$) and the R-matrix of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, based on the fact that solutions of KZ($\mathfrak{sl}_2$) can be written in integral form. In particular, using the quantum R-matrix we are able to compute also monodromy of other systems of linear partial differential equations whose solutions have similar integral expressions. Systems of PDE's of this kind arise for instance in the theory of Schramm-Loewner evolutions.

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1 Analytic continuation

We recall some complex analysis needed for studying monodromy of linear ODE's. For more details the reader may consult e.g. [Rud87] or [NP69]. The most important tool for us is analytic continuation of holomorphic functions. From the monodromy theorem we obtain the crucial result that analytic continuation of solutions of linear ODE's with meromorphic coefficients defines a linear representation of the fundamental group of the domain where the equation and its solutions are analytically defined, acting on the space of local solutions. Moreover, the unique analytic continuation of a solution is again a solution, and continuations of two linearly independent solutions are also linearly independent. This will be proved in section 2.3, where we concentrate on Fuchsian equations, our main interest being the hypergeometric equation.

We denote the complex plane as usual by $\mathbb{C}$. We call a piecewise smooth map from the unit interval $[0,1] \subset \mathbb{R}$ to the complex plane a *path*. By a *domain* we mean a nonempty, open and connected subset of the complex plane, or, more generally, the Riemann sphere $\overline{\mathbb{C}}$. A domain $D \subset \overline{\mathbb{C}}$ is said to be *simply connected* if the complement $\overline{\mathbb{C}} \setminus D$ is connected. Equivalently, D is simply connected if its fundamental group is trivial.

We denote the space of holomorphic functions defined on a domain D by $Hol(D)$, and use the words *holomorphic* and *analytic* interchangeably. The reader should have some experience in the fundamentals of complex analysis, covering basics of analytic functions and the most common examples of multivalued functions, such as the complex power function. Knowledge of the theory of linear ODE's will also be useful.

The following properties of analytic and meromorphic functions are worthwhile to recall. The proofs can be found in any book of basic complex analysis, for example in [NP69].

Theorem 1.1.
(i) Let f be analytic in $B(z_0, R)$, $z_0 \in \mathbb{C}$, $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \ \text{where}$$
$$c_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} dw$$

and $\gamma : [0,1] \to B(z_0, R)$ is a path $t \mapsto z_0 + re^{\mathrm{i}t}$ with $0 < r < R$. The series has a radius of convergence up to the nearest singularity of f.
(ii) If above f is analytic in the punctured disc $B(z_0, R) \setminus \{z_0\}$, then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

and the series converges in the punctured disc $B(z_0, R') \setminus \{z_0\}$, where $R' \geq R$ is the distance to the nearest singularity of f other than z_0.

Notice in particular that the power series expansions are uniquely determined by the coefficients in their domain of convergence. The power series expansions of analytic and meromorphic functions are usually called the *Taylor* and *Laurent expansions*, respectively.

Another important tool is the fact that analytic and meromorphic functions cannot have accumulated zeros.

Proposition 1.2. *Let f be meromorphic in the domain D. Then either f is identically zero or the zeros of f are isolated. Moreover, the set of singularities of f is a set of isolated points in D.*

1.1 Continuation of analytic functions

Next we shall define rigorously the concept of continuation of analytic functions along chains of open discs, and along paths covered by open discs.

Definition 1.3. A *function element* is an ordered pair (f, B), where $B \subset \mathbb{C}$ is an open circular disc and f is holomorphic in B. Two function elements (f_1, B_1), (f_2, B_2) are *direct continuations* of each other if $B_1 \cap B_2 \neq \emptyset$ and $f_1 = f_2$ in $B_1 \cap B_2$.

Naturally, a given function element (f_0, B_0) can be thought of to be continued along a chain of circular discs $\{B_1, \ldots, B_n\}$ if there exists functions $f_1, \ldots, f_n$ so that the function elements (f_i, B_i) and (f_{i+1}, B_{i+1}) are direct continuations of each other for every $i = 0, \ldots, n-1$. The function f_n is said to be the *analytic continuation* of f_0 along the chain.

Proposition 1.4. *Let (f, B) be a function element and suppose f has an analytic continuation f_n along the chain $\{B_1, \ldots, B_n\}$. Then f_n is uniquely determined.*

Proof. If f_1 and g_1 are two analytic continuations of f to B_1 then by definition $f_1 = f = g_1$ on $B \cap B_1$. By proposition 1.2, since $B \cap B_1$ is open and connected, $f_1 = g_1$ on B_1. The assertion follows by induction on the number of terms of the chain. □

Definition 1.5. Let $\gamma : [0, 1] \to \mathbb{C}$ be a path. We say that a chain $\{B_1, \ldots, B_n\}$ of circular discs *covers* γ if there are numbers

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

such that $\gamma(0)$ is the center of B_0, $\gamma(1)$ is the center of B_n and

$$\gamma([t_i, t_{i+1}]) \subset B_i$$

for every $i = 0, \dots, n-1$. If a function element (f, B) can be continued along the chain $\{B_1, \dots, B_n\}$ we say that the function f_n is the *analytic continuation* of f *along the path* γ.

It is important to notice that also analytic continuation along paths is uniquely determined. The reader may consult [Rud87] for a detailed proof; by proposition 1.4 one only needs to check that the value of the continuation does not depend on the choice of the chain covering γ.

The concept of analytic continuation can be defined also for meromorphic functions. This is called *meromorphic continuation*, and the resulting functions are naturally assumed to be meromorphic. Meromorphic continuations are also unique, and similar results as presented in the next sections hold also in the meromorphic case.

1.2 Existence of analytic continuation

Analytic continuations along arbitrary paths do not always exist, but if a function f has an analytic continuation along a path γ, the continuation is uniquely determined and independent of the choice of the path within the same homotopy class. However, if we start from a fixed function element (f, B) and are able to continue it analytically along every path in a domain D, this might not produce an analytic function defined in the whole D. For instance, this is the case in the next example, where multivalued functions are analytically continued.

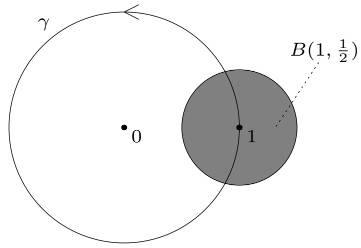


Figure 2: The loop $\gamma(t) = e^{2\pi \mathrm{i} t}$

Example 1.6. Recall the complex logarithm

$$\ln z := \ln |z| + i \arg z + 2\pi i n.$$

It is crucial that the complex logarithm is a *multivalued* function, that is, different choices of $n \in \mathbb{N} = \{1, 2, \dots\}$ produce different branches of ln. When the function $\ln z = \ln |z| + i \arg z + 2\pi i n$ with a particular choice of branch is analytically continued along the loop $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = e^{2\pi \mathrm{i} t}$, starting from $B(1, \frac{1}{2})$, the resulting continuation is

$$\ln |z| + i \arg z + 2\pi i(n + 1).$$

So the original function gets an *increment* of $2\pi i$ in the continuation. This is nothing but a change of branch.

Similarly, the analytic continuation along γ of the power function

$$z^a := \exp(a \ln z) = \exp(a(\ln|z| + i \arg z + 2\pi i n)),$$

which as well is multivalued, with a particular choice of branch gets a *"phase factor"* $e^{2\pi \mathrm{i} a}$, whence the analytic continuation of z^a along γ is $z^a e^{2\pi \mathrm{i} a}$.

Actually, both of the above functions can be analytically continued along any path in the domain $\mathbb{C} \setminus \{0\}$, but they are not well defined as analytic functions on $\mathbb{C} \setminus \{0\}$. This is because there is no continuous choice of branch for the complex logarithm nor for the power function in the whole $\mathbb{C} \setminus \{0\}$; see e.g. [Rud87].

Example 1.7. Since by theorem 1.1 an analytic function f can be represented as a power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

around any point $z_0 \in \mathbb{C}$ where it is analytic, with a radius of convergence R up to the nearest singularity, analytic continuations of f exist along every path in $B(z_0, R)$. By uniqueness these coincide with f.

Example 1.8. Consider the analytic continuation of a function of the form

$$f(z) = g(z)h(z),$$

where g and h are analytic functions in a disc B_0. Clearly f is analytic. Suppose f has an analytic continuation along a chain $\{B_1, \ldots, B_n\}$ such that h is analytic in an open set $D \subset \mathbb{C}$,

$$\bigcup_{k=0}^{n} B_k \subset D,$$

and g has an analytic continuation g_n along the same chain. Then the analytic continuation of f along the chain $\{B_1, \ldots, B_n\}$ is by uniqueness

$$f_n(z) = g_n(z)h(z).$$

In the above example it could happen that g would not have an analytic continuation along the chain $\{B_1, \ldots, B_n\}$. An important special case when g has an analytic continuation along "almost" every path γ is the following, concerning solutions of linear ODE's. More details about ODE's of the second order will be explained in the next chapter.

Suppose g is an analytic solution of a linear ordinary differential equation (ODE) having rational coefficients in its normal form, defined in a circular disc $B_0 \subset \mathbb{C}$. Let $\gamma : [0,1] \to \mathbb{C}$ be a loop starting from the center of B_0, avoiding all the singular points of the ODE. The reader can find the general definition of singular points of ODE's in [Dub07]. Under these assumptions we have the following.

Proposition 1.9. *The function g has an analytic continuation along γ.*

Proof. By the theory of ODE's each point $\gamma(t) \in \gamma([0,1])$ has an open neighbourhood U_t in which there exists a basis for the analytic solutions of the ODE. Since $\gamma([0,1])$ is compact the open cover $\{U_t\}_{t\in[0,1]}$ of $\gamma([0,1])$ has a finite subcover $\{U_{t_1}, \dots, U_{t_n}\}$, $0 \le t_1 < \cdots < t_n \le 1$.

Let $\{f_1^{(i)}, \dots, f_m^{(i)}\}$, $i = 0, \dots, n$, be a basis for the solutions of the ODE in U_{t_i}, where we write $U_0 := B_0$. We may assume that $f_j^{(0)} = f_j^{(n)}$ for $i = 0, \dots, m$ since γ is a loop. Let $M_i \in \mathbb{C}^{m\times m}$ be the matrix associated to the change of basis in the intersection $U_{t_i} \cap U_{t_i+1}$, $i = 0, \dots, n-1$, so that

$$f^{(i+1)} = M_i f^{(i)},$$

where we denote $f^{(i)} = (f_1^{(i)} \dots f_m^{(i)})^T$. Write

$$g = af^{(0)} = (a_1 f_1^{(0)} \cdots a_n f_n^{(0)})^T,$$

where $a = (a_1 \dots a_n) \in \mathbb{C}^m$. Since $0 \ne det(M_0) = det(M_0^T)$ the system

$$a = x^{(0)} M_0,$$

where $x^{(0)} = (x_1^{(0)} \dots x_n^{(0)})$, has a unique solution $x^{(0)}$. In other words, there exists $x^{(0)} \in \mathbb{C}^m$ so that

$$g = af^{(0)} = x^{(0)} M_0 f^{(0)} = x^{(0)} f^{(1)}$$

in $U_0 \cap U_{t_1}$. Now $x^{(0)} f^{(1)}$ is the analytic continuation of g to U_{t_1}. The same procedure in each intersection $U_{t_i} \cap U_{t_i+1}$, $i = 0, \dots, n-1$, yields the analytic continuation

$$\widetilde{g} := x^{(n-1)} f^{(n)} = x^{(n-1)} f^{(0)}$$

of g along γ, where $x^{(i)} \in \mathbb{C}^m$ is the unique solution of the equation

$$x^{(i-1)} = x^{(i)} M_i,$$

and $x^{(-1)} := a$. Moreover, solving the equations we obtain an explicit expression for the analytic continuation of g along γ with respect to the chosen bases $\{f_1^{(i)}, \dots, f_m^{(i)}\}$, namely

$$\widetilde{g} = a(M_{n-1}^{-1} \cdots M_0^{-1}) f^{(0)} = a(M_{n-1} \cdots M_0)^{-1} f^{(0)}.$$

□

In particular, from the proof of the previous proposition we see that for solutions of linear ODE's with meromorphic coefficients the analytic continuation along paths avoiding singularities is indeed unique.

1.3 Homotopy invariance

Recall that two paths $\gamma_1, \gamma_2 : [0,1] \to D$ on a domain $D \subset \overline{\mathbb{C}}$ are called *(path)homotopic* if there exists a continuous map $h : [0,1] \times [0,1] \to D$ such that for every $t \in [0,1]$

$$h(0,t) = \gamma_1(t), \quad h(1,t) = \gamma_2(t), \quad \text{and}$$
$$h(s,0) = \gamma_1(0) = \gamma_2(0), \quad h(s,1) = \gamma_1(1) = \gamma_2(1) \text{ for all } s \in [0,1].$$

The map h is called a *homotopy* from γ_1 to γ_2. By a *loop* we mean a path with the same initial and end point, which we call the *base point* of the loop. Path homotopy is an equivalence relation in the set of loops with a fixed base point. The equivalence classes of loops on D with a chosen base point $z_0 \in D$ form the *fundamental group* $\pi_1(D, z_0)$ of the domain D, where the group operation is concatenation of representatives of the equivalence classes. If γ is a path we denote the reverse path by $\overleftarrow{\gamma}$.

Remark. We will always read concatenation of paths *from right to left*, meaning that the rightmost path is the first.

A useful observation is that for two homotopic paths the homotopy h can be obtained using elementary deformations in small convex domains. For a detailed proof the reader may consult [Sak10].

Definition 1.10. Let $\gamma_1, \gamma_2 : [0,1] \to D$ be two paths. We say that γ_1 is obtained from γ_2 by an *elementary deformation* if there exists a homotopy $h : [0,1] \times [0,1] \to D$, and $0 \le a < b \le 1$, and a convex domain $C \subset D$ such that for every $s \in [0,1]$

$$h(s,t) \in C \text{ for every } t \in [a,b],$$
$$h(s,t) = \gamma_1(t) = \gamma_2(t) \text{ for every } t \in [0,1] \setminus [a,b].$$

Lemma 1.11. *If γ, γ' are two homotopic paths in a domain D then there exists a sequence $\{\gamma_1, \dots, \gamma_n\}$ of paths in D such that $\gamma = \gamma_1$, $\gamma' = \gamma_n$ and for $i = 2, \dots, n$ the path γ_i is obtained from γ_{i-1} by an elementary deformation.*

Analytic continuation along paths depends only on the homotopy class. This is the *monodromy theorem* which we state without a proof. The assertion is quite intuitive but the proof somewhat technical. A detailed proof can be found in [Rud87].

Theorem 1.12. *Let D be a simply connected domain and (f, B) a function element such that $B \subset D$ and f can be analytically continued along every path in D starting from the center of B. Then there exists an analytic function g defined in D such that $g|_B = f$.*

In particular, we have the following.

Corollary 1.13. *Let (f, B) be a function element and γ_1, γ_2 two homotopic paths starting from the center of B. Suppose f is analytic in the domain enclosed by the two paths γ_1 and γ_2. If f has analytic continuations g_1, g_2 along γ_1, γ_2, respectively, then $g_1 = g_2$.*

Proof. It is enough to show that the analytic continuation of f along the path $\gamma := \gamma_1 \overleftarrow{\gamma_2}$ equals f, which means that the analytic continuation along a homotopically trivial loop is the same as along the constant path. By lemma 1.11 there exists a chain of elementary deformations in arbitrarily small convex sets from γ to the constant path, and the assertion follows from example 1.7, theorem 1.12 and uniqueness of the analytic continuation. □

1.4 Monodromy of linear ODE's

In proposition 1.9 we showed that solutions of linear ODE's having only finitely many singular points always have analytic continuations along paths which do not meet the singularities. Moreover, by the theory of ODE's, under certain conditions specified in the sequel, there exist local multivalued solutions around non-singular points. We will denote by $Sol(B(z_0, r))$ the space of solutions of the ODE defined in the vicinity $B(z_0, r)$ of z_0. Now the following linear operator can be consistently defined.

Definition 1.14. Assume that z_0 is a non-singular point of a linear ODE and let $\gamma : [0, 1] \to \mathbb{C}$ be a loop avoiding all the singular points of the ODE such that $\gamma(0) = \gamma(1) = z_0$. The *monodromy operator* along γ (based at z_0) is the linear map

$$M_\gamma \in Aut(Sol(B(z_0, r)))$$

which associates to every solution f of the ODE the analytic continuation of f along γ.

For a linear ODE the set $Sol(B(z_0, r))$ is a complex vector space, and thus with a choice of basis we can identify

$$Aut(Sol(B(z_0, r))) \cong GL_n(\mathbb{C}),$$

where $n \in \mathbb{N}$ is the order of the ODE. Then the monodromy operators form a subgroup of $GL_n(\mathbb{C})$ called the *monodromy group* of the ODE, based at z_0. By corollary 1.13 it is a linear representation of the fundamental group of the space of non-singular points of the ODE. Namely, for two loops γ, η with the same base point, avoiding the singularities,

$$M_\gamma M_\eta = M_{\gamma\eta}.$$

Naturally, the inverse operator of M_γ is the operator $M_{\overleftarrow{\gamma}}$ associated to the reverse loop, and the unit operator M_1 corresponds to the homotopy class of the constant path.

From the proof of proposition 1.9 it can also be seen that the monodromy groups at different base points are conjugate to each other. In particular, they are isomorphic, which implies that the choice of a base point does not affect the structure of the group. This is to be expected, since the same holds for the fundamental group of any connected domain. We will only consider ODE's whose set of non-singular points is connected, and denote the fundamental group of a connected domain D with any base point by $\pi_1(D)$.

2 The hypergeometric equation

We compute the monodromy of the hypergeometric equation (HGE), which is a second order linear ordinary differential equation of great importance. The equation has two singularities in the complex plane, and third at infinity. Hence the domain where HGE is analytically defined is $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$, and its fundamental group has two generators, namely the homotopy classes of loops around zero and one. In a way, HGE is the most general second order linear ODE having three regular singularities on the Riemann sphere $\overline{\mathbb{C}}$. Namely, any second order linear ODE with three regular singularities on $\overline{\mathbb{C}}$ can be converted into HGE by a suitable change of variables.

Solutions of HGE are long studied and relatively simple. We present two methods for solving the equation and computing its monodromy. The reason for this is that the monodromy of HGE is easy to compute by simply continuing analytically the power series solutions obtained from the Fröbenius method, whereas in computing the monodromy of the KZ-equations later on we will use another method based on "contour deformation". For that we in addition introduce integral expressions of solutions of HGE, presented by Leonhard Euler in 1748 in his book "Introduction to analysis of the infinite", and compute their monodromy by deforming and cutting up the paths of integration of each solution in a suitable way.

The purpose of this section is both to provide the first example of computing monodromy of linear ODE's and to introduce useful methods in solving the KZ-equations and computing their monodromy. It is crucial that solutions of KZ can also be written in integral form, moreover very similar to the Euler's solutions of HGE. The reader is not expected to have background from the theory of linear ODE's although some previous experience will be helpful.

2.1 Second order linear ODE's

Consider first the standard form of a second order linear homogeneous ordinary differential equation in one complex variable

$$\left\{ \frac{d^2}{dz^2} + p_1(z)\frac{d}{dz} + p_2(z) \right\} w(z) = 0, \tag{1}$$

where w is the unknown function and p_1 and p_2 are given functions. The analyticity of the solution of (1) is completely determined by the analyticity of the coefficients p_1 and p_2. Suppose p_1 and p_2 are single-valued analytic functions of z in a certain domain $D \subset \mathbb{C}$, except for a finite number of isolated points. Then the points in D can be classified into the following categories.

Ordinary points of the equation (1) are such points $z_0 \in D$ that both p_1 and p_2 are analytic at z_0 and its vicinity.

Singular points of the equation (1) are such points $z_0 \in D$ that either p_1 or p_2 is not analytic at z_0. Singular points can further be classified into regular and irregular singularities.

The singular point $z_0 \in D$ is a *regular singularity* if p_1 has a pole of order at most one and p_2 has a pole of order at most two at z_0. Otherwise z_0 is called an *irregular singularity.* An equation with all its singular points being regular is called *Fuchsian.*

The existence and uniqueness of solutions of (1) with fixed initial conditions around ordinary points is well known. The proof can be found in [WG89].

Theorem 2.1. *If p_1 and p_2 are single-valued and analytic in $B(z_0, r)$ for some $r > 0$, the equation (1) has a unique solution $w(z)$ in $B(z_0, r)$ satisfying the initial conditions*

$$w(z_0) = c_0, \quad w'(z_0) = c_1,$$

where c_0 and c_1 are arbitrary constants. In particular, $w(z)$ is single-valued and analytic in $B(z_0, r)$.

The singular points of (1) may also be singular points of the solutions. However, at regular singularities the solutions are of certain form. These are called *regular solutions.* They are multivalued functions which can be written as linear combinations of

$$w_1(z) = (z - z_0)^{\rho_1} \sum_{n=0}^{\infty} c_n (z - z_0)^n, \tag{r1}$$

where $c_0 \neq 0$, or

$$w_2(z) = \alpha w_1(z) \ln(z - z_0) + (z - z_0)^{\rho_2} \sum_{n=0}^{\infty} d_n (z - z_0)^n, \tag{r2}$$

where $d_n \neq 0$, in the vicinity of a point z_0. The coefficients ρ_1, ρ_2, c_n, d_n and α are to be determined. Moreover, regular solutions for (1) exist around a point z_0 only if z_0 is either ordinary or a regular singularity; see [WG89].

Theorem 2.2. *The necessary and sufficient conditions for (1) to have two regular linearly independent solutions in the vicinity $B(z_0, r)$ of its singular point z_0 are that*

$$(z - z_0)p_1(z) \ \text{ and } \ (z - z_0)^2 p_2(z)$$

are analytic in $B(z_0, r)$.

A convenient way of solving the equation (1) around regular singularities is the *Fröbenius method*. Then the obtained solutions can be analytically continued to all the ordinary points of (1). The procedure is standard; see e.g. [WG89]. The idea is to use an ansatz

$$w(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^{\rho+n},$$

with a possible branch point or pole at the regular singularity z_0. Differentiating w term by term and substituting to (1), recurrence relations for the coefficients are obtained. We present an example of this in the next section, solving the hypergeometric equation.

The equation (1) can naturally be extended into the Riemann sphere $\overline{\mathbb{C}}$. When considering the nature of the point ∞ we have to make the transformation of variables $t = \frac{1}{z}$. Then the equation (1) becomes

$$\left\{ \frac{d^2}{dt^2} + \left(\frac{2}{t} - \frac{1}{t^2} p_1 \left(\frac{1}{t} \right) \right) \frac{d}{dt} + \frac{1}{t^4} p_2 \left(\frac{1}{t} \right) \right\} u(t) = 0,$$

where $u(t) = u(\frac{1}{z}) = w(z)$.

2.2 Solutions of the hypergeometric equation

Next we will have a closer look at the hypergeometric equation (HGE). The solution of HGE around zero can be written as the *hypergeometric series* already introduced in 1655 by John Wallis in his book "Arithmetica Infinitorum", and further investigated among others by Leonhard Euler, Carl Friedrich Gauss, Ernst Kummer, Bernhard Riemann and Hermann Schwarz during the eighteenth and nineteenth centuries. We shall first solve the equation using the Fröbenius method, and compute the monodromy of the power series solutions.

The *hypergeometric equation*

$$\left\{ z(1-z)\frac{d^2}{dz^2} + (c - (a+b+1)z)\frac{d}{dz} - ab \right\} w(z) = 0 \qquad \text{(HGE)}$$

of parameters $a, b, c \in \mathbb{C}$, is the prototype of a Fuchsian equation with at most three regular singularities, namely 0, 1, and ∞. The singularities 0 and 1 can be easily found from the normal form of the equation

$$\left\{ \frac{d^2}{dz^2} + \frac{c - (a+b+1)z}{z(1-z)} \frac{d}{dz} - \frac{ab}{z(1-z)} \right\} w(z) = 0,$$

since the functions

$$p_1(z) = \frac{c - (a+b+1)z}{z(1-z)} \quad \text{and}$$

$$p_2(z) = -\frac{ab}{z(1-z)}$$

are clearly meromorphic in $\mathbb{C}$ with possible poles at 0 and 1. Moreover, the orders of the poles are at most one.

We use the Fröbenius method to solve the equation HGE around these points. Consider first the neighbourhood of zero.

2.2.1 Solutions around 0

Let

$$w(z) = \sum_{n=0}^{\infty} d_n z^{\rho+n},$$

where $d_0 \neq 0$. Differentiating w term by term and substituting to HGE we obtain

$$w'(z) = \sum_{n=0}^{\infty} d_n(\rho+n)z^{\rho+n-1} \quad \text{and}$$
$$w''(z) = \sum_{n=0}^{\infty} d_n(\rho+n)(\rho+n-1)z^{\rho+n-2}$$

and HGE becomes

$$0 = z(1-z)\sum_{n=0}^{\infty} d_n(\rho+n)(\rho+n-1)z^{\rho+n-2}$$
$$+ (c-(a+b+1)z)\sum_{n=0}^{\infty} d_n(\rho+n)z^{\rho+n-1} - ab\sum_{n=0}^{\infty} d_n z^{\rho+n}.$$

This reduces to

$$\begin{aligned}
0 = &\sum_{n=0}^{\infty} d_n(\rho+n)(\rho+n-1+c)z^{\rho+n-1} \\
&- \sum_{n=1}^{\infty} d_{n-1}\{(\rho+n-1)(\rho+n-2)+(a+b+1)(\rho+n-1)+ab\}z^{\rho+n-1} \\
= &\, d_0\rho(\rho-1+c)z^{\rho-1} + \sum_{n=1}^{\infty}\{d_n(\rho+n)(\rho+n-1+c) \\
&- d_{n-1}((\rho+n-1)(\rho+n-1+a+b)+ab)\}z^{\rho+n-1}.
\end{aligned}$$

By uniqueness of the power series expansion of analytic functions (compare with theorem 1.1) all the coefficients of different powers of z have to be identically zero,

$$\begin{cases} d_0\rho(\rho-1+c) = 0 \\ d_n(\rho+n)(\rho+n-1+c) - d_{n-1}((\rho+n-1)(\rho+n-1+a+b)+ab) = 0. \end{cases}$$

Vanishing of the coefficient of the lowest order term implies the *indicial equation*

$$\rho(\rho - 1 + c) = 0$$

which determines the values of the exponent, $\rho \in \{0, 1 - c\}$. Using the other equation we obtain a recurrence relation for the coefficients d_n, $n > 0$

$$d_n = \frac{(\rho + n + a - 1)(\rho + n + b - 1)}{(\rho + n)(\rho + n + c - 1)} d_{n-1},$$

for $c \notin \{\dots, -2, -1, 0\}$. If c is a nonpositive integer the recurrence relation might not have a solution. Hence, suppose $c \notin \{\dots, -2, -1, 0\}$.

Choose $\rho = 0$. If $a, b, c \notin \{\dots, -2, -1, 0\}$, we find that when $n \to \infty$,

$$\frac{d_{n-1}}{d_n} = \frac{n(n - 1 + c)}{(n - 1 + a)(n - 1 + b)} = \frac{(1 - \frac{1+c}{n})}{(1 - \frac{1+a}{n})(1 - \frac{1+b}{n})} \longrightarrow 1.$$

Hence the radius of convergence of the series is 1, that is, $w(z) = \sum_{n=0}^{\infty} d_n z^n$ is analytic in the open disc $B(0, 1)$. If a or $b \in \{\dots, -2, -1\}$, the series degenerates into a polynomial, which has infinite radius of convergence, and if a or $b = 0$ then the solution corresponding to $\rho = 0$ is constant. The function w is called the *hypergeometric series* and denoted by

$$F(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n, \tag{2}$$

where $(m)_n := m(m + 1) \cdots (m + n - 1) = \frac{\Gamma(m+n)}{\Gamma(m)}$.

Choosing $\rho = 1 - c$ we obtain the other solution in terms of the hypergeometric series,

$$w_{\rho=1-c}(z) = z^{1-c} F(a - c + 1, b - c + 1, 2 - c; z).$$

By proposition 1.9 the hypergeometric series can be analytically continued to the domain $\mathbb{C} \setminus \{1\}$ as a multivalued function. The analytic continuation is called the *hypergeometric function*, and denoted by the same symbol. It can also be thought of as a multivalued function on $\overline{\mathbb{C}} \setminus \{1, \infty\}$, with the two singularities 1 and ∞. In [WG89] it is shown that if $a, b \notin \{\dots, -2, -1, 0\}$, these two points are the branch points of $F(a, b, c; z)$. Hence $F(a, b, c; z)$ is a single-valued analytic function in $\overline{\mathbb{C}} \setminus [1, \infty]$.

Remark. It is clear that if the difference of the roots of the indicial equation is not an integer then the Fröbenius method produces two linearly independent (multivalued) solutions $\{w_1, w_2\}$ of HGE. These are obtained with different choices of the exponent, that is, the root of the indicial equation. Moreover, these solutions are regular solutions of the form (r1).

If the difference of the roots of the indicial equation happens to be an integer then existence of two solutions of the form (r1) is not guaranteed.

However, also in this case two linearly independent regular solutions are obtained. One of them is of the form (r1) and the other of the form (r2). The idea is to consider derivatives of the coefficients of the ansatz, which implies arise of logarithms; for details the reader may consult [WG89].

Consider a solution of HGE of the form (r2),

$$w(z) = \alpha \ln(z - z_0)(z - z_0)^{\rho_1} \sum_{n=0}^{\infty} c_n (z - z_0)^n + (z - z_0)^{\rho_2} \sum_{n=0}^{\infty} d_n (z - z_0)^n,$$

where $d_n \neq 0$, ρ_1, ρ_2, c_n, d_n and α are complex numbers, in the vicinity of a point z_0. Because of the logarithm appearing in the expression, analytic continuation of w around the singularity z_0 yields also a *linear increment* in the solution, whereas solutions of the form (r1) only get *"phase factors"* (recall example 1.6).

We will only consider the generic case, that is when the difference of the roots of the indicial equation is not an integer. This ensures that no linear increments arise in analytic continuation of the solutions around singularities, and in particular that the monodromy representation, which we will compute in the next section, is diagonalizable.

2.2.2 Solutions around 1

The procedure is identical to the previous case. The Fröbenius solutions are obtained using the ansatz

$$w(z) = \sum_{n=0}^{\infty} d_n (z - 1)^{\rho + n},$$

where $d_0 \neq 0$. The indicial equation turns out to be

$$\rho(\rho + a + b - c) = 0,$$

with the roots $\rho_1 = 0$ and $\rho_2 = c - a - b$. The analysis of linear independence of the solutions is as above, the generic case being $c - a - b \notin \mathbb{Z}$. The radius of convergence of the generic solutions is at least one.

2.2.3 Solutions around ∞

We make the transformation of variables $t = \frac{1}{z}$. Then HGE becomes

$$\left\{ \frac{d^2}{dt^2} + \frac{1}{t} \left(2 - \frac{ct - (a + b + 1)}{t - 1} \right) \frac{d}{dt} + \frac{ab}{t^2(1 - t)} \right\} u(t) = 0, \qquad \text{(HGE')}$$

where $u(t) = u(\frac{1}{z}) = w(z)$. The functions

$$q_1(t) = \frac{1}{t}\left(2 - \frac{ct - (a+b+1)}{t-1}\right) \quad \text{and}$$
$$q_2(t) = \frac{ab}{t^2(1-t)}$$

are meromorphic in $\mathbb{C}$, and have possible poles at 0 of orders at most one and two, respectively. So also the point ∞ is at most a regular singularity of HGE.

The solutions for HGE are obtained by analysing the solutions of HGE' around 0. The indicial equation has the roots $\rho_1 = a$ and $\rho_2 = b$, and the solution obtained by the Fröbenius method is of the form

$$w(z) = \sum_{n=0}^{\infty} d_n z^{-(\rho+n)},$$

where $d_0 \neq 0$. Again, the analysis of linear independence of the solutions is as before, the generic case being $a - b \notin \mathbb{Z}$. The domain of convergence of the generic solutions is at least $\mathbb{C} \setminus \overline{B(0,1)}$.

2.3 Monodromy of the hypergeometric equation

In the sequel we assume that $1-c, c-a-b, a-b \notin \mathbb{Z}$. This ensures that for the different choices ρ_1 and ρ_2 of the exponent ρ we obtain linearly independent solutions of the form

$$w(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^{\rho+n},$$

and that the recurrence relation for the coefficients has a unique solution. Moreover, as we will see, the monodromy of HGE is diagonalizable in this case.

Let $\{w_1^{(0)}, w_2^{(0)}\}$, $\{w_1^{(1)}, w_2^{(1)}\}$ and $\{w_1^{(\infty)}, w_2^{(\infty)}\}$ be the linear bases of solutions of HGE obtained by the Fröbenius method around the points 0, 1 and ∞, respectively. We want to study what happens to the solutions when they are continued analytically along closed loops in $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. We first note that HGE is invariant under analytic continuation.

Proposition 2.3. *Let $z_0 \in \mathbb{C} \setminus A$ and let w be a solution of a Fuchsian equation of type (1) in the disc $B(z_0, r)$, where A is the set of regular singular points of the equation. Assume that w has an analytic continuation $\tilde{w}$ along a path γ in $\mathbb{C} \setminus A$. Then $\tilde{w}$ is a solution of the equation in the domain where it is defined.*

Proof. Define the function

$$G(z) := \left\{ \frac{d^2}{dz^2} + p_1(z)\frac{d}{dz} + p_2(z) \right\} w(z)$$

which is meromorphic in $B(z_0, r)$. Since the poles of G are in A, it has a meromorphic continuation $\tilde{G}$ along γ. Moreover, since $G \equiv 0$ in $B(z_0, r)$, we have that also $\tilde{G} \equiv 0$, by meromorphicity and proposition 1.2. But

$$\tilde{G}(z) = \left\{ \frac{d^2}{dz^2} + p_1(z)\frac{d}{dz} + p_2(z) \right\} \tilde{w}(z),$$

from which the assertion follows. □

It can also similarly be shown that the analytic continuations of two linearly independent solutions of a Fuchsian equation of type (1) are linearly independent. Notice also that the above proof generalises to Fuchsian ODE's of any order.

In particular, the previous theorem holds for HGE. Moreover, we can compute the analytic continuations of the bases $\{w_1^{(0)}, w_2^{(0)}\}$, $\{w_1^{(1)}, w_2^{(1)}\}$ and $\{w_1^{(\infty)}, w_2^{(\infty)}\}$ around the points 0, 1 and ∞, respectively. Define the loops

$$\gamma_0, \gamma_1, \gamma_\infty : [0,1] \to \mathbb{C}$$

around the singularities as in the picture. Notice that the composed path $\gamma_\infty\gamma_1\gamma_0$ is homotopic to the constant path. Notice also that by proposition 1.9 and theorem 1.12 monodromy is homotopy invariant, whence the paths $\gamma_0, \gamma_1, \gamma_\infty$ need not necessarily lie in the domains of convergence of the corresponding power series solutions of HGE. In particular, the loop γ_∞ in the picture represents the homotopy class of a loop around the point ∞ in $\overline{\mathbb{C}} \setminus \overline{B(0,1)}$.

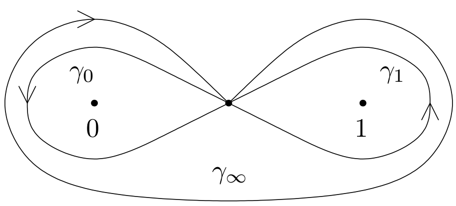


Figure 3: Loops representing elements of the fundamental group of the domain $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$

Using examples 1.6 and 1.8 we obtain the following.
1. The analytic continuations of

$$w_1^{(0)}(z) := \sum_{n=0}^{\infty} (d_n)^{(0)}_{\rho=0} z^n, \quad w_2^{(0)}(z) := z^{1-c} \sum_{n=0}^{\infty} (d_n)^{(0)}_{\rho=1-c} z^n$$

around γ_0 are

$$\widetilde{w_1^{(0)}}(z) = w_1^{(0)}(z), \quad \widetilde{w_2^{(0)}}(z) = e^{2\pi \mathrm{i}(1-c)} w_2^{(0)}(z),$$

2. the analytic continuations of

$$w_1^{(1)}(z) := \sum_{n=0}^{\infty} (d_n)^{(1)}_{\rho=0} (z-1)^n, \quad w_2^{(1)}(z) := (z-1)^{c-a-b} \sum_{n=0}^{\infty} (d_n)^{(1)}_{\rho=c-a-b} (z-1)^n$$

around γ_1 are

$$\widetilde{w_1^{(1)}}(z) = w_1^{(1)}(z), \quad \widetilde{w_2^{(1)}}(z) = e^{2\pi \mathrm{i}(c-a-b)} w_2^{(1)}(z),$$

3. and the analytic continuations of

$$w_1^{(\infty)}(z) := z^{-a} \sum_{n=0}^{\infty} (d_n)^{(\infty)}_{\rho=a} z^{-n}, \quad w_2^{(\infty)}(z) := z^{-b} \sum_{n=0}^{\infty} (d_n)^{(\infty)}_{\rho=b} z^{-n}$$

around γ_∞ are

$$\widetilde{w_1^{(\infty)}}(z) = e^{2\pi \mathrm{i} a} w_1^{(\infty)}(z), \quad \widetilde{w_2^{(\infty)}}(z) = e^{2\pi \mathrm{i} b} w_2^{(\infty)}(z).$$

To shorten the notation, define

$$A := e^{\pi \mathrm{i} a}, \quad B := e^{\pi \mathrm{i} b}, \quad C := e^{\pi \mathrm{i} c}.$$

In this notation the eigenvalues of the monodromy operators M_0, M_1 and M_∞, corresponding to the loops γ_0, γ_1 and γ_∞, are $\{1, C^{-2}\}$, $\{1, A^{-2}B^{-2}C^2\}$ and $\{A^2, B^2\}$, respectively. Notice that if $1-c, c-a-b, a-b \notin \mathbb{Z}$ then the monodromy matrices are diagonalizable.

The fundamental group of $\mathbb{C} \setminus \{0,1\}$ is the free group of two generators, which are the homotopy classes of the loops γ_0 and γ_1. If we consider instead the domain $\overline{\mathbb{C}} \setminus \{0,1,\infty\}$, it is easy to see that the fundamental group can be presented by the generators $[\gamma_0]$ and $[\gamma_1]$, and the relation

$$[\gamma_\infty][\gamma_1][\gamma_0] = 1.$$

The monodromy group has the same relations, whence

$$M_\infty M_1 M_0 = id \in Aut(Sol(U)),$$

where U is a simply connected domain where the solutions are defined. This information is sufficient for computing the monodromy of HGE exactly in the generic case $1-c, c-a-b, a-b \notin \mathbb{Z}$. Next we will compute the monodromy.

Choose the basis $\{w_1^{(\infty)}, w_2^{(\infty)}\}$ for the space of the solutions of HGE, in which the operator M_∞ is diagonal,

$$M_\infty = \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix}$$

Write for $j = 0, 1$

$$M_j = \begin{pmatrix} u_j & v_j \\ w_j & x_j \end{pmatrix}$$

Since we know the eigenvalues of M_0 and M_1, we have the relations

$$\begin{cases} 1 + C^{-2} = TrM_0 = u_0 + x_0 \\ 1 + A^{-2}B^{-2}C^2 = TrM_1 = u_1 + x_1 \end{cases}$$

$$\begin{cases} C^{-2} = \det M_0 = u_0x_0 - v_0w_0 \\ A^{-2}B^{-2}C^2 = \det M_1 = u_1x_1 - v_1w_1 \end{cases}$$

From the relation $M_\infty M_1 M_0 = id$ we obtain

$$\begin{cases} A^2(u_1u_0 + v_1w_0) = 1 \\ w_1u_0 + x_1w_0 = 0 \\ u_1v_0 + v_1x_0 = 0 \\ B^2(w_1v_0 + x_1x_0) = 1 \end{cases}$$

We have eight unknown complex constants u_j, v_j, w_j, x_j, $j = 0, 1$, and eight relations between them. It turns out that the equations have a solution, with respect to a scaling parameter $\lambda \in \mathbb{R}\backslash\{0\}$. The parameter λ depends on the choice of basis for the local solutions $Sol(U)$. It corresponds to mutual normalisation of the two basis vectors around ∞.

The solution can be written as

$$M_0 = \frac{1}{C^2(A^2 - B^2)} \begin{pmatrix} (B^2(A^2 - 1 - C^2) + C^2) & (A^2 - 1)(B^2 - C^2)\lambda^{-1} \\ (B^2 - 1)(C^2 - A^2)\lambda & A^2(1 - B^2 + C^2) - C^2 \end{pmatrix}$$

$$M_1 = \frac{1}{A^2 - B^2} \begin{pmatrix} 1 - B^2 + C^2(1 - A^{-2}) & (A^{-2} - 1)(B^2 - C^2)\lambda^{-1} \\ (A^2 - C^2)(1 - B^{-2})\lambda & A^2 - 1 + C^2(B^{-2} - 1) \end{pmatrix}$$

We have now computed the monodromy group of HGE in the basis where M_∞ is diagonal.

2.4 Integral representations of the hypergeometric function

Since for $a, b \notin \{\ldots, -2, -1, 0\}$ the radius of convergence of the hypergeometric series is 1, applications of the series representation of the hypergeometric function are quite limited. Luckily, the function has various integral representations, which are more useful. One of them is the *Euler integral*

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt. \tag{3}$$

Making the change of variables $w = zt$, where $z \in (0,1)$, the integral (3) can be written as

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^z z^{1-c}w^{b-1}(z-w)^{c-b-1}(1-w)^{-a}dw. \tag{4}$$

Note that the integral is convergent at $w = 0$ only if $Re(b-1) > -1$ and at $w = z$ only if $Re(c-b-1) > -1$, which means that (4) is well defined for $Re(c) > Re(b) > 0$. Moreover, convergence of the integral (3) requires also that $Re(-a) > -1$, that is $Re(a) < 1$. These restrictions can however be dropped, using another path of integration. We will prove this in the next section.

For $Re(a) < 1$ and $Re(c) > Re(b) > 0$ the integral (3) is indeed a representation of the hypergeometric function; by existence and uniqueness of ODE's one only needs to check that it satisfies HGE and that it tends to 1 as $z \to 0$. The limit at zero is easy to check, since

$$\begin{aligned}
&\lim_{z\to 0} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \int_0^1 t^{b-1}(1-t)^{c-b-1}dt \\
&= B(b, c-b) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(b+(c-b))} = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}
\end{aligned}$$

by properties of the beta function.

To see that the integral (3) is a solution of HGE, observe that

$$\begin{aligned}
&\left\{z(1-z)\frac{d^2}{dz^2} + (c-(a+b+1)z)\frac{d}{dz} - ab\right\} \\
&\cdot \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt \\
&= \int_0^1 \left\{z(1-z)\frac{d^2}{dz^2} + (c-(a+b+1)z)\frac{d}{dz} - ab\right\} \\
&\cdot(t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a})dt \\
&= \int_0^1 a(1-t)^{c-b-1}t^{b-1}(1-tz)^{-2-a} \\
&\cdot(b(tz-1) + t(c+(1+a)(t-1)z - ctz))dt \\
&= \int_0^1 \frac{d}{dt}\left(-a(1-t)^{c-b}t^b(1-tz)^{-1-a}\right)dt \\
&= (-a(1-t)^{c-b}t^b(1-tz)^{-1-a})_{|t=1} - (-a(1-t)^{c-b}t^b(1-tz)^{-1-a})_{|t=0} = 0.
\end{aligned}$$

For a general procedure of integral solutions one may consult [WG89].

We will also need another integral solution of HGE, namely

$$\widetilde{F}(a,b,c;z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_z^1 z^{1-c} w^{b-1} (z-w)^{c-b-1} (1-w)^{-a} dw.$$

For this expression, the integral is convergent at $w = z$ only if $Re(a) < 1$ and at $w = 1$ only if $Re(c) > Re(b)$. These restrictions can also be dropped, by similar reasoning as for $F(a,b,c;z)$.

It can be shown that the functions $F(a,b,c;z)$ and $\widetilde{F}(a,b,c;z)$ are linearly independent for generic values of the parameters $a,b,c \in \mathbb{C}$. The linear independence of F and $\widetilde{F}$ is considered later, since it follows from the monodromy computation which is presented in the next section.

2.5 Monodromy of the hypergeometric equation revisited

In this section we denote the integrand of (4), which is a multivalued function in the domain $\{(z,w) \in \mathbb{C}^2 | z \neq 0, w \neq 0, w \neq 1, z \neq w\}$, by

$$\varphi(z,w) := z^{1-c} w^{b-1} (z-w)^{c-b-1} (1-w)^{-a},$$

and the two solutions of HGE by

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^z \varphi(z,w) dw$$

and

$$\widetilde{F}(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_z^1 \widetilde{\varphi}(z,w) dw,$$

where the tilde in the latter expression reminds that the branches of φ and $\widetilde{\varphi}$ may be differently chosen. We shall compute the monodromy of these functions using a “contour deformation“ method. The “phase factors“ are obtained using the result of example 1.6 concerning the power function, and we use the shorthand notation $A = e^{\pi \mathrm{i} a}, B = e^{\pi \mathrm{i} b}, C = e^{\pi \mathrm{i} c}$.

Suppose first that $a, b, c \in \mathbb{R}$. The branch of $\varphi(z,w)$ is chosen to be such that $\varphi(z,w) > 0$ when $w \in [0,z]$. The function $\widetilde{\varphi}(z,w)$ is obtained from $\varphi(z,w)$ by interchanging the points $w \in [0,z]$ and z so that w moves below z, and dividing the result by $e^{-\pi \mathrm{i}(c-b-1)}$ in order to have $\widetilde{\varphi}(z,w) > 0$ when $w \in [z,1]$. These choices of branches guarantee that the functions $F(a,b,c;z)$ and $\widetilde{F}(a,b,c;z)$ are positive and real on the interval $[0,1]$.

Define the loops $\omega_0, \omega_z^{\pm}, \omega_1 : [0,1] \to \mathbb{C} \setminus \{0,1\}$ by

$$\begin{aligned} \omega_0(s) &:= re^{2\pi \mathrm{i} s}, \\ \omega_z(s)^{\pm} &:= z \pm re^{\pi \mathrm{i} s}, \\ \omega_1(s) &:= 1 - re^{2\pi \mathrm{i} s}, \end{aligned}$$

where $r > 0$ is small and $0 < z < 1$.

Lemma 2.4. *Suppose $a, b, c \in \mathbb{C}$ satisfy $Re(c) > Re(b) > 0$ and $Re(a) < 1$. Then as $r \to 0$*

$$\int_{\omega_0} \varphi(z,w)dw \to 0,$$
$$\int_{\omega_1} \widetilde{\varphi}(z,w)dw \to 0,$$
$$\int_{\omega_z^{\pm}} \varphi(z,w)dw \to 0.$$

Proof. When r is small enough, there exist bounded functions $C_0(r), C_z(r)$ and $C_1(r)$ such that

$$\int_{\omega_0} \varphi(z,w)dw \leq 2\pi r \left|C_0(r)\right| r^{b-1},$$
$$\int_{\omega_z^{\pm}} \varphi(z,w)dw \leq \pi r \left|C_z(r)\right| r^{c-b-1},$$
$$\int_{\omega_1} \widetilde{\varphi}(z,w)dw \leq 2\pi r \left|C_1(r)\right| r^{-a}.$$

Taking $r \to 0$ the right hand sides tend to zero, since $Re(c-b), Re(b) > 0$ and $Re(1-a) > 0$. □

Next we extend definition of the solutions $F(a, b, c; z)$ and $\widetilde{F}(a, b, c; z)$ to “almost all“ values of the parameters $a, b, c \in \mathbb{C}$.

Instead of the line segment $[0, z]$ in the integral (4) take for path of integration the *Pochammer contour*

$$\gamma := (\tau_\epsilon^{(0)})^{-1}\sigma^{-1}(\tau_\epsilon^{(z)})^{-1}\sigma\tau_\epsilon^{(0)}\sigma^{-1}\tau_\epsilon^{(z)}\sigma,$$

where $0 < \epsilon << z < 1$ and $\sigma, \tau_\epsilon^{(0)}, \tau_{z-\epsilon} : [0,1] \to \mathbb{C}$ are the paths

$$\sigma(s) := \epsilon(1-s) + (z-\epsilon)s,$$
$$\tau_\epsilon^{(0)}(s) := \epsilon e^{2\pi \mathrm{i} s},$$
$$\tau_\epsilon^{(z)}(s) := z - \epsilon e^{2\pi \mathrm{i} s}.$$

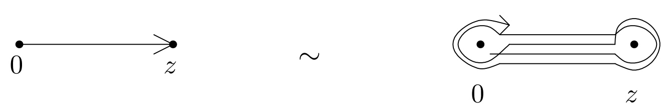


Figure 4: The integration segment $[0, z]$ corresponds to the Pochammer contour γ.

Replacing the segment $[0, z]$ by the Pochammer contour γ the value of the function $F(a, b, c; z)$ does not change up to a multiplicative constant. Namely, by lemma 2.4 the integrals over $\tau_\epsilon^{(0)}$ and $\tau_\epsilon^{(z)}$ vanish when $\epsilon \to 0$, and it follows that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_\gamma \varphi(z,w)dw\\&= (1 - e^{2\pi \mathrm{i}(c-b-1)} + e^{2\pi \mathrm{i}(c-b-1+b-1)} - e^{2\pi \mathrm{i}(b-1)})F(a,b,c;z)\\&= (1 - B^{-2})(C^2 - B^2)F(a,b,c;z) \neq 0\end{aligned}$$

for $b, c - b \notin \mathbb{Z}$. Moreover, the value of the integral does not depend on ϵ. Hence

$$\begin{aligned}F(a,b,c;z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^z \varphi(z,w)dw\\&= \frac{1}{(1-B^{-2})(C^2-B^2)}\lim_{\epsilon\to 0}\int_\gamma \varphi(z,w)dw,\end{aligned}$$

and by analytic continuation on the variables $a, b, c \in \mathbb{C}$ the function $F(a, b, c; z)$ is well defined for $b, c - b \notin \mathbb{Z}$. Similarly one can check that $\widetilde{F}(a, b, c; z)$ is well defined for $a, c - b \notin \mathbb{Z}$.

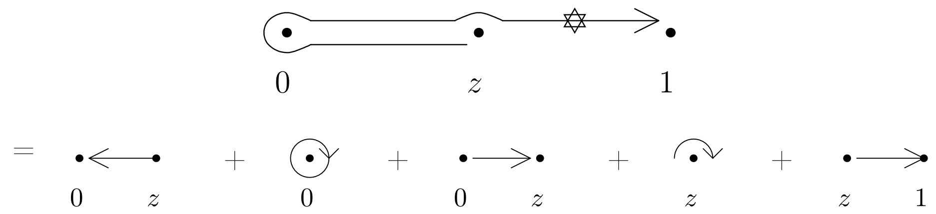
Consider now the effect of monodromy on the solutions $F(a, b, c; z)$ and $\widetilde{F}(a, b, c; z)$ of HGE, with the choices of branches explained above.

Monodromy around 0. Let $\gamma_0 : [0, 1] \to \mathbb{C} \setminus \{0, 1\}$ be the path $\gamma_0(s) := ze^{2\pi \mathrm{i} s}$. We compute the monodromy operator M_0 in the basis $\{F(a, b, c; z), \widetilde{F}(a, b, c; z)\}$. Later we will see that the solutions $F(a, b, c; z)$ and $\widetilde{F}(a, b, c; z)$ of HGE are linearly independent, and indeed form a basis for the solutions.

Since $F(a, b, c; z)$ is analytic at zero

$$M_0F(a,b,c;z) = F(a,b,c;z).$$

For $\widetilde{F}(a, b, c; z)$, we cut the path of integration into small pieces where the phase of the integrand remains constant. The effect of M_0 on the path of integration $[z, 1]$ is as in the following picture. The star indicates the point where the branch of the integrand is chosen, before the action of γ_0.



Taking into account all the "phase factors" the picture represents the following function.

$$
\begin{aligned}
&- e^{2\pi \mathrm{i}(1-c)} e^{\pi \mathrm{i}(c-b-1)} e^{2\pi \mathrm{i}(b-1)} \int_r^z \varphi(z,w)dw - \int_{\omega_0} \varphi(z,w)dw \\
&+ e^{2\pi \mathrm{i}(1-c)} e^{\pi \mathrm{i}(c-b-1)} \int_r^{z-r} \varphi(z,w)dw - \int_{\omega_z^+} \varphi(z,w)dw \\
&+ e^{2\pi \mathrm{i}(1-c)} \int_{z+r}^1 \widetilde{\varphi}(z,w)dw \\
= {} & C^{-1}B \int_r^z \varphi(z,w)dw - \int_{\omega_0} \varphi(z,w)dw - C^{-1}B^{-1} \int_r^{z-r} \varphi(z,w)dw \\
&- \int_{\omega_z^+} \varphi(z,w)dw + C^{-2} \int_{z+r}^1 \widetilde{\varphi}(z,w)dw \\
\underset{r\to 0}{\longrightarrow} {} & C^{-1}(B - B^{-1})F(a,b,c;z) + C^{-2}\widetilde{F}(a,b,c;z) = M_0 \widetilde{F}(a,b,c;z).
\end{aligned}
$$

Hence

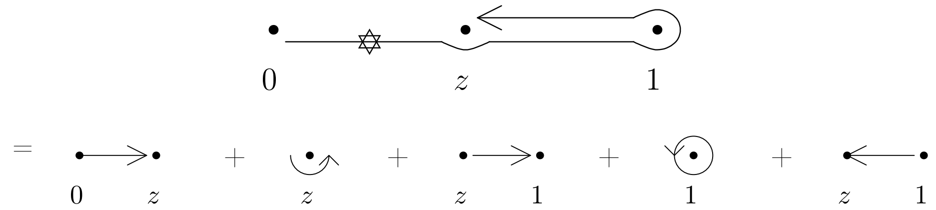
$$
M_0 = \begin{pmatrix} 1 & (B - B^{-1})C^{-1} \\ 0 & C^{-2} \end{pmatrix}
$$

Notice that if $b \notin \mathbb{Z}$, then $\widetilde{F}(a,b,c;z)$ is not an eigenvector of M_0. But $F(a,b,c;z)$ is always an eigenvector of M_0, and all vectors that are linearly dependent on $F(a,b,c;z)$ are its scalar multiples. Hence we deduce that if $b \notin \mathbb{Z}$, the integral solutions $F(a,b,c;z)$ and $\widetilde{F}(a,b,c;z)$ of HGE are linearly independent.

Notice also that for $c \in \mathbb{Z}$ and $b \notin \mathbb{Z}$ the matrix M_0 is not diagonalizable, and in fact then the above expression of M_0 is in the Jordan normal form. Recall that when $c \in \mathbb{Z}$, one of the linearly independent solutions of HGE contains logarithmic terms, whence it gets a linear increment in analytic continuation. This implies that the monodromy operator around the singularity zero is not diagonalizable.

Monodromy around 1. Let $\gamma_1 : [0,1] \to \mathbb{C} \setminus \{0,1\}$ be the path $\gamma_1(s) := 1 - (1-z)e^{2\pi \mathrm{i} s}$. As above, we cut the path of integration into small pieces where the phase of the integrand remains constant.

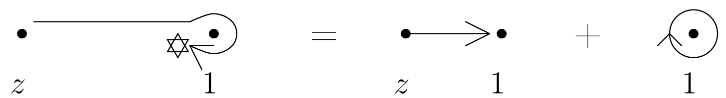
For $F(a,b,c;z)$, the effect of M_1 on the path of integration $[0,z]$ is as in the following picture. The star indicates the point where the branch of the integrand is chosen, before the action of γ_1.



Taking into account all the "phase factors" the picture represents the following function.

$$\begin{aligned}
&\int_0^{z-r}\varphi(z,w)dw+\int_{\omega_z^-}\varphi(z,w)dw+e^{\pi i(c-b-1)}\int_{z+r}^{1-r}\widetilde{\varphi}(z,w)dw\\
&+\int_{\omega_1}\widetilde{\varphi}(z,w)dw-e^{\pi i(c-b-1)}e^{-2\pi i a}\int_z^{1-r}\widetilde{\varphi}(z,w)dw\\
=&\int_0^{z-r}\varphi(z,w)dw+\int_{\omega_z^-}\varphi(z,w)dw-CB^{-1}\int_{z+r}^{1-r}\widetilde{\varphi}(z,w)dw\\
&+\int_{\omega_1}\widetilde{\varphi}(z,w)dw+CB^{-1}A^{-2}\int_z^{1-r}\widetilde{\varphi}(z,w)dw\\
\underset{r\to 0}{\longrightarrow}&\ F(a,b,c;z)+CB^{-1}(A^{-2}-1)\widetilde{F}(a,b,c;z)=M_1F(a,b,c;z).
\end{aligned}$$

For $\widetilde{F}(a,b,c;z)$, the effect of M_1 on the path of integration $[z,1]$ is the following.



The "phase factors" yield

$$\begin{aligned}
&e^{2\pi i(c-b-1)}e^{-2\pi i a}\int_z^{1-r}\widetilde{\varphi}(z,w)dw-\int_{\omega_1}\widetilde{\varphi}(z,w)dw\\
=&\ C^2B^{-2}A^{-2}\int_z^{1-r}\widetilde{\varphi}(z,w)dw-\int_{\omega_1}\widetilde{\varphi}(z,w)dw\\
\underset{r\to 0}{\longrightarrow}&\ C^2B^{-2}A^{-2}\widetilde{F}(a,b,c;z)=M_1\widetilde{F}(a,b,c;z).
\end{aligned}$$

Hence

$$M_1=\begin{pmatrix}1 & 0\\ (A^{-2}-1)B^{-1}C & A^{-2}B^{-2}C^2\end{pmatrix}$$

Notice that if $a \notin \mathbb{Z}$ then $F(a,b,c;z)$ is not an eigenvector of M_1, and as before, we deduce that if $a \notin \mathbb{Z}$, then the integral solutions $F(a,b,c;z)$ and $\widetilde{F}(a,b,c;z)$ of HGE are linearly independent. Moreover, for $c-b-a \in \mathbb{Z}$ and $a \notin \mathbb{Z}$ the matrix M_1 is not diagonalizable, and then the above expression of M_1 is in the lower diagonal Jordan normal form.

Monodromy around ∞. The effect of M_∞ on the functions $F(a,b,c;z)$ and $\widetilde{F}(a,b,c;z)$ can either be computed similarly as above, using a loop around ∞, or using the relation

$$[\gamma_\infty][\gamma_1][\gamma_0] = 1$$

in the fundamental group of $\overline{\mathbb{C}} \setminus \{0,1,\infty\}$. The result is

$$M_\infty = \begin{pmatrix} A+B-AB & A^2BC^{-1}(1-B^2) \\ (A^2-1)BC & A^2B^2 \end{pmatrix}$$

Conclusion. We notice that the eigenvalues of the monodromy matrices obtained using “contour deformation“ are the same as in section 2.3. In fact, the monodromy matrices are exactly the same, which can be seen by change of basis. In a basis such that M_∞ is diagonal, the monodromy matrices are

$$M_0 = \frac{1}{C^2(A^2-B^2)} \begin{pmatrix} (B^2(A^2-1-C^2)+C^2) & (A^2-1)(B^2-C^2)\lambda^{-1} \\ (B^2-1)(C^2-A^2)\lambda & A^2(1-B^2+C^2)-C^2 \end{pmatrix}$$
$$M_1 = \frac{1}{A^2-B^2} \begin{pmatrix} 1-B^2+C^2(1-A^{-2}) & (A^{-2}-1)(B^2-C^2)\lambda^{-1} \\ (A^2-C^2)(1-B^{-2})\lambda & A^2-1+C^2(B^{-2}-1) \end{pmatrix}$$
$$M_\infty = \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix}$$

where λ is a scaling parameter. For M_∞ to be diagonalizable we need to require that $a-b \notin \mathbb{Z}$.

Recall that we assumed the parameters a,b,c to be real valued. Since the solutions $F(a,b,c;z)$ and $\widetilde{F}(a,b,c;z)$ of HGE are analytic functions in each of the variables $a,b,c \in \mathbb{C}$, analytic continuation in a,b and c yields well defined solutions for $a,b,a-b,b-c \notin \mathbb{Z}$. Moreover, by example 1.7 the above matrices represent the monodromy of the hypergeometric equation not only for $a,b,c \in \mathbb{R}$ but also for $a,b,a-b,b-c \in \mathbb{C} \setminus \mathbb{Z}$. However, since $\mathbb{Z}$ is very sparse in $\mathbb{C}$, we deduce that the result is valid for “almost all“ $a,b,c \in \mathbb{C}$.

3 Connections and Pfaffian systems

In this section we introduce connections associated to Pfaffian systems, which are systems of linear partial differential equations defined analytically on a complex manifold M. Usually the manifold is a domain in $\mathbb{C}$, and the complex structure is natural. A Pfaffian system defines a connection on a trivial vector bundle over M. The hypergeometric equation is a Pfaffian system on the one dimensional manifold $\mathbb{C} \setminus \{0, 1\}$, and the KZ-equations provide a less trivial example of a Pfaffian system.

The main results of this section are the following. Firstly, a Pfaffian system which induces a flat connection over M always has local fundamental solutions. Conversely, existence of a fundamental solution implies flatness of the induced connection. Secondly, in this case a monodromy representation associated to the equation can be consistently defined via parallel transport of solutions along smooth paths. Namely, consistency of the monodromy representation requires homotopy invariance of the parallel transport, which is guaranteed for flat connections. In the one dimensional case parallel transport is simply analytic continuation along paths, and the idea is to be able to analytically continue solutions of systems in higher dimensions as well.

For the reader it is useful to be somewhat familiar with smooth manifolds and their tangent and cotangent bundles. We recall the definitions of complex manifolds and holomorphic vector bundles, in analogy with the theory of smooth manifolds found e.g. in [Lee97] and [Lee03]. The reader might also want to take a look at [For91] for more details on complex structures. Knowledge of the theory of connections on vector bundles is not required, and we shall prove the used properties of connections in detail. After the background being covered, we will finally define monodromy of flat holomorphic vector bundles as a representation of the fundamental group acting on the fibers of the bundle.

Definition of monodromy associated to vector bundles enables us to consider monodromy of Pfaffian systems and especially of KZ. It turns out that the system KZ has local fundamental solutions, and that it indeed induces a flat connection over the complex manifold where it is defined, which further guarantees that the monodromy of KZ is well defined. This allows us to even speak of computing the monodromy of solutions of KZ.

3.1 Complex manifolds

Recall that a *topological n-manifold M* is a topological Hausdorff space whose topology has a countable basis, such that M is locally homeomorphic to $\mathbb{R}^n$. An *atlas* on M is a collection $\{U_i, x_i\}_{i \in I}$ of charts, where $\{U_i\}$ is an open cover of M and for every $i \in I$ the map

$$x_i = (x_i^1, \ldots, x_i^n) : U_i \to x_i(U_i)$$

is a homeomorphism into an open subset $x_i(U_i) \subset \mathbb{R}^n$.

If M admits a smooth structure, that is, if whenever $U_i \cap U_j \neq \emptyset$, the transition maps

$$(x_i \circ x_j^{-1})|_{x_j(U_i \cap U_j)} : x_j(U_i \cap U_j) \to x_i(U_i \cap U_j)$$

are C^∞-diffeomorphisms in $\mathbb{R}^n$, we say that M is a *smooth manifold.* Holomorphic manifolds with a complex structure can be defined in a similar manner.

Definition 3.1. A *complex (holomorphic) n-manifold* M is a topological Hausdorff space whose topology has a countable basis, such that M is locally homeomorphic to $\mathbb{C}^n$. Notice that as a real manifold M has dimension $2n$.

A *complex (holomorphic) atlas* on M is a collection $\{U_i, z_i\}_{i \in I}$ of charts, where $\{U_i\}$ is an open cover of M and for every $i \in I$ the map

$$z_i = (z_i^1, \ldots, z_i^n) : U_i \to z_i(U_i)$$

is a homeomorphism into an open subset $z_i(U_i) \subset \mathbb{C}^n$, such that the following compatibility condition holds:
whenever $U_i \cap U_j \neq \emptyset$, the component functions of the transition maps

$$(z_i \circ z_j^{-1})|_{z_j(U_i \cap U_j)} : z_j(U_i \cap U_j) \to z_i(U_i \cap U_j)$$

are holomorphic.

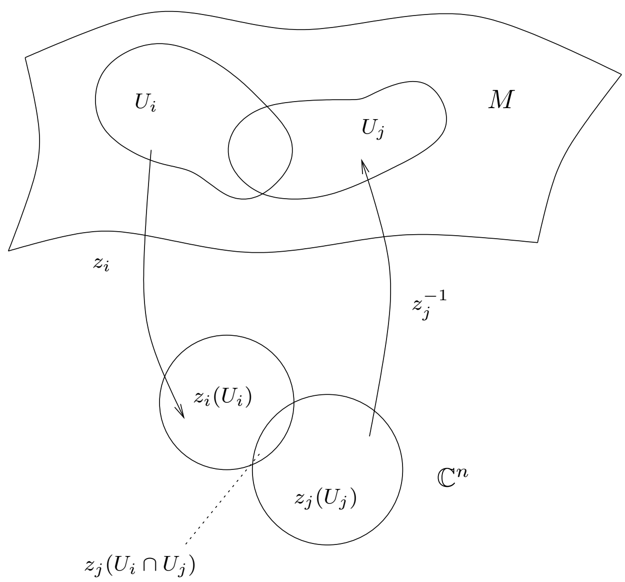
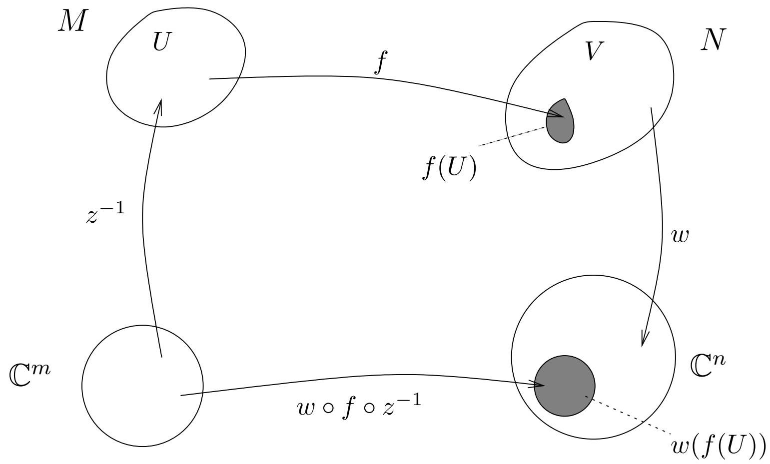


Figure 5: Two intersecting charts

We call a map $f : M \to N$ between two complex manifolds *holomorphic* if the corresponding functions

$$w \circ f \circ z^{-1} : z(U) \to w(f(U)),$$

where (U, z) is a chart on M such that $f(U) \subset V$, and (V, w) is a chart on N, has holomorphic component functions.



The tangent and cotangent spaces (T_pM, T_p^*M), as well as holomorphic differential forms can be defined analogously to the smooth case. The reader may find the detailed theory of smooth manifolds in [Lee03], and the basics of complex manifolds in [For91].

3.2 Integrable Pfaffian systems

Next we will define Pfaffian systems on complex manifolds and study the existence of solutions for them. We also provide some examples of how these equations look like.

Consider the system of first order homogeneous linear partial differential equations

$$\frac{\partial}{\partial z^i}\phi_k(z^1, \ldots, z^m) = \sum_{j=1}^{n} f_{(i)kj}(z^1, \ldots, z^m)\phi_j(z^1, \ldots, z^m), \tag{5}$$

where $i = 1, \ldots, m$, $k = 1, \ldots, n$, and

$$\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{C}^n$$

is a unknown function defined on an open subset $U \subset M$ of a complex $m-$manifold M,

$$z = (z^1, \ldots, z^m) : U \to \mathbb{C}^m$$

being the chart function on U. The coefficients $f_{(i)kj} : U \to \mathbb{C}$ are holomorphic functions.

The equation (5) consists of m differential equations, which can be written in matrix form as follows. Define for each $i = 1, \ldots, m$ the $n \times n$-matrix-valued function $A_{(i)} = (a_{(i)kj})_{j,k=1,\ldots,n}$ by

$$a_{(i)kj} := f_{(i)kj}(z^1, \ldots, z^m).$$

Then (5) reads

$$\frac{\partial}{\partial z^i}\phi(z^1, \ldots, z^m) = A_{(i)}\phi(z^1, \ldots, z^m).$$

Notice that the operators $(\frac{\partial}{\partial z^i})_p$ form a natural basis of the tangent space T_pM of M at any point $p \in U$, and the *differentials* $(dz^i)_p$ a natural basis of the cotangent space T_p^*M. Recall also that *one-forms* on M are precisely the elements of the cotangent bundle T^*M. Holomorphic one-forms can be written as

$$f(z^1, \ldots, z^m)dz = \sum_{k=1}^{m} f_k(z^1, \ldots, z^m)dz^k,$$

where $f_k : U \to \mathbb{C}$ are holomorphic.

Example 3.2. Let c_i, $i = 1, \ldots, n$, be complex valued functions. Let

$$M = \mathbb{C} \setminus \{w_1, \ldots, w_k\},$$

where $\{w_1, \ldots, w_k\}$ are the zeros of c_n.

The manifold M is naturally equipped with the complex structure inherited from $\mathbb{C}$, z being the chart function. Hence we may consider the equation

$$c_n(z)f^{(n)}(z) + c_{n-1}(z)f^{(n-1)}(z) + \cdots + c_1(z)f'(z) + c_0(z)f(z) = 0,$$

where $f : M \to \mathbb{C}$ is a unknown analytic function and $f^{(i)}$ is the i:th derivative of f. This is nothing but a linear ODE of n :th order.

To reduce this to equation (5) one can make the following change of variables. Define

$$f_i(z) = f^{(i-1)}(z),$$

$i = 1, \ldots, n$, where we denote $f^{(0)} = f$. Then we obtain a linear system of first order ODE's

$$\begin{cases} \frac{d}{dz}f_i = f_{i+1}, \quad i = 1, \ldots, n-1 \\ \frac{d}{dz}f_n = -\frac{1}{c_n(z)}(c_0(z)f_1(z) + c_1(z)f_2(z) + \cdots + c_{n-1}(z)f_n(z)). \end{cases}$$

In matrix form, this is

$$\frac{d}{dz}F(z) = A(z)F(z),$$

where $F(z) = (f_1(z) \cdots f_n(z))^T$ and $A(z)$ is the matrix

$$A(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{c_0(z)}{c_n(z)} & -\frac{c_1(z)}{c_n(z)} & -\frac{c_2(z)}{c_n(z)} & \cdots & -\frac{c_{n-1}(z)}{c_n(z)} \end{pmatrix}$$

Example 3.3. Let $V = V_1 \otimes \cdots \otimes V_N$, where V_i, $i = 1, \ldots, N$, are linear finite dimensional complex representations of a semisimple Lie algebra $\mathfrak{g}$. Let

$$M = \{(z^1, \ldots, z^N) \in \mathbb{C}^N\} \setminus \bigcup_{i<j} \{z^i = z^j\}$$

be a complex manifold and $\phi : M \to V$ a unknown function defined on M taking values in the vector space V. Let also $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ be an element whose representation is an operator acting on $V_i \otimes V_j$ for any $i, j \in \{1, \ldots, N\}$, and let $\kappa \in \mathbb{C}$. By Ω_{ij} we mean the operator Ω acting on the i:th and j:th tensor component of V. The parameter κ and the operator Ω depend on the Lie algebra $\mathfrak{g}$; for a precise definition see chapter 5.

The *Knizhnik-Zamolodchikov equations*

$$\frac{\partial}{\partial z^i} \phi(z^1, \ldots, z^N) = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z^i - z^j} \phi(z^1, \ldots, z^N), \tag{KZ}$$

$i = 1, \ldots, N$, arise from the Wess-Zumino-(Novikov)-Witten-model of conformal field theory, where they describe the conformal blocks of the model.

We observe that the KZ-equations are of form (5) but the unknown function ϕ takes values in the vector space V instead of $\mathbb{C}^n$. However, choosing a suitable basis we see that V is isomorphic to $\mathbb{C}^{\dim(V)}$. The matrices associated to the system KZ are $n \times n$ -matrices involving the operators Ω_{ij},

$$A_{(i)} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z^i - z^j}.$$

We denote the set of holomorphic k-forms defined on an open subset U of a complex manifold M by $\Omega^k(U)$. The *wedge product* of forms connects a k-form α and an l-form β, resulting a $(k+l)$-form $\alpha \wedge \beta$. It is defined as follows. If $X_1, \ldots, X_{k+l}$ are holomorphic vector fields on U, let

$$\begin{aligned} &\alpha \wedge \beta(X_1, \ldots, X_{k+l}) \\ &:= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sgn(\sigma) \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}), \end{aligned}$$

where S_{k+l} is the symmetric group.

Recall that in smooth, or complex, manifolds, there exists an exterior differentiation d connecting smooth, or holomorphic, n-forms to $(n+1)$-forms. The exterior differentiation satisfies the following properties, which are proved for the smooth case in [Lee03].

Theorem 3.4. *Let M be a complex manifold and $U \subset M$ open. For every integer $k \geq 0$ there exists a unique $\mathbb{C}$-linear operator*

$$d = d_U^k : \Omega^k(U) \to \Omega^{k+1}(U)$$

such that

(i) d is a $\wedge$-antiderivation, that is, for every $\alpha \in \Omega^k(U)$, $\beta \in \Omega^l(U)$ it satisfies

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta;$$

(ii) for $k = 0$ the operator $d = d_U^0$ is the complex differential

$$d : Hol(U) \to \Omega^1(U), \;\; f \mapsto df;$$

(iii) $d^2 = d \circ d = 0$

(iv) if $V \subset U \subset M$ are open and $\alpha \in \Omega^k(U)$, then $d(\alpha|_V) = (d\alpha)|_V$.

Using a matrix of one-forms the linear system (5) can be written in a more compact form.

Lemma 3.5. *The linear system (5) can on any open $U \subset M$ be written in the form*

$$d\phi(z^1, \dots, z^m) = \Lambda\phi(z^1, \dots, z^m), \tag{PFA}$$

where Λ is a matrix Pfaffian one-form*, that is a $n \times n$-matrix whose entries are holomorphic one-forms on U.*

Proof. Write the differential of ϕ_k, $k = 1, \dots, n$, in the natural basis of one-forms

$$d\phi_k(z^1, \dots, z^m) = \sum_{i=1}^{m} \frac{\partial}{\partial z^i}\phi_k(z^1, \dots, z^m)dz^i.$$

By (5) this can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f_{(i)kj}(z^1, \dots, z^m)\phi_j(z^1, \dots, z^m)dz^i.$$

The assertion follows defining the matrix $\Lambda := (\lambda_{kj})_{k,j=1,\dots,n}$ by

$$\lambda_{kj} := \sum_{i=1}^{m} f_{(i)kj}(z^1, \dots, z^m)dz^i.$$

□

Next we study under which conditions Pfaffian systems have local solutions. The main result is that a local basis of solutions exists if and only if the Pfaffian matrix satisfies a condition which is called integrability.

Definition 3.6. The system PFA are said to be *integrable*, if the matrix Λ satisfies the integrability condition

$$d\Lambda = \Lambda \wedge \Lambda. \tag{INT}$$

In terms of matrix elements, this reads

$$d\lambda_{ij} = (\Lambda \wedge \Lambda)_{ij} = \sum_{k=1}^{n} \lambda_{ik} \wedge \lambda_{kj}, \quad i, j = 1, \ldots, n.$$

It is shown in lemma 3.9 that the integrability condition is necessary for the existence of a local fundamental solution of the system PFA. From the Fröbenius theorem it follows that this is also sufficient; see theorem 3.10.

Definition 3.7. Any n pointwise linearly independent solutions

$$\phi^{(j)} = (\phi_1^{(j)}, \ldots, \phi_n^{(j)}) : U \to \mathbb{C}^n,$$

$j = 1, \ldots, n$, of the equation (5), defined on an open subset U of M, form a *fundamental system* of solutions of (5). In particular they form a basis for the vector space of the local solutions of (5). Written as columns of the matrix

$$X = (x_{ij})_{i,j=1,\ldots,n},$$

where $x_{ij} := \phi_i^{(j)}$, they form a *fundamental solution* for the system PFA. Notice that since X is pointwise invertible, it is a map $X : U \to GL_n(\mathbb{C})$.

Lemma 3.8. *The fundamental solution X satisfies the system PFA in matrix form, that is*

$$dX = \Lambda X,$$

where the differentiation is for each matrix component separately.

Proof. If $\Lambda = (\lambda_{ij})_{i,j=1,\ldots,n}$, $X = (x_{ij})_{i,j=1,\ldots,n}$ are $n \times n$-matrices, their product is $\Lambda X = (\sum_{k=1}^{n} \lambda_{ik} x_{kj})_{i,j=1,\ldots,n}$. Hence

$$dX = \Lambda X \iff dx_{ij} = \sum_{k=1}^{n} \lambda_{ik} x_{kj} \text{ for all } i, j = 1, \ldots, n.$$

By the choice of the matrix elements x_{ij}, this is equivalent to

$$d\phi_i^{(j)} = \sum_{k=1}^{n} a_{ik} \phi_k^{(j)} \text{ for all } i, j = 1, \ldots, n.$$

But every function $\phi^{(j)} = (\phi_1^{(j)}, \ldots, \phi_n^{(j)})$, $j = 1, \ldots, n$, satisfies the system PFA, whence for every $j = 1, \ldots, n$,

$$d\phi^{(j)}(z^1, \ldots, z^n) = \Lambda\phi^{(j)}(z^1, \ldots, z^n)$$
$$\iff d\phi_i^{(j)} = \sum_{k=1}^{n} \lambda_{ik}\phi_k^{(j)} \quad \text{for all } i, j = 1, \ldots, n.$$

□

Lemma 3.9. *Suppose there exists a fundamental solution X of the system PFA. Then the matrix Λ satisfies the integrability condition INT.*

Proof. We first notice that X is pointwise invertible, and

$$0 = dI_{n\times n} = d(X^{-1}X) = (dX^{-1})X + X^{-1}dX,$$

where $I_{n\times n} \in GL_n(\mathbb{C})$ is the constant function whose value is the unit matrix. By lemma 3.8 this implies

$$dX^{-1} = -X^{-1}(dX)X^{-1} = -X^{-1}(\Lambda X)X^{-1} = -X^{-1}\Lambda.$$

Hence

$$\Lambda = -XdX^{-1},$$

and differentiating Λ, using theorem 3.4 and lemma 3.8 we obtain

$$d\Lambda = -dX \wedge dX^{-1} - (-1)^0 X \wedge d^2X^{-1} = -dX \wedge dX^{-1}.$$

This is equivalent to

$$\begin{aligned} d\lambda_{ij} &= \sum_{k,l,m=1}^{n} \lambda_{ik}x_{kl} \wedge y_{lm}\lambda_{mj} = \sum_{k,l,m=1}^{n} (x_{kl}y_{lm})\lambda_{ik} \wedge \lambda_{mj} \\ &= \sum_{k,m=1}^{n} (XX^{-1})_{kl}\lambda_{ik} \wedge \lambda_{mj} = \sum_{k,m=1}^{n} \delta_{km}\lambda_{ik} \wedge \lambda_{mj} \\ &= \sum_{k=1}^{n} \lambda_{ik} \wedge \lambda_{mj} = (\Lambda \wedge \Lambda)_{ij}, \end{aligned}$$

where $(y_{ij})_{i,j=1,\ldots,n} = X^{-1}$. □

After the previous lemma the following important result follows from the Fröbenius theorem, which is proved in detail in [AF02]. We only state the following and refer to [AF02] for the proof, which is beyond the scope of this thesis.

Theorem 3.10. *A fundamental solution of the system PFA exists if and only if the matrix Λ satisfies the integrability condition INT.*

Proof. Necessity follows from lemma 3.9, and sufficiency from the Fröbenius theorem. A detailed proof can be found in [AF02]. □

Notice that the fundamental solution $X : U \to GL_n(\mathbb{C})$ is not necessarily a global solution for the system PFA. If the manifold M is not simply connected there might not exist a global single-valued solution for PFA. However, the integrability condition guarantees existence of local solutions defined on an open subset U of M.

3.3 Holomorphic vector bundles

Another tool needed for studying monodromy of Pfaffian systems is the concept of holomorphic vector bundles on complex manifolds, and connections on these bundles. Pfaffian systems define connections on trivial vector bundles, and solutions of the systems can be thought of as horizontal sections.

A complex vector bundle of rank n over a topological manifold M is a topological manifold which is built from "cylinders" $U_i \times \mathbb{C}^n$, where U_i is a chart on M. The structure added to M is the linear structure along the fibers $\{p\} \times \mathbb{C}^n$, $p \in M$. Using trivial vector bundles and connections one can define another formulation for the integrability condition INT.

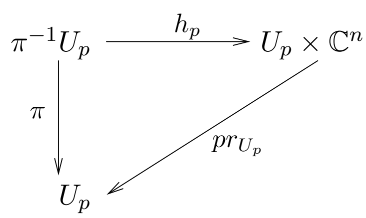
Let M and E be topological spaces, and $\pi : E \to M$ a continuous map. Suppose every *fiber* $\pi^{-1}(p)$, $p \in M$, has the structure of an n-dimensional vector space over $\mathbb{C}$.

Definition 3.11. The pair (E, π) is a *complex vector bundle of rank n over M* if for every $p \in M$ there exists an open neighbourhood U_p of p in M and a homeomorphism (*a local trivialisation*)

$$h_p : \pi^{-1}U_p \to U_p \times \mathbb{C}^n$$

so that the following conditions hold.

(i) h_p is *fiber-preserving*, that is the diagram



commutes, where $pr_{U_p} : U_p \times \mathbb{C}^n \to U_p$ is projection on the first component,

(ii) for every $x \in U_p$ the map

$$h_p|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{C}^n$$

is an isomorphism of vector spaces.

If $\{U_i\}_{i\in I}$ is an open cover of M and $h_i : \pi^{-1}U_i \to U_i \times \mathbb{C}^n$ are local trivialisations, then the family $\{\pi^{-1}U_i, h_i\}_{i\in I}$ is called an *atlas* of E. A vector bundle is *trivial* if it admits a global atlas, that is a global trivialisation $h : E \to M \times \mathbb{C}^n$.

Definition 3.12. Let $U \subset M$ be open. A *section* of E over U is a continuous map $s : U \to E$ such that

$$\pi \circ s = id_U.$$

Notice that a section s can be defined either *locally* on an open subset U of M, or *globally* on the whole space M.

If $h_i : \pi^{-1}U_i \to U_i \times \mathbb{C}^n$ is a local trivialisation of E then one can associate to the section s a unique continuous function

$$\begin{aligned} &s_i : U_i \cap U \to \mathbb{C}^n \text{ such that} \\ &h_i(s(x)) = (x, s_i(x)) \end{aligned}$$

for every $x \in U_i \cap U$. The function s_i is called a *representation* of s with respect to the local trivialisation h_i.

Definition 3.13. A *local frame* on an open subset U of M is a n-tuple $\{e_1, \ldots, e_n\}$ of sections such that for every $x \in U$ the set $\{e_1(x), \ldots, e_n(x)\}$ is a basis of $\pi^{-1}(x)$. If one may choose $U = M$ there exists a *global frame* on M.

Definition 3.14. The vector bundle E over a complex manifold M is *holomorphic* if $\pi : E \to M$ and the transition functions associated to an atlas of E are holomorphic.

A section is said to be holomorphic if its representation with respect to every local trivialisation is holomorphic. We denote the space of holomorphic sections defined on an open subset U of M by $\Gamma(U, E)$. When $U = M$, we have the space of global holomorphic sections $\Gamma(M, E)$. Notice that when considering sections defined only locally we interpret the bundle E over M to be restricted to a bundle $\pi^{-1}(U)$ over U.

Examples of complex vector bundles are the tangent and cotangent bundles TM and T^*M of a complex manifold M. The sections of the tangent bundle are called *vector fields*, and the sections of the cotangent bundle *covector fields*. *Coordinate vector fields* form a basis of the space of holomorphic vector fields. Similarly, *differential one-forms* form a basis of the space of holomorphic covector fields. Proofs and examples of these facts can be found in [Lee97].

Recall that the *n:th exterior power* of a vector space V is defined as the subspace of V spanned by elements of the form $v_1 \wedge \cdots \wedge v_n$,

$$\wedge^n V = span\{v_1 \wedge \cdots \wedge v_n : v_i \in V, i = 1, \ldots, n\}.$$

The elements of $\wedge^n V$ are called *multivectors*. A differential k-form on M can be thought of as a holomorphic section of the k:th exterior power of the cotangent bundle T^*M of M.

If M is a complex manifold and (E, π) a holomorphic vector bundle of rank n over M, sections of the tensor product bundle

$$E \otimes \wedge^k T^*M$$

are called *E-valued differential k-forms* on M. The structure of the tensor product bundle is natural, and the fibers

$$\pi^{-1}(p) \otimes \wedge^k T_p^*M$$

are tensor products of the fibers $\pi^{-1}(p)$ of E and the fibers $\wedge^k T_p^* M$ of the extrerior power bundle

$$\wedge^k T^*M = \sqcup_{p \in M} \wedge^k T_p^* M.$$

We denote the set of E-valued holomorphic differential k-forms defined on an open subset U of M by $\Omega^k(U, E)$.

Definition 3.15. Let M be a complex manifold and (E, π) a holomorphic vector bundle of rank n over M. A *connection* on E is a $\mathbb{C}$-linear map

$$\nabla : \Gamma(M, E) \to \Omega^1(M, E)$$

such that for any section $s \in \Gamma(M, E)$ and any $f \in Hol(M)$

$$\nabla(fs) = (df)s + f(\nabla s).$$

Usually, the bundle E is the trivial bundle $M \times \mathbb{C}^n$, whence the E-valued differential forms are differential forms on M taking values in $\mathbb{C}^n$. If $n = 1$, these are the ordinary differential forms on M.

Remark. The connection can be interpreted as a map

$$\nabla : Vect(M) \times \Gamma(M, E) \to \Gamma(M, E),$$

defined by the formula

$$\nabla(X, s) = \nabla_X s := (\nabla s)(X),$$

where $X \mapsto \nabla_X$ is $Hol(M)$-linear, and $Vect(M)$ is the space of smooth, or holomorphic, vector fields on M. Above the E-valued one-form ∇s is evaluated on the vector field X, resulting a section $(\nabla s)(X)$. Notice that by definition the connection satisfies

$$\nabla_X(fs) = (Xf)s + f(\nabla_X s)$$

for every $f \in Hol(M)$, $X \in Vect(M)$ and $s \in \Gamma(M, E)$.

Let $\{e_1, \dots, e_n\}$ be a local frame on $U \subset M$. Any section $s \in \Gamma(U, E)$ defined on U can be written in $\pi^{-1}(U)$ as

$$s = \sum_{i=1}^{n} f_i e_i,$$

where $f_i \in Hol(U)$, $i = 1, \dots, n$, are holomorphic functions. A very useful fact is that the value of the connection depends only the values of the vector field $X \in Vect(M)$ and the section $s \in \Gamma(M, E)$ *locally*.

Proposition 3.16. *Let $p \in M$, $X \in Vect(M)$ and $s \in \Gamma(M, E)$. The value of $\nabla_X s \in \Gamma(M, E)$ at p depends only on $X_p \in T_pM$ and the values of s along a smooth path $\gamma : [0, 1] \to M$ such that $\gamma'(0) = X_p$.*

Proof. Let (U, z) be a chart on M, $z : M \to \mathbb{C}^N$, such that $p \in U$, and let $\{\partial_1 \dots, \partial_N\}$ be the corresponding coordinate vector fields. Let also $\{e_1, \dots, e_n\}$ be a local frame on U. Write

$$X = \sum_{j=1}^{N} x_j \partial_j, \quad s = \sum_{i=1}^{n} f_i e_i,$$

where $x_j, f_i \in Hol(U)$. Then

$$\begin{aligned}(\nabla_X s)_p &= \sum_{i=1}^{n} (\nabla_X(f_i e_i))_p = \sum_{i=1}^{n} ((X_p f_i) e_i(p) + f_i(p)(\nabla_X e_i)_p) \\ &= \sum_{i=1}^{n} \left((X_p f_i) e_i(p) + \sum_{j=1}^{N} f_i(p) x_j(p) (\nabla_{\partial_j} e_i)_p \right)\end{aligned}$$

where $f_i(p)x_j(p)$ depend only on $X_p \in T_pM$ and $s(p) \in E$, and $X_p f_i$ depend only on the values of s along a smooth path $\gamma : [0, 1] \to M$ such that $\gamma'(0) = X_p$. □

Remark. By proposition 3.16 the definition of a connection as a mapping $\nabla : \Gamma(M, E) \to \Omega^1(M, E)$ makes sense also for every open subset U of M, meaning that ∇ can be defined on U as a linear map $\nabla : \Gamma(U, E) \to \Omega^1(U, E)$.

Lemma 3.17. *Any connection ∇ on E can be locally written as*

$$\nabla = d - \Lambda : \Gamma(U, E) \to \Omega^1(U, E),$$

where $U \subset M$ is open, d is the exterior differentiation and Λ is a $n \times n$-matrix of E-valued differential one-forms on U.

Proof. Let $\{e_1, \dots, e_n\}$ be a local frame on $U \subset M$ and write a section in the frame as $s = \sum_{i=1}^n f_i e_i \in \Gamma(U, E)$. Write

$$\nabla e_i := \sum_{j=1}^n \Gamma_i^j e_j,$$

where $\Gamma_i^j \in \Omega^1(U)$. By definition, the connection ∇ satisfies

$$\begin{aligned} \nabla s = \nabla(\sum_{i=1}^n f_i e_i) &= \sum_{i=1}^n \nabla(f_i e_i) = \sum_{i=1}^n ((df_i)e_i + f_i(\nabla e_i)) \\ &= \sum_{i=1}^n ((df_i)e_i + f_i \sum_{j=1}^n \Gamma_i^j e_j) = \sum_{j=1}^n (df_j + \sum_{i=1}^n f_i \Gamma_i^j) e_j. \end{aligned}$$

The assertion follows defining $\Lambda := (\lambda_{ji}) = (-\Gamma_i^j)_{j,i=1,\dots,n}$. □

Definition 3.18. A section $s \in \Gamma(M, E)$ is *horizontal* for the connection ∇ if

$$\nabla s = 0.$$

In particular, horizontal sections solve locally the equation

$$ds = \Lambda s.$$

Remark. In the case $E = M \times \mathbb{C}^n$ this is the system PFA, and if we take the trivial bundle $E = M \times W^{\otimes n}$, the KZ-equations with $V_1 = \dots = V_N =: W$ are of this form. In particular, solutions of KZ are horizontal sections with respect to the connection defined by the system KZ. The explicit form of this connection is considered in the next section.

Connections can also be naturally extended to linear maps connecting E-valued holomorphic differential forms on M. We interpret 0-forms as functions and write $\Omega^0(M) = Hol(M)$, and E-valued 0-forms as sections and write $\Omega^0(M, E) = \Gamma(M, E)$. We have

$$\Omega^k(M, E) \cong \Omega^k(M) \otimes \Gamma(M, E)$$

as modules over $Hol(M)$. Hence, we may identify these spaces. For every $k \geq 0$, define $\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ by

$$\nabla(\varphi \otimes s) := d\varphi \otimes s + \nabla s \wedge \varphi,$$

where $\varphi \in \Omega^k(M)$, $s \in \Gamma(M, E)$.

3.4 Flat connections

Next we introduce conditions under which local horizontal sections exist for a connection defined by a Pfaffian system. This corresponds to finding local solutions of the system. We will need a notion of curvature of connections. The scalar curvature associated to the affine connection on the tangent bundle of a manifold illustrates in a way the shape of the manifold; for a sphere the scalar curvature is positive, and a prototype of a manifold with negative scalar curvature is the hyperboloid. Connections of manifolds with zero curvature are called flat. It turns out that flat connections always have local horizontal sections.

Consider the system PFA. The matrix Λ of one-forms defines a connection

$$\nabla^{(\Lambda)} := d - \Lambda$$

on the trivial vector bundle $E = M \times \mathbb{C}^n$ of rank n over M. By definition, local solutions of PFA are the horizontal sections for the connection $\nabla^{(\Lambda)}$. The integrability condition INT can be written in terms of connections.

Recall that for a smooth, or holomorphic, manifold M, the *curvature tensor field*

$$R : Vect(M) \times Vect(M) \times Vect(M) \to Vect(M)$$

associated to an affine connection ∇ (that is a connection defined on the tangent bundle TM) is defined by the formula

$$R(X,Y)Z := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z,$$

where $[\cdot,\cdot]$ is the Lie bracket on $Vect(M)$ and the connection is interpreted as a map

$$\nabla : Vect(M) \times Vect(M) \to Vect(M),$$

defined by the formula

$$\nabla(X,Z) = \nabla_X Z := (\nabla Z)(X),$$

where the one-form ∇Z is evaluated on the vector field X, resulting a vector field. The manifold is said to be *flat* if the curvature tensor field is identically zero. This motivates the following definition.

Definition 3.19. Let M be a complex manifold, (E,π) a holomorphic vector bundle of rank n over M and ∇ a connection on E. The multilinear map

$$R : Vect(M) \times Vect(M) \times \Gamma(M,E) \to \Gamma(M,E)$$

defined by the formula

$$R(X,Y)s := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s$$

is called the *curvature* of the connection. Linearity of the right hand side over $Hol(M)$ in the first two arguments and over $\mathbb{C}$ in the third argument can be shown by a direct computation.

The map R can be thought of as a linear map

$$R : \Gamma(M, E) \to \Omega^2(M, E)$$

and extended in a similar manner as ∇ to a linear map

$$R : \Omega^k(M, E) \to \Omega^{k+2}(M, E),\ k \geq 0.$$

Definition 3.20. The connection ∇ is a *flat* if its curvature is identically zero.

Actually, there is a more convenient definition for the flatness of connections. First we note a simple observation, which can be proved using the properties of the exterior differentiation. The proof of the following property is found in [Lee03].

Lemma 3.21. *If $\varphi \in \Omega^1(M)$, $X, Y \in Vect(M)$ then*

$$d\varphi(X, Y) = X(\varphi(Y)) - Y(\varphi(X)) - \varphi([X, Y]).$$

In the next lemma we identify the isomorphic $Hol(M)$-modules $\Omega^k(M, E)$ and $\Omega^k(M) \otimes \Gamma(M, E)$.

Lemma 3.22. *Let $\omega = \varphi \otimes s \in \Omega^1(M, E)$, where $\varphi \in \Omega^1(M)$, $s \in \Gamma(M, E)$. Then for all holomorphic vector fields $X, Y \in Vect(M)$*

$$\nabla\omega(X, Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y]).$$

Proof.

$$\begin{aligned}
\nabla\omega(X, Y) &= \nabla(\varphi \otimes s)(X, Y) = (d\varphi \otimes s + \nabla s \wedge \varphi)(X, Y) \\
&= (X(\varphi(Y)) - Y(\varphi(X)) - \varphi([X, Y])) \otimes s \\
&\quad + ((\nabla s)(X))\varphi(Y) - ((\nabla s)(Y))\varphi(X) \\
&= ((d(\varphi(Y)))X - (d(\varphi(X)))Y - \varphi([X, Y])) \otimes s \\
&\quad + ((\nabla s)(X))\varphi(Y) - ((\nabla s)(Y))\varphi(X) \\
&= (d(\varphi(Y)) \otimes s + \varphi(Y)(\nabla s))(X) \\
&\quad - (d(\varphi(X)) \otimes s + \varphi(X)(\nabla s))(Y) - \varphi([X, Y]) \otimes s \\
&= (\nabla(\varphi(Y) \otimes s))(X) - (\nabla(\varphi(X) \otimes s))(Y) - \omega([X, Y]) \\
&= \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y]),
\end{aligned}$$

where we used the definition of the wedge product of one-forms

$$\alpha \wedge \beta(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X),$$

and the definition of the differential acting on vector fields

$$(dX)Y = YX.$$

□

Proposition 3.23. *For any section $s \in \Gamma(M, E)$*

$$(\nabla \circ \nabla)s = R(s).$$

Proof. Substitute $\omega = \nabla s$ in lemma 3.22 to obtain

$$\begin{aligned}\nabla(\nabla s)(X, Y) &= \nabla_X(\nabla s(Y)) - \nabla_Y(\nabla s(X)) - \nabla s([X, Y]) \\ &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = R(X, Y)s.\end{aligned}$$

□

Hence a connection ∇ is flat if and only if $\nabla \circ \nabla = 0$. This enables us to prove the following important result.

Theorem 3.24. *Let M ba a complex m-manifold and $\Lambda = (\lambda_{ij})_{i,j=1,\dots,n}$ a matrix of one-forms on M. Then the system PFA associated to Λ is integrable if and only if the connection $\nabla^{(\Lambda)}$ is flat.*

Proof. Let $\{e_1, \dots, e_n\}$ be a local frame on $U \subset M$. Let $s \in \Gamma(U, M \times \mathbb{C}^n)$, and write $s = \sum_{i=1}^n f_i e_i$. By lemma 3.17

$$\nabla^{(\Lambda)} s = \sum_{j=1}^n (df_j - \sum_{i=1}^n f_i \lambda_{ji}) e_j.$$

By properties of the exterior differentiation

$$\nabla^{(\Lambda)} \circ \nabla^{(\Lambda)} s = d\left(\sum_{j=1}^n df_j - \sum_{i,j=1}^n f_i \lambda_{ji}\right) e_j - \sum_{j,k=1}^n \lambda_{jk} \wedge \left(df_k - \sum_{i=1}^n f_i \lambda_{ki}\right) e_j$$

$$= \sum_{j=1}^n \left(d^2 f_j - \sum_{i=1}^n ((d\lambda_{ji}) f_i - \lambda_{ji} \wedge df_i) - \sum_{k=1}^\lambda \lambda_{jk} \wedge df_k + \sum_{i,k=1}^n \lambda_{jk} \wedge f_i \lambda_{ki}\right) e_j$$

$$= -\sum_{i,j=1}^n (d\lambda_{ij}) f_i e_j + \sum_{j=1}^n \left(\sum_{i=1}^n \lambda_{ji} \wedge df_i - \sum_{k=1}^n (\lambda_{jk} \wedge df_k)\right) e_j$$

$$+ \sum_{i,j,k=1}^n \lambda_{jk} \wedge \lambda_{ki} f_i e_j$$

$$= -(d\Lambda)s + \sum_{k=1}^n \left(\lambda_{ik} \wedge ds^k - \lambda_{ik} \wedge ds^k\right) + (\Lambda \wedge \Lambda)s$$

$$= -(d\Lambda - \Lambda \wedge \Lambda)s$$

and the assertion follows from proposition 3.23. □

Thus the question of existence of local fundamental solutions of PFA can be considered as the question of flatness of the connection associated to PFA. Also, finding a linearly independent set of local solutions reduces to finding a basis of the space of local horizontal sections associated to the trivial bundle over M.

Example 3.25. If $m = 1$, the integrability condition $d\Lambda = \Lambda \wedge \Lambda$ is trivial. Namely, there are no nonzero two-forms on a one-dimensional manifold, and $d\Lambda$ and $\Lambda \wedge \Lambda$ would be matrices of two-forms.

In example 3.2 the manifold was $M = \mathbb{C} \setminus \{w_1, \ldots, w_k\}$, and the fundamental solution always exists. Remember also the existence and uniqueness theorem for linear ODE's.

Example 3.26. The hypergeometric equation HGE is of the Pfaffian form. The manifold under consideration is $M = \mathbb{C} \setminus \{0, 1\}$, and the Pfaffian matrix can be written as

$$\Lambda(z) = \begin{pmatrix} 0 & dz \\ -p_2(z)dz & -p_1(z)dz \end{pmatrix}$$

where

$$p_1(z) = \frac{c - (a + b + 1)z}{z(1 - z)}$$

and

$$p_2(z) = -\frac{ab}{z(1 - z)}.$$

Also in this case the underlying manifold is one-dimensional, and the existence of local solutions guaranteed.

Example 3.27. The KZ-equations of example 3.3 can be written in the form suggested by lemma 3.5

$$d\phi(z^1, \ldots, z^N) = \frac{1}{\kappa} \sum_i \sum_{j \neq i} \frac{\Omega_{ij}}{z^i - z^j} \phi(z^1, \ldots, z^N) dz^i \qquad \text{(KZ')}$$

with the Pfaffian $n \times n$-matrix

$$\Lambda = \frac{1}{\kappa} \sum_i \sum_{j \neq i} \frac{\Omega_{ij}}{z^i - z^j} dz^i.$$

A more convenient way to write Λ is

$$\Lambda = \frac{1}{\kappa} \sum_{1 \leq i < j \leq N} \frac{\Omega_{ij}}{z^i - z^j} (dz^i - dz^j).$$

It can be shown that the connection defined by Λ is flat, but the computation is a bit tedious, and can be found in [Kas95]. Hence there exists a local fundamental solution of the KZ-equations.

3.5 Monodromy by parallel transport

Another advantage of flat connections is that the parallel transport of sections along paths is independent of the choice of the path within the same homotopy class. Hence by parallel transport we obtain a well defined monodromy representation, acting on the fibers of the bundle. Let in this section M be a complex manifold, (E, π) a holomorphic vector bundle of rank n over M and ∇ a connection on E.

Definition 3.28. If $\gamma : [0,1] \to M$ is a smooth path on M, we say that a holomorphic map $\rho : [0,1] \to E$, interpreted so that ρ extends to a holomorphic function in the vicinity of $[0,1]$, is a holomorphic *section along* γ if $\pi \circ \rho = \gamma$. By definition, any section $s \in \Gamma(M, E)$ can be restricted to a holomorphic section $\rho = s \circ \gamma$ along γ, since

$$\pi \circ \rho = \pi \circ s \circ \gamma = id_M \circ \gamma = \gamma.$$

However, not every section ρ along γ is induced by a section $s \in \Gamma(M, E)$. If ρ is induced by $s \in \Gamma(M, E)$, that is $\rho = s \circ \gamma$, then we call ρ *extendible*.

Definition 3.29. A section $s \in \Gamma(M, E)$ is said to be *parallel* if

$$\nabla_X s = 0$$

for all $X \in Vect(M)$.

Now that we have defined the notion of sections along smooth paths, we would like to define a directional derivative for sections. This is actually the original motivation of defining connections, namely to be able to differentiate sections along smooth paths. The following result, which enables differentiation along paths, is proved in [Lee97] for affine connections on smooth manifolds. The general case is proved similarly.

Proposition 3.30. *For each smooth path $\gamma : [0,1] \to M$ the connection ∇ determines a unique $\mathbb{C}$-linear operator D_t acting on the space of holomorphic sections along γ satisfying the following properties.*

(i) For every $f \in Hol([0,1])$, interpreted so that f extends to a holomorphic function in the vicinity of $[0,1]$, and every holomorphic section $\rho : [0,1] \to E$ along γ the operator D_t satisfies the product rule

$$D_t(f\rho) = f'\rho + fD_t\rho.$$

(ii) If ρ is extendible, then for any $s \in \Gamma(M, E)$ such that $s \circ \gamma = \rho$

$$D_t\rho = \nabla_{\gamma'} s.$$

The operator D_t is called the *covariant derivative* along γ. We can now define parallel sections along paths. Notice that by the property (ii) of the covariant derivative the following definition is consistent with the previous definition of parallel sections.

Definition 3.31. Let $\gamma : [0,1] \to M$ be a smooth path on M. A holomorphic section $\rho : [0,1] \to E$ along γ is said to be *parallel along* γ if

$$D_t \rho = 0$$

for every $t \in [0,1]$.

From the existence and uniqueness theorem for linear ODE's we obtain the following result. The proof, again for affine connections on smooth manifolds, is found in [Lee97].

Theorem 3.32. *Let $\gamma : [0,1] \to M$ be a smooth path and $s_0 \in \pi^{-1}(\gamma(0))$. Then there exists a unique holomorphic parallel section $\rho : [0,1] \to E$ along γ such that $\rho(0) = s_0$. The parallel section ρ is called the* parallel translation of s_0 along γ *with respect to the connection ∇.*

Before proving the main result, we state a useful lemma proved in detail in [Mee05].

Lemma 3.33. *The connection ∇ is flat if and only if every $p \in M$ has an open neighbourhood on which there exists a local parallel frame $\{e_1, \ldots, e_n\}$, that is, a local frame satisfying*

$$\nabla_X e_j = 0$$

for all $X \in Vect(M)$, $j = 1, \ldots, n$.

Proof. The sufficiency is easy. Namely, if $\{e_1, \ldots, e_n\}$ is a local parallel frame then one can write any $s \in \Gamma(M, E)$ as

$$s = \sum_{j=1}^{n} f_j e_j$$

and compute for any vector fields $X, Y \in Vect(M)$ locally

$$\begin{aligned}
R(X,Y)s = & \ (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) \sum_{j=1}^{n} f_j e_j \\
= & \sum_{j=1}^{n} (\nabla_X ((Y f_j) e_j + f_j \nabla_Y e_j) - \nabla_Y ((X f_j) e_j + f_j \nabla_X e_j) \\
& - (([X,Y] f_j) e_j + f_j \nabla_{[X,Y]} e_j)) \\
= & \sum_{j=1}^{n} (((XY f_j) e_j + (Y f_j) \nabla_X e_j) - ((YX f_j) e_j + (X f_j) \nabla_Y e_j) - ([X,Y] f_j) e_j) \\
= & \sum_{j=1}^{n} ((XY - YX - [X,Y]) f_j) e_j = 0
\end{aligned}$$

by definition of the commutator of vector fields. To prove the necessity, assuming that ∇ is flat one can construct a local parallel frame. For the construction the reader may consult [Mee05]. □

From this we get the following result, which enables us to define monodromy in language of flat connections.

Theorem 3.34. *Parallel transport along smooth paths depends only on the homotopy class if and only if ∇ is flat.*

Proof. Suppose first that ∇ is flat. As in the proof of corollary 1.13 it suffices to show that parallel transport along a homotopically trivial loop is the same as along a constant path. So, let γ be a homotopically trivial loop. By the previous lemma there exists a local parallel frame $\{e_1, \ldots, e_n\}$ around $\gamma(t)$ for each $t \in [0, 1]$, and by compactness, there is a finite open cover $\{U_i\}_{i=1}^n$ of γ such that on every U_i there exists a local parallel frame. Now the proof is similar to the proof of corollary 1.13, making use of elementary deformations in the sets U_i on the manifold M, resulting that the parallel transport $\rho : [0, 1] \to E$ along γ satisfies $\rho(0) = \rho(1)$.

On the other hand, suppose parallel transport depends only on the homotopy class. Let $p \in M$. We will show that there exists a local parallel frame $\{e_1, \ldots, e_n\}$ on an open neighbourhood U of p. Then the assertion follows from the previous lemma.

Let U be an open simply connected subset of M (which means that the fundamental group of U is trivial) such that $p \in U$, and let $\{e_1(p), \ldots, e_n(p)\}$ be a basis of $\pi^{-1}(p)$. For any $p' \in U$, denote by $e_i(p')$ the end point of the parallel transport of $e_i(p)$ along an arbitrary smooth path $\gamma : [0, 1] \to U$ such that $\gamma(0) = p$, $\gamma(1) = p'$. Since parallel transport depends only on the homotopy class and U is simply connected, the value of $e_i(p')$ does not depend on the choice of the path γ. Moreover, since $\{e_1(p), \ldots, e_n(p)\}$ are linearly independent, also $\{e_1(p'), \ldots, e_n(p')\}$ are linearly independent. It remains to show that $\nabla_X e_j = 0$ for all $X \in Vect(M)$, $j = 1, \ldots, n$.

For that, by proposition 3.16 it is enough to show that $\nabla_{X_{p'}} e_j = 0$ for any $p' \in U$, $j = 1, \ldots, n$. Choose a smooth path $\gamma : [0, 1] \to U$ such that $\gamma(0) = p$, $\gamma(1) = p'$, and $\gamma'(1) = X_{p'}$. Then by definition

$$0 = \nabla_{\gamma'(1)} e_j = \nabla_{X_{p'}} e_j,$$

which proves the claim. □

Hence parallel transport along loops associated to a flat connection induces a well defined linear representation of the fundamental group of the manifold M acting on the fibers of E. It is called *the monodromy representation*

$$\rho : \pi_1(M, p) \to Aut(\pi^{-1}(p)) \cong GL_n(\mathbb{C}),$$

and defined by

$$\rho : \gamma \mapsto M_\gamma,$$

where $M_\gamma : \pi^{-1}(p) \to \pi^{-1}(p)$ is the monodromy operator acting on the fibres $\pi^{-1}(p)$, induced by parallel transports of holomorphic sections along the loop γ. Notice that by theorem 3.34 the monodromy representation is well defined if the associated connection is flat.

4 Bialgebras and quantum $\mathfrak{sl}_2$

Next we present some definitions needed for a closer look on the KZ-equations and in particular on the monodromy representation in the space of local solutions of KZ. Recall from example 3.3 that solutions of the KZ-equations are functions taking values in a linear complex tensor product representation of a semisimple Lie algebra. Moreover, the equations contain a linear operator acting on the tensor components.

Our main interest is the KZ-equations for the Lie algebra $\mathfrak{sl}_2$. Thus, before studying the monodromy of KZ we will recall briefly the structure of $\mathfrak{sl}_2$, the notion of bialgebras and Hopf algebras, and definition of the so called quantum $\mathfrak{sl}_2$, denoted by $U_q(\mathfrak{sl}_2)$, which is a q-deformation of the universal enveloping algebra of $\mathfrak{sl}_2$. We also recall the necessary representation theory of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$, in the extent of irreducible highest weight representations and semisimplicity. We will omit most of the proofs, since they are not needed in the sequel, and some of the proofs are also quite long. However, we state the results with careful references. For more details on Hopf algebras and quantum groups the reader may consult e.g. [Kas95] or [Kyt11], and on the theory of Lie algebras e.g. [Hum72] or [Kna04]. All the structures below can be defined for an arbitrary field, but in most applications only complex structures are used.

The reader need not to have background from the theory of Lie algebras, representations or quantum groups although some experience in the basics of algebra will be useful. Nevertheless, knowledge of linear algebra is necessary, and we expect also that the reader is somewhat familiar with the concept of tensor products. If V and W are two complex vector spaces, their *tensor product* $V \otimes W$ can be thought of as a vector space generated by simple tensors of the form $v \otimes w$, where $v \in V$, $w \in W$. The tensor product is a bilinear map $V \times W \to V \otimes W$ in the sense that for $\alpha \in \mathbb{C}$, $v_1, v_2 \in V$, $w_1, w_2 \in W$

$$\begin{aligned}
&(v_1 + v_2) \otimes w_1 = v_1 \otimes w_1 + v_2 \otimes w_1 \\
&v_1 \otimes (w_1 + w_2) = v_1 \otimes w_1 + v_1 \otimes w_2 \\
&(\alpha v_1) \otimes w_1 = v_1 \otimes (\alpha w_1) = \alpha(v_1 \otimes w_1).
\end{aligned}$$

More details on tensor products and the basics of algebra needed in this chapter can be found in [Kas95] and [Kyt11].

4.1 Bialgebras and Hopf algebras

Definition 4.1. A *bialgebra* is a $\mathbb{C}$-vector space A with the following properties:

(i) (A, μ, η) is a unital associative algebra, with the product $\mu : A \otimes A \to A$ and the unit $\eta : \mathbb{C} \to A$, $\eta(1) = 1_A$, which are linear maps satisfying

$$\mu \circ (\mu \otimes id_A) = \mu \circ (id_A \otimes \mu), \tag{H1}$$

$$\mu \circ (\eta \otimes id_A) = id_A = \mu \circ (id_A \otimes \eta), \tag{H2}$$

(ii) (A, Δ, ϵ) is a counital coassociative *coalgebra*, with the coproduct $\Delta : A \to A \otimes A$ and the counit $\epsilon : A \to \mathbb{C}$, which are linear maps satisfying

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta, \tag{H1'}$$

$$(\epsilon \otimes id_A) \circ \Delta = id_A = (id_A \otimes \epsilon) \circ \Delta, \tag{H2'}$$

(iii) the maps $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{C}$ are morphisms of algebras.

The properties (H1) and (H1') are called *associativity* and *coassociativity*, whereas the properties (H2) and (H2') are called *unitality* and *counitality.*

Definition 4.2. A bialgebra A is a *Hopf algebra* if it admits an *antipode*, that is a linear map $\gamma : A \to A$ such that

$$\mu \circ (\gamma \otimes id_A) \circ \Delta = \eta \circ \epsilon = \mu \circ (id_A \otimes \gamma) \circ \Delta. \tag{H3}$$

A *Hopf subalgebra* of a Hopf algebra A is a vector subspace of A which is stable under the product, coproduct and antipode. A *Hopf ideal* is a two sided ideal which also is a coideal and stable under the antipode. In other words, if A is a Hopf algebra, a Hopf ideal $J \subset A$ satisfies by definition

$$\begin{aligned}
&\mu(J \otimes A), \quad \mu(A \otimes J) \subset J \quad (\textit{ideal}),\\
&\Delta(J) \subset J \otimes A + A \otimes J, \quad \epsilon(J) = 0 \quad (\textit{coideal}),\\
&\gamma(J) \subset J.
\end{aligned}$$

If $J \subset A$ is a Hopf ideal then the quotient space A/J admits a natural Hopf algebra structure.

Definition 4.3. For any two vector spaces V and W the *tensor flip*

$$\tau_{V,W} : V \otimes W \to W \otimes V$$

is defined by its action on simple tensors

$$\tau_{V,W}(v \otimes w) := w \otimes v.$$

For a bialgebra A, we denote the *opposite product* by $\mu^{op} : A \otimes A \to A$,

$$\mu^{op} = \mu \circ \tau_{A,A},$$

and the *co-opposite coproduct* by $\Delta^{cop} : A \to A \otimes A$,

$$\Delta^{cop} = \tau_{A,A} \circ \Delta.$$

Definition 4.4. A linear *representation of a group* G on a vector space V is a group homomorphism $G \to Aut(V)$. A linear *representation of an algebra* A on a vector space V is an algebra morphism $A \to End(V)$. The vector space V is said to be a (left) G- or A-*module.* The representation is *irreducible* if it does not contain any proper invariant subspaces other than $\{0\}$. Representations of bialgebras are their representations as algebras.

Let A be a bialgebra and let ρ_V, ρ_W be two representations of A on the vector spaces V and W. It is easy to see that

$$\rho_{V\otimes W} := (\rho_V \otimes \rho_W) \circ \Delta : A \to End(V \otimes W)$$

is a representation of A on $V \otimes W$.

Moreover, for $n > 2$ the vector space $V^{\otimes n}$ is equipped with the representation

$$\rho_{V^{\otimes n}} := (\rho_V \otimes \rho_V \otimes \cdots \rho_V) \circ \Delta^{(n)} : A \to End(V^{\otimes n}),$$

where $\Delta^{(n)}$ is the $(n-1)$-fold coproduct

$$\Delta^{(n)} := (\Delta \otimes id_A \otimes id_A \otimes \cdots \otimes id_A) \circ \cdots \circ (\Delta \otimes id_A) \circ \Delta.$$

4.2 The quantum enveloping algebra $U_q(\mathfrak{sl}_2)$

A complex *Lie algebra* $\mathfrak{g}$ is a complex vector space with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie product, which is bilinear, antisymmetric, and satisfies the Jacobi identity

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0.$$

The reader may find a detailed presentation concerning the theory of semisimple Lie algebras in [Hum72] or [Kna04].

Recall that the (semi)simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by the elements e, f, h, subject to the relations

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

The element h generates the Cartan subalgebra, which is a maximal commutative subalgebra of $\mathfrak{sl}_2$. The generators e and f correspond to lowering and raising operators for eigenvalues of h, respectively, in $\mathfrak{sl}_2$-modules. For semisimple Lie algebras all finite dimensional irreducible representations are classified by highest weights, as is proved e.g. in [Hum72] and [Kna04].

The *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a unital associative algebra which consists of formal linear combinations of words whose letters are elements of $\mathfrak{g}$, with the relations $xy - yx - [x, y] = 0$, and the obvious multilinearity relations. It is uniquely determined by a universality property, and the Lie algebra $\mathfrak{g}$ can be embedded to $U(\mathfrak{g})$. It can also be shown that there is an one-to-one correspondence between representations of $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$.

Lemma 4.5. *Let $\mathfrak{g}$ be a complex Lie algebra. Its universal enveloping algebra $U(\mathfrak{g})$ admits a bialgebra structure*

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \epsilon(x) = 0$$

for all $x \in \mathfrak{g} \subset U(\mathfrak{g})$.

Notice that since by definition $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra, it is enough to define the coproduct and counit for elements of $\mathfrak{g}$ only. The proof is a simple computation using the definition of a bialgebra. It follows from the relation

$$\Delta(xy-yx) = \Delta(x)\Delta(y)-\Delta(y)\Delta(x) = (xy-yx)\otimes 1+1\otimes(xy-yx) = \Delta([x,y])$$

that the coproduct is well defined in $U(\mathfrak{g})$.

It is obvious that the universal enveloping algebra $U(\mathfrak{sl}_2)$ can be defined as follows.

Definition 4.6. The *universal enveloping algebra* $U(\mathfrak{sl}_2)$ is the algebra generated by the elements e, f, h, subject to the relations

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

The element

$$c := ef + fe + \frac{1}{2}h^2 \in U(\mathfrak{sl}_2)$$

is called the *Casimir element* of $\mathfrak{sl}_2$.

The next lemma is a direct computation, and for details one may consult [Kas95].

Lemma 4.7. *The Casimir element belongs to the center of $U(\mathfrak{sl}_2)$.*

Remark. For any semisimple Lie algebra the Casimir element can be defined as follows. Define the *adjoint action*

$$ad_x : \mathfrak{g} \to \mathfrak{g}$$

for every $x \in \mathfrak{g}$ by $ad_x(z) := [x, z]$, $z \in \mathfrak{g}$. Let $\{x_i\}$ be a basis of $\mathfrak{g}$, and $\{\tilde{x_i}\}$ the dual basis with respect to the *Killing form*

$$B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C},$$

defined by $B(x, y) := Tr(ad_x ad_y)$. Notice that since $\mathfrak{g}$ is a finite dimensional vector space, the operators $ad_x \in End(\mathfrak{g})$ have well defined trace. We define the *Casimir element* by

$$c := \sum_i x_i \tilde{x_i} \in U(\mathfrak{g}).$$

It is shown in [Kna04] that the Casimir element is a central element in $U(\mathfrak{g})$ which is independent of the choice of basis of $\mathfrak{g}$.

We are now ready to define the quantum enveloping algebra of $\mathfrak{sl}_2$, which is a q-deformation of the universal enveloping algebra $U(\mathfrak{sl}_2)$. Actually, q-deformations of universal enveloping algebras can be defined for all complex Lie algebras; see [Kas95].

Definition 4.8. Let $q \in \mathbb{C}\backslash\{0, \pm 1\}$. The *quantum enveloping algebra* $U_q(\mathfrak{sl}_2)$ is the algebra generated by the elements E, F, K, K^{-1} satisfying the relations

$$KK^{-1} = 1 = K^{-1}K, \qquad KEK^{-1} = q^2E,$$

$$EF - FE = \frac{1}{q - q^{-1}}(K - K^{-1}), \qquad KFK^{-1} = q^{-2}F.$$

In $U_q(\mathfrak{sl}_2)$ the *quantum Casimir element* is a central element defined by

$$C := EF + \frac{1}{(q - q^{-1})^2}(q^{-1}K + qK^{-1}).$$

The quantum enveloping algebra has the following Hopf algebra structure, proved in [Kyt11].

Lemma 4.9. *$U_q(\mathfrak{sl}_2)$ admits a unique Hopf algebra structure determined by the coproduct*

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1.$$

The Hopf algebra structure is the following.

$$\epsilon(K) = 1, \quad \epsilon(E) = 0 = \epsilon(F),$$

$$\gamma(K) = K^{-1}, \quad \gamma(E) = -EK^{-1}, \quad \gamma(F) = -KF.$$

Central elements act as scalars in finite dimensional irreducible representations of algebras. This follows from the next *Schur's lemma.*

Lemma 4.10. (Schur) *Let V and W be irreducible representations of an algebra A and $f : V \to W$ an A-module map, that is a linear map such that*

$$f(x.v) = x.f(v)$$

for all $x \in A$, $v \in V$. Then

(i) either $f \equiv 0$ or f is an isomorphism of A-modules, and
(ii) if $V = W$ then $f = \lambda id_V$ for some $\lambda \in \mathbb{C}$.

Proof. By irreducibility of V, either $Ker(f) = V$, in which case $f \equiv 0$, or $Ker(f) = \{0\}$. If $Ker(f) = \{0\}$, then f is injective, and by irreducibility of W, $Im(f) = W$. This proves (i).

Suppose then $V = W$. Since $\mathbb{C}$ is algebraically complete, there exists an eigenvalue $\lambda \in \mathbb{C}$ of f. Now the map $f - \lambda id_V : V \to V$ is an A-module map with a nontrivial kernel, whence by irreducibility $Ker(f - \lambda id_V) = V$. This implies $f - \lambda id_V = 0$ and proves (ii). □

Corollary 4.11. *Let $\rho : A \to End(V)$ be an irreducible finite dimensional representation of an algebra A. Then for every central element $a \in A$*

$$\rho(a) = \lambda_a id_V$$

for some $\lambda_a \in \mathbb{C}$.

Proof. Let $a \in A$ be central. Then $\rho(a) \in End(V)$ is an A-module map, since

$$\rho(a)(x.v) = \rho(a)\rho(x)v = \rho(ax)v = \rho(xa)v = \rho(x)\rho(a)v = x.\rho(a)v$$

for all $x \in A$, $v \in V$. The assertion follows from the case (ii) in Schur's lemma 4.10. □

By Poincare-Birkhoff-Witt theorem (see [Hum72]) the ordered words consisting of the basis elements of a Lie algebra form a basis for the universal enveloping algebra. Similarly, we may choose a Poincare-Birkhoff-Witt -type basis $\{F^m K^k E^n\}_{m,n \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}}$ for the q-deformation $U_q(\mathfrak{sl}_2)$.

Remark. In certain sense, when $q \to 1$, the quantum enveloping algebra becomes the universal enveloping algebra. However, this seems impossible in view of definition 4.8. Let us think of K as the element $q^h = \exp(h \ln q)$, where h is the generator of $\mathfrak{sl}_2$. Then it is easy to see that

$$\lim_{q \to 1} \frac{q^h - q^{-h}}{q - q^{-1}} = h.$$

So in this limit we get $[E, F] = h$. Similarly, the relations $KEK^{-1} = q^2 E$ and $KFK^{-1} = q^{-2}F$ lead to $[h, E] = 2E$ and $[h, F] = -2F$. Thus we recover the standard commutation relations of $\mathfrak{sl}_2$.

There is also a more rigorous relation between $U_q(\mathfrak{sl}_2)$ and $U(\mathfrak{sl}_2)$, which is studied in detail in [Kas95].

4.3 Representations of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$

The rest of this chapter is devoted to representation theory of the Lie algebra $\mathfrak{sl}_2$ and its quantum version, $U_q(\mathfrak{sl}_2)$. We will state the main results without proofs, for which the reader may consult e.g. [Kas95] or [Kyt11]. Remarkably, when the parameter $q \in \mathbb{C} \setminus \{0\}$ is not a root of unity, the irreducible representations of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$ are of similar form. It is convenient to define the *q-integers*, *q-factorials* and *q-binomial coefficients*

$$\begin{aligned} [n]_q &:= \frac{q^n - q^{-n}}{q - q^{-1}}, \\ [n]_q! &:= [n]_q [n-1]_q \cdots [1]_q, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &:= \frac{[n]_q!}{[k]_q! [n-k]_q!}, \end{aligned}$$

where $q \in \mathbb{C} \setminus \{0, \pm 1\}$. For $q \to 1$ these become the usual integers and factorials. Write $q = e^\epsilon \simeq 1 + \epsilon$, where $\epsilon > 0$ is small. Then, as $\epsilon \to 0$,

$$\begin{aligned} [n]_q = &\frac{q^n - q^{-n}}{q - q^{-1}} = \frac{e^{\epsilon n} - e^{-\epsilon n}}{e^\epsilon - e^{-\epsilon}} \\ \simeq &\frac{(1 + \epsilon n) - (1 - \epsilon n)}{(1 + \epsilon) - (1 - \epsilon)} = \frac{2\epsilon n}{2\epsilon} = n. \end{aligned}$$

Definition 4.12. Let V be a $\mathfrak{sl}_2$-module. A *highest weight vector* $v_0 \in V$ of weight $\mu \in \mathbb{C}$ is defined by the conditions $v_0 \neq 0$,

$$h.v_0 = \mu v_0 \quad \text{and} \quad e.v_0 = 0.$$

The finite dimensional irreducible representations of $\mathfrak{sl}_2$ are classified by highest weights. The following theorem is proved in [Kas95] and [Kyt11].

Theorem 4.13. *For any integer $d > 0$ there exists a d-dimensional irreducible representation V_d of $\mathfrak{sl}_2$ with basis $\{v_j\}_{j=0}^{d-1}$ such that*

$$\begin{aligned} h.v_j &= (d - 1 - 2j)v_j, \\ f.v_j &= v_{j+1}, \\ e.v_j &= j(d - j)v_{j-1}. \end{aligned}$$

There are no other finite dimensional irreducible $\mathfrak{sl}_2$-modules. Here v_0 is the highest weight vector of weight $d - 1$, and the basis vectors are of the form $v_j = f^j.v_0$ for $j = 0, \ldots, d-1$, and $v_d = 0$.

Remark. From the proof of theorem 4.13 it follows that if an irreducible representation V of $\mathfrak{sl}_2$ contains a highest weight vector of weight μ, then it follows that $V = V_{\mu+1}$.

The irreducible representations of $U_q(\mathfrak{sl}_2)$ correspond to the $\mathfrak{sl}_2$-modules V_d up to choice of sign, which is also proved in [Kas95]. The highest weight vectors w_0 of $U_q(\mathfrak{sl}_2)$-modules can be defined by the conditions $w_0 \neq 0$,

$$K.w_0 = \lambda w_0 \text{ and } E.w_0 = 0.$$

Theorem 4.14. *Let $q \in \mathbb{C} \setminus \{0\}$ not be a root of unity. For any integer $d > 0$ and $\epsilon \in \{\pm 1\}$ there exists a d-dimensional irreducible representation W_d^ϵ of $U_q(\mathfrak{sl}_2)$ with basis $\{w_j\}_{j=0}^{d-1}$ such that*

$$\begin{aligned} K.w_j &= \epsilon q^{d-1-2j} w_j, \\ F.w_j &= w_{j+1}, \\ E.w_j &= \epsilon [j]_q [d-j]_q w_{j-1}. \end{aligned}$$

There are no other finite dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules.

Semisimple Lie algebras have the property that all their finite dimensional modules can be written as direct sums of finitely many irreducible modules. In [Kyt11] it is shown that when q is not a root of unity the modules of $U_q(\mathfrak{sl}_2)$ have also this property. However, when q is a root of unity $U_q(\mathfrak{sl}_2)$ has indecomposable representations, which are not irreducible. In particular, semisimplicity fails in this case.

4.4 The Clebsch-Gordan formula

Given two finite dimensional irreducible $\mathfrak{sl}_2$- or $U_q(\mathfrak{sl}_2)$-modules, consider the tensor product of them equipped with the action defined by the coproduct $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{sl}_2$, or the coproduct of $U_q(\mathfrak{sl}_2)$ from lemma 4.9. We assume that $q \in \mathbb{C} \setminus \{0\}$ is not a root of unity. By semisimplicity the tensor product can be decomposed into a finite direct sum of irreducible modules. This decomposition has a nice form known as the Clebsch-Gordan formula. In order to prove the formula we state first a very useful lemma.

Lemma 4.15. *Let $V = V_{d_1} \otimes V_{d_2}$ be a $\mathfrak{sl}_2$-module, and denote by $\{v_i\}_{i=0}^{d_1-1}$ and $\{v_i'\}_{i=0}^{d_2-1}$ the bases of V_{d_1} and V_{d_2}, respectively, as in theorem 4.13. Then for all $p = 0, \ldots, min\{d_1, d_2\} - 1$ in V there exists a highest weight vector of weight $d_1 + d_2 - 2p - 2$, which is*

$$v^{(p)} = \sum_{s=0}^{p} \frac{(-1)^s}{s!} \frac{p!}{(p-s)!} \frac{(d_1-1-s)!}{(d_1-1)!} \frac{(d_2-p-1+s)!}{(d_2-p-1)!} v_s \otimes v'_{p-s}.$$

Proof. By a direct computation we see that for all $p = 0\dots, min\{d_1, d_2\} - 1$ the vector $v^{(p)}$ is an eigenvector of h with eigenvalue $d_1 + d_2 - 2p - 2$, and e annihilates $v^{(p)}$, that is

$$e.v^{(p)} = \sum_{s=0}^{p} \frac{(-1)^s}{s!} \frac{p!}{(p-s)!} \frac{(d_1 - 1 - s)!}{(d_1 - 1)!} \frac{(d_2 - p - 1 + s)!}{(d_2 - p - 1)!} (e.v_s \otimes 1.v'_{p-s} + 1.v_s \otimes e.v'_{p-s}) = 0.$$

For more details the reader may consult [Kas95]. □

The existence of highest weight vectors of certain weights implies the *Clebsch-Gordan formula.*

Proposition 4.16. (Clebsch-Gordan) *Let $d_1, d_2 > 0$ be two integers. Then there exists an isomorphism of $\mathfrak{sl}_2$-modules*

$$V_{d_1} \otimes V_{d_2} \cong V_{d_1+d_2-1} \oplus V_{d_1+d_2-3} \oplus \cdots \oplus V_{|d_1-d_2|+1}.$$

Proof. By lemma 4.15 the module $V_{d_1} \otimes V_{d_2}$ contains a highest weight vector of weight $d_1 + d_2 - 2p - 2$ for all $p = 0, \dots, min\{d_1, d_2\} - 1$. It follows that it contains the irreducibles $V_{d_1+d_2-2p-1}$ as submodules. That is, there exist nonzero morphisms of modules

$$f_p : V_{d_1+d_2-2p-1} \to V_{d_1} \otimes V_{d_2},$$

$p = 0, \dots, min\{d_1, d_2\} - 1$, and moreover, because $V_{d_1+d_2-2p-1}$ are irreducible, $Ker(f_p) = \{0\}$. So each f_p is an embedding of modules. Since the submodules V_p are distinct for different $p = 0, \dots, d_2 - 1$, their sum is direct. Namely, if for $r \neq s$ the intersection $V_r \cap V_s \neq \{0\}$, it would be a submodule of both V_r and V_s, and then by irreducibility $V_r \cap V_s = V_s = V_r$, contradicting the fact that V_r and V_s are distinct. It remains to notice that the dimensions of the vector spaces $V_{d_1} \otimes V_{d_2}$ and $V_{d_1+d_2-1} \oplus V_{d_1+d_2-3} \oplus \cdots \oplus V_{d_1-d_2+1}$ are equal. Indeed,

$$\begin{aligned} dim(V_{d_1} \otimes V_{d_2}) = d_1 d_2 &= \sum_{p=0}^{d_2-1} (d_1 + d_2 - 2p - 1) \\ &= dim(V_{d_1+d_2-1} \oplus V_{d_1+d_2-3} \oplus \cdots \oplus V_{d_1-d_2+1}). \end{aligned}$$

□

The quantum Clebsch-Gordan formula for $U_q(\mathfrak{sl}_2)$-modules is proved in the same way. The corresponding highest weight vectors are of the form

$$w^{(p)} = \sum_{s=0}^{p} \frac{(-1)^s}{[s]_q!} \frac{[p]_q!}{[p-s]_q!} \frac{[d_1 - 1 - s]_q!}{[d_1 - 1]_q!} \frac{[d_2 - p - 1 + s]_q!}{[d_2 - p - 1]_q!} q^{s(2p-d_2-s)} w_s \otimes w'_{p-s},$$

$p = 0, \dots, min\{d_1, d_2\} - 1.$

Proposition 4.17. (Quantum Clebsch-Gordan)
(i) Let $d > 0$ be an integer, $\epsilon, \epsilon' \in \{\pm 1\}$. Then the irreducible $U_q(\mathfrak{sl}_2)$-modules

$$W_d^{\epsilon'} \otimes W_1^{\epsilon} \cong W_d^{\epsilon\epsilon'} \cong W_1^{\epsilon} \otimes W_d^{\epsilon'}$$

as $U_q(\mathfrak{sl}_2)$-modules.
(ii) Let $d_1, d_2 > 0$ be two integers. Then there exists an isomorphism of $U_q(\mathfrak{sl}_2)$-modules

$$W_{d_1}^1 \otimes W_{d_2}^1 \cong W_{d_1+d_2-1}^1 \oplus W_{d_1+d_2-3}^1 \oplus \cdots \oplus W_{|d_1-d_2|+1}^1.$$

Example 4.18. The $\mathfrak{sl}_2$-module $V_2 \otimes V_2 \cong V_1 \oplus V_3$, where the highest weight vectors are

$$\begin{aligned} v^{(1)} &= v_0 \otimes v_1 - v_1 \otimes v_0 \ \in V_1 \subset V_2 \otimes V_2, \quad \text{and} \\ v^{(0)} &= v_0 \otimes v_0 \ \in V_3 \subset V_2 \otimes V_2, \end{aligned}$$

with the obvious notation for the basis vectors of $V_2 \otimes V_2$.

The corresponding $U_q(\mathfrak{sl}_2)$-module is $W_2^1 \otimes W_2^1 \cong W_1^1 \oplus W_3^1$, with the highest weight vectors

$$\begin{aligned} w^{(1)} &= w_0 \otimes w_1 - q^{-1} w_1 \otimes w_0 \ \in W_1^1 \subset W_2^1 \otimes W_2^1, \quad \text{and} \\ w^{(0)} &= w_0 \otimes w_0 \ \in W_3^1 \subset W_2^1 \otimes W_2^1. \end{aligned}$$

Example 4.19. The $\mathfrak{sl}_2$-module $V_3 \otimes V_3 \cong V_1 \oplus V_3 \oplus V_5$, where the highest weight vectors are

$$\begin{aligned} v^{(2)} &= v_0 \otimes v_2 - v_1 \otimes v_1 + v_2 \otimes v_0 \ \in V_1, \\ v^{(1)} &= v_0 \otimes v_1 - v_1 \otimes v_0 \ \in V_3, \quad \text{and} \\ v^{(0)} &= v_0 \otimes v_0 \ \in V_5. \end{aligned}$$

For $U_q(\mathfrak{sl}_2)$-modules, $W_3^1 \otimes W_3^1 \cong W_1^1 \oplus W_3^1 \oplus W_5^1$, with the highest weight vectors

$$\begin{aligned} w^{(2)} &= w_0 \otimes w_2 - w_1 \otimes w_1 + q^{-2} w_2 \otimes w_0 \ \in W_1^1, \\ w^{(1)} &= w_0 \otimes w_1 - q^{-2} w_1 \otimes w_0 \ \in W_3^1, \quad \text{and} \\ w^{(0)} &= w_0 \otimes w_0 \ \in W_5^1. \end{aligned}$$

5 Monodromy of KZ($\mathfrak{sl}_2$)

Now that we have introduced the theory of analytic continuation in the language of flat connections on vector bundles over complex manifolds, and the basics of bialgebras, together with recalling the necessary results on representations of the Lie algebra $\mathfrak{sl}_2$ and its quantum counterpart $U_q(\mathfrak{sl}_2)$, we are finally ready to consider the KZ-equations associated to $\mathfrak{sl}_2$. We will next explain how to find the fundamental group $\pi_1(Y_N)$ of the complex manifold

$$Y_N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N\} \setminus \bigcup_{i<j} \{z_i = z_j\},$$

where the KZ-equations are analytically defined, and how to compute the monodromy representation of $\pi_1(Y_N)$ associated to the solutions of KZ in the case of the (semi)simple Lie algebra $\mathfrak{sl}_2$. Our treatment of the monodromy of KZ for $\mathfrak{sl}_2$ can be generalised to any semisimple Lie algebra following the same lines. The reason why we have chosen to present the case of $\mathfrak{sl}_2$ is that for $\mathfrak{sl}_2$ the solutions of KZ are the simplest to obtain, although even for that case they include all generalised hypergeometric functions studied in the last century.

We first introduce a group closely related to the symmetric group S_N, called the braid group B_N, which is a finitely generated group intuitively easy to picture. It turns out that a subgroup of this group is isomorphic to the fundamental group $\pi_1(Y_N)$. We will construct not only a monodromy representation of $\pi_1(Y_N)$ but also a representation of B_N, using parallel transport in a quotient bundle having a flat connection, obtained from the trivial vector bundle and the connection defined by the Pfaffian system KZ. The advantage of this is twofold. Naturally, we obtain by restriction the monodromy of KZ. Moreover, as we will see in section 6.8, it turns out that the monodromy representation of B_N is equivalent to a representation arising from the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. This is a remarkable and suprising relation between two apparently distinct ways to construct representations of the braid group.

After introducing the fundamental group of $\pi_1(Y_N)$ and the monodromy representation of B_N, we will solve KZ($\mathfrak{sl}_2$) first in the simplest low dimensional cases, and then in general. We will not prove in detail all the results concerning solutions of KZ, but refer to [EFK98], since the proofs are not very illuminating. The crucial point about the solutions is that they can be written in integral form, indeed very similar to the Euler's solutions of HGE. Moreover, this enables us to use the "contour deformation" method in computing the monodromy of these solutions. The "contour deformation" method produces a monodromy representation of B_N which is later seen to be equivalent to the representation arising from $U_q(\mathfrak{sl}_2)[\sqrt{K}]$.

5.1 The KZ-equations for $\mathfrak{sl}_2$

Let V be a linear finite dimensional complex representation of a semisimple Lie algebra $\mathfrak{g}$, and $W := V^{\otimes N}$ the tensor product representation. Let

$$Y_N = \{(z_1, \dots, z_N) \in \mathbb{C}^N\} \setminus \bigcup_{i<j} \{z_i = z_j\}$$

be a complex manifold with the natural structure, and $\phi : Y_N \to W$ a unknown function defined on Y_N taking values in the vector space W. Let also $\Omega \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be as defined below, acting on $V \otimes V$, and let $\kappa \in \mathbb{C}$. By Ω_{ij} we mean the operator Ω acting on the i:th and j:th tensor component of W.

Denote the coproduct on the universal enveloping algebra $U(\mathfrak{g})$ by

$$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}),$$

defined on the generators $x \in \mathfrak{g}$ of $U(\mathfrak{g})$ by

$$\Delta(x) := 1 \otimes x + x \otimes 1.$$

Let $c \in U(\mathfrak{g})$ be the Casimir element, which is central in $U(\mathfrak{g})$ (see lemma 4.7). The operator Ω defined by

$$\Omega := \frac{1}{2}(\Delta(c) - 1 \otimes c - c \otimes c)$$

is called the *symmetric invariant tensor*. For the (semi)simple Lie algebra $\mathfrak{sl}_2$ with the basis $\{h, e, f\}$ it is of the form

$$\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e.$$

If $v_1, \dots, v_N \in V$, the action of Ω_{ij} on $W = V^{\otimes N}$ for $\mathfrak{sl}_2$ can be written on simple tensors as

$$\begin{aligned} \Omega_{ij}(v_1 \otimes \cdots \otimes v_N) = \; & \frac{1}{2} v_1 \otimes \cdots \otimes h.v_i \otimes \cdots \otimes h.v_j \otimes \cdots \otimes v_N \\ & + v_1 \otimes \cdots \otimes e.v_i \otimes \cdots \otimes f.v_j \otimes \cdots \otimes v_N \\ & + v_1 \otimes \cdots \otimes f.v_i \otimes \cdots \otimes e.v_j \otimes \cdots \otimes v_N, \end{aligned}$$

where $0 < i < j \leq N$.

Remark. Clearly the operator Ω for $\mathfrak{sl}_2$ is symmetric. In general, let $\{x_i\}$ be a basis of $\mathfrak{g}$ and $\{\tilde{x_i}\}$ the dual basis with respect to the *Killing form* $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$, defined by $B(x, y) := Tr(ad_x ad_y)$, where the *adjoint action* $ad_x : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad_x(z) := [x, z]$. Then Ω can be written using the expression $c = \sum_i x_i \tilde{x_i}$ for the Casimir element as

$$\Omega = \frac{1}{2}(\Delta(c) - 1 \otimes c - c \otimes c) = \frac{1}{2} \sum_i (x_i \otimes \tilde{x_i} + \tilde{x_i} \otimes x_i),$$

and symmetricity follows immediately.

Consider the KZ-equations

$$\frac{\partial}{\partial z_i}\phi(z_1,\dots,z_N) = \frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i - z_j}\phi(z_1,\dots,z_N), \tag{KZ($\mathfrak{sl}_2$)}$$

$i = 1,\dots,N$, associated to $\mathfrak{sl}_2$. Recall that $U(\mathfrak{sl}_2)$ acts on $V^{\otimes N}$ by

$$(\rho_V \otimes \cdots \otimes \rho_V) \circ \Delta^{(N)}(x) : V^{\otimes N} \to V^{\otimes N},$$

where $x \in \mathfrak{sl}_2 \subset U(\mathfrak{sl}_2)$ and $\Delta^{(N)}$ is the $(N-1)$-fold coproduct. That is, if $v_1,\dots,v_N \in V$, the action of $x \in \mathfrak{sl}_2$ on simple tensors looks like

$$x.(v_1 \otimes \cdots \otimes v_N) = \sum_{i=1}^{N} v_1 \otimes \cdots \otimes x.v_i \otimes \cdots \otimes v_N.$$

The next proposition says that the operator Ω is invariant with respect to the action of $U(\mathfrak{sl}_2)$.

Proposition 5.1. *The element*

$$\Omega = \frac{1}{2}h \otimes h + e \otimes f + f \otimes e \ \in U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$$

satisfies

$$[\Delta(x), \Omega] = 0$$

for all $x \in U(\mathfrak{sl}_2)$.

Proof. Since by lemma 4.7 the element $c = ef + fe + \frac{1}{2}h^2$ is central in $U(\mathfrak{sl}_2)$ and the coproduct $\Delta : U(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$ is a morphism of algebras, the identity

$$[\Delta(x), \Delta(c)] = \Delta([x, c]) = 0$$

holds for every $x \in U(\mathfrak{sl}_2)$. Hence if $x \in \mathfrak{sl}_2$,

$$\begin{aligned}&[\Delta(x), 1 \otimes c + c \otimes c] = [1 \otimes x + x \otimes 1, 1 \otimes c + c \otimes c]\\ &= [1 \otimes x, 1 \otimes c] + [1 \otimes x, c \otimes c] + [x \otimes 1, 1 \otimes c] + [x \otimes 1, c \otimes c] = 0.\end{aligned}$$

But by definition

$$\Omega = \frac{1}{2}(\Delta(c) - 1 \otimes c - c \otimes c),$$

and the assertion follows from the fact that $\mathfrak{sl}_2$ generates $U(\mathfrak{sl}_2)$ as an algebra. □

Remark. Notice that the above proof shows the invariance of the symmetric tensor $\Omega = \frac{1}{2}(\Delta(c) - 1 \otimes c - c \otimes c)$, where c is the Casimir element, also for a general semisimple Lie algebra. We only used the fact that the Casimir element belongs to the center of the Lie algebra, which is true for any semisimple Lie algebra [Kna04].

The invariance of Ω means that it commutes with the action of $U(\mathfrak{sl}_2)$ on $V \otimes V$, and hence for every $i, j = 1, \ldots, N$ the operator Ω_{ij} commutes with the action of $U(\mathfrak{sl}_2)$ on $V^{\otimes N}$. This implies in particular the following.

Corollary 5.2. *For any solution $\phi : Y_N \to V^{\otimes N}$ of KZ($\mathfrak{sl}_2$) and for any $x \in U(\mathfrak{sl}_2)$ also the function $x.\phi : Y_N \to V^{\otimes N}$ defined by*

$$x.\phi(z_1, \ldots, z_N) := ((\rho_V \otimes \cdots \otimes \rho_V) \circ \Delta^{(N)}(x))(\phi(z_1, \ldots, z_N))$$

is a solution of KZ($\mathfrak{sl}_2$).

In the sequel we will consider the case when V is irreducible. Then by theorem 4.13 it is of the form $V = V_d$. By semisimplicity any finite dimensional $\mathfrak{sl}_2$-module is a direct sum of irreducible modules, whence it is enough to solve the monodromy of KZ($\mathfrak{sl}_2$) in a tensor product $V^{\otimes N}$ of irreducible modules.

5.2 Braids

Recall that the monodromy representation associated to KZ is defined as a linear representation of the fundamental group of the manifold Y_N, by parallel transport along loops on Y_N. Hence in order to compute the monodromy of KZ we first have to find out what the fundamental group of Y_N is. Thus we will in this section define the braid group, which can be thought of as paths (not necessarily loops) on the manifod Y_N. It turns out that a subgroup of the braid group, consisting of elements that correspond loops on Y_N, called the pure braid group, is isomorphic to the fundamental group of Y_N.

5.2.1 The braid group

Definition 5.3. A *simple polygonal arc* in $\mathbb{C} \times [0, 1]$ is the union

$$L = \bigcup_{i=1}^{n-1} [P_i, P_{i+1}]$$

of a finite number of line segments $[P_i, P_{i+1}] \subset \mathbb{C} \times [0, 1]$ such that

$$(P_i, P_{i+1}) \cap (P_j, P_{j+1}) = \emptyset \quad \text{and} \quad P_i \neq P_j \quad \text{for } i \neq j,$$

except possibly for the first and last points $\{i, j\} = \{1, n\}$.

The points $P_i \in \mathbb{C} \times [0, 1]$ are called the *vertices* of L and the line segments $[P_i, P_{i+1}]$ its *edges*. The arc is said to be *closed* if $P_1 = P_n$, and *oriented* if it has been given an orientation in the usual sense of $\mathbb{C} \times [0, 1]$.

Notice that for an oriented arc the notion of a starting and an end point is meaningful. Fix an integer $n > 0$.

Definition 5.4. A *braid b with n strands* is the union of a finite number of pairwise disjoint simple oriented polygonal arcs in $\mathbb{C} \times [0,1]$ such that

(i) $b \cap (\mathbb{C} \times \{0,1\}) = (\{1,\dots,n\} \times \{0\}) \cup (\{1,\dots,n\} \times \{1\})$, that is, b intersects the boundary planes of $\mathbb{C} \times [0,1]$ transversally,

(ii) every point in $\{1,\dots,n\} \times \{0\}$ is a starting point of an arc in b, and every point in $\{1,\dots,n\} \times \{1\}$ is an end point of an arc in b,

(iii) b contains no closed arc, and

(iv) for all $t \in [0,1]$ the intersection of b with the plane $\mathbb{C} \times \{t\}$ consists of exactly n points.

Figure 6 shows a plane diagram of a braid with five strands. The order of the braids composed is from bottom to top, as explained below in detail.

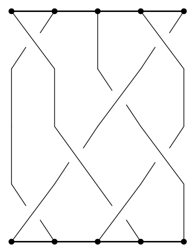


Figure 6: A braid with five strands

Definition 5.5. An *isotopy* of the space $\mathbb{C} \times [0,1]$ is a piecewise linear map

$$h : [0,1] \times \mathbb{C} \times [0,1] \to \mathbb{C} \times [0,1]$$

such that for any $t \in [0,1]$ the map

$$h_t : \mathbb{C} \times [0,1] \to \mathbb{C} \times [0,1]$$

is a homeomorphism,

$$h_0 = id_{\mathbb{C} \times [0,1]},$$

and h_t restricts to the identity on the boundary, that is

$$(h_t)_{|\mathbb{C} \times \{0,1\}} = id_{\mathbb{C} \times \{0,1\}}.$$

Two braids b_1, b_2 are *isotopic* if there exists an isotopy h of $\mathbb{C} \times [0,1]$ such that $h_1(b_1) = b_2$ and for every $t \in [0,1]$ the set $h_t(b_1)$ is a braid with n strands.

Isotopy is an equivalence relation on the set of braids with n strands. From now on we identify braids with their equivalence classes.

We define the *composition* of braids as follows. Let b_1, b_2 be two braids with n strands such that the end points of b_1 are the starting points of b_2. Place b_2 on top of b_1 to obtain a braid $b_2 b_1$ in $\mathbb{C} \times [0,2] = \mathbb{C} \times ([0,1] \cup [1,2])$, and "shrink" $b_2 b_1$ into $\mathbb{C} \times [0,1]$. We read the composed braids *from right to left* so that in plane diagrams such as figure 6 the braids are placed on top of each other starting from the bottom of the diagram and from the right of the composition $b_1 \cdots b_k$.

In [Kas95] it is shown that the composition of braids is compatible with isotopy. Moreover, we have the following result which is in fact quite intuitive. For the proof the reader may consult [Boh47] or [Art47].

Proposition 5.6. *Denote by B_n the isotopy classes of braids with n strands. The composition of braids induces a product on B_n. Moreover, B_n is a group which can be presented by the generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations*

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1, \tag{6}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{7}$$

for $i, j = 1, \ldots, n-1$, $n > 2$. Also, B_1 is the trivial group, and $B_2 \cong \mathbb{Z}$.

The generators and relations are illustrated by figure 7 below.

Since transpositions of the symmetric group S_n satisfy the braid group relations, there is a homomorphism between these groups. Indeed, every braid $b \in B_n$ defines a unique permutation $\tau(b) \in S_n$ of the set of its starting and end points.

Proposition 5.7. *The map $b \mapsto \tau(b)$ induces a surjective homomorphism from the braid group B_n to the symmetric group S_n.*

Proof. Clearly isotopic braids induce the same permutation. The map is a homomorphism since it respects the composition of braids. By proposition 5.6 the group B_n is generated by σ_i, $i = 1, \ldots, n-1$, which induce the permutations $(i, i+1) \in S_n$. Since S_n is generated by these transpositions the map is surjective. □

The kernel of the homomorphism $b \mapsto \tau(b)$ is a normal subgroup of B_n, and it is called the *pure braid group* P_n. It consists of braids which preserve the order of their starting and end points.

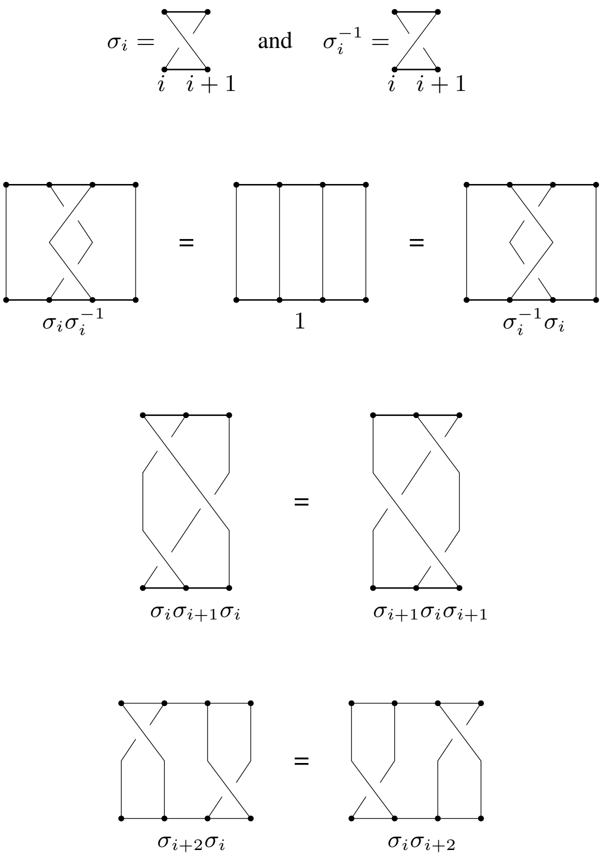


Figure 7: Braid group relations

5.2.2 Representing braids as loops

We are now ready to determine the fundamental group of the manifold

$$Y_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n\} \setminus \bigcup_{i<j} \{z_i = z_j\},$$

where $n > 0$ is an integer. The group $\pi(Y_n)$ is actually isomorphic to the pure braid group. We will also find that the whole braid group is isomorphic to the fundamental group of the *configuration space* $\mathcal{C}_n := Y_n/S_n$ of n distinct points in $\mathbb{C}$, where the symmetric group S_n acts on Y_n by permutation of coordinates.

Consider loops on $\mathcal{C}_n$ representing different homotopy classes. It is clear that every homotopy class contains a piecewise linear path. Hence a loop on $\mathcal{C}_n$ with the base point $p := [(z_1, \dots, z_n)] \in \mathcal{C}_n$ can be thought of as a piecewise linear map $f = (f_1, \dots, f_n) : [0, 1] \to \mathbb{C}^n$ such that for all $t \in [0, 1]$,

$i, j = 1, \ldots, n$, $i \neq j$,

$$f_i(t) \neq f_j(t), \quad \text{and}$$
$$f(0) = (f_1(0), \ldots, f_n(0)) = (z_1, \ldots, z_n),$$
$$\{f_1(1), \ldots, f_n(1)\} = \{z_1, \ldots, z_n\}.$$

Homotopic loops in $\mathcal{C}_n$ correspond to isotopic braids with n strands.

Proposition 5.8. *The braid group B_n is isomorphic to the fundamental group of $\mathcal{C}_n$.*

Proof. The proof consists of two observations. Firstly, we show that loops in $\mathcal{C}_n$ correspond to braids with n strands.

Let $f : [0,1] \to \mathbb{C}^n$ be a loop as above. By connectedness we may assume that $f(0) = (1, \ldots, n) \in \mathbb{C}^n$. Then for any $i = 1, \ldots, n$ the map $f_i : [0,1] \to \mathbb{C}$ is a path that moves the point $i \in \mathbb{C}$ to one of the points $\{1, \ldots, n\}$, and it defines a simple oriented polygonal arc in $\mathbb{C} \times [0,1]$. By definition $\{f_1, \ldots, f_n\}$ forms a collection of pairwise disjoint simple oriented polygonal arcs as in definition 5.4, that is a braid with n strands.

On the other hand, for any braid b with n strands, define for all $t \in [0,1]$ and $i = 1, \ldots, n$ the map $f_i(t)$ as the projection pr_1 of the intersection of $\mathbb{C} \times \{t\}$ with the connected component of b starting at $(i, 0) \in \mathbb{C} \times [0,1]$ onto $\mathbb{C}$. Then the map $f = (f_1, \ldots, f_n) : [0,1] \to \mathbb{C}^n$ is a piecewise linear map satisfying the conditions of a loop on $\mathcal{C}_n$.

Secondly, we show that homotopy of loops on $\mathcal{C}_n$ corresponds to isotopy of braids with n strands.

Let f and g be two homotopic loops on $\mathcal{C}_n$. Then by definition of the homotopy of paths there exists a piecewise linear map

$$H = (H_1, \ldots, H_n) : [0,1] \times [0,1] \to \mathbb{C}^n$$

such that the following holds. We denote by $s \in [0,1]$ the deformation parameter associated to the homotopy H, and by $t \in [0,1]$ the parameter (time) of the loops.

(i) For all $s, t \in [0,1] \times [0,1]$, $i, j = 1, \ldots, n$, $i \neq j$

$$H_i(s,t) \neq H_j(s,t),$$

(ii) for all $s \in [0,1]$, $i = 1, \ldots, n$

$$H_i(s,0) = f_i(0) = g_i(0) \text{ and } H_i(s,1) = f_i(1) = g_i(1),$$

that is $H_s(0) = f(0) = g(0)$, and

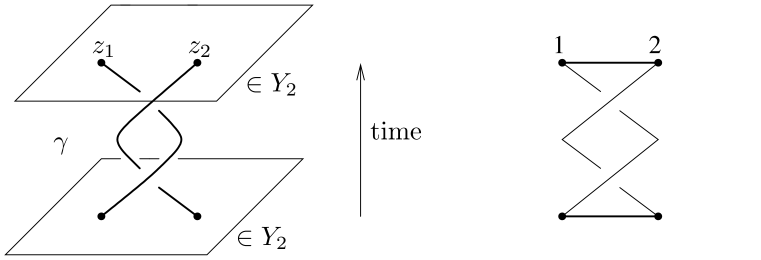
(iii) for all $t \in [0,1]$, $i = 1, \ldots, n$

$$H_i(0,t) = f_i(t) \text{ and } H_i(1,t) = g_i(t),$$

that is $H(0,t) = f(t)$ and $H(1,t) = g(t)$.

The assertion follows from analogy with definition 5.5. □

It is also intuitively clear that the fundamental group of Y_n is isomorphic to the pure braid group P_n, that is braids that preserve the order of the points $\{z_1, \ldots, z_n\}$. This fact can be proved similarly as the previous proposition.



A loop γ in Y_2 — The braid $\sigma_1^{-1}\sigma_1^{-1} \in B_2$

Figure 8: The fundamental group of Y_2 is isomorphic to the pure braid group P_2.

5.2.3 The monodromy representation of the braid group

Recall that the connection $\nabla^{(\Lambda)} = d - \Lambda$, where

$$\Lambda = \frac{1}{\kappa}\sum_i \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} dz_i = \frac{1}{\kappa} \sum_{1 \leq i < j \leq N} \frac{\Omega_{ij}}{z_i - z_j}(dz_i - dz_j),$$

associated to the KZ-equations, is flat. Hence by theorem 3.34 it defines a representation of the fundamental group of Y_N acting on the fibres $V^{\otimes N}$ of the trivial bundle $Y_N \times V^{\otimes N}$. This monodromy representation is motivated by analytic continuation of solutions in the case of ODE's in one complex variable. In higher dimensions the corresponding "continuation" is parallel transport of local horizontal sections, which are local solutions of KZ, along loops on Y_N with respect to the flat connection $\nabla^{(\Lambda)}$. Also in this case analytic continuations of solutions are again solutions of KZ.

On the other hand, the fundamental group of the manifold Y_N is isomorphic to the pure braid group P_N with the generators $\sigma_i^2 \in P_N$, where $\sigma_i \in B_N$, $i = 1, \ldots, N-1$, are the generators of the braid group with N strands. So the flat connection $\nabla^{(\Lambda)}$ defines a monodromy representation

$$\rho_N^{(\Lambda)} : P_N \to Aut(V^{\otimes N})$$

of the pure braid group.

However, we are interested in representations of the whole braid group B_N. Since P_N is a subgroup of B_N, the monodromy of KZ follows from restricting the monodromy representation of B_N (which will be defined shortly) to the set P_N.

Let the symmetric group S_N act naturally on Y_N, that is by

$$(z_1, \dots, z_N).\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(N)})$$

for all $(z_1, \dots, z_N) \in Y_N$, $\sigma \in S_N$, and let

$$\mathcal{C}_N := Y_N / S_N$$

be the configuration space. Consider also the left group action of S_N on $V^{\otimes N}$ by

$$\sigma.(u_1 \otimes \cdots \otimes u_N) = (u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(N)})$$

for all $u_1 \otimes \cdots \otimes u_N \in V^{\otimes N}$, $\sigma \in S_N$.

Combining these we obtain a right action of S_N on the bundle $Y_N \times V^{\otimes N}$ by

$$(z; w).\sigma = (z.\sigma; \sigma^{-1}.w),$$

for all $z := (z_1, \dots, z_N) \in Y_N$, $w = u_1 \otimes \cdots \otimes u_N \in V^{\otimes N}$ and $\sigma \in S_N$.

Since in $\mathcal{C}_N$ the points $(z_1, \dots, z_N)$ and $(z_{\sigma(1)}, \dots, z_{\sigma(N)})$ represent equal elements, the bundle projection $p : Y_N \times V^{\otimes N} \to Y_N$ factors through the quotient space $(Y_N \times V^{\otimes N})/S_N$, which hence becomes a vector bundle over $\mathcal{C}_N$ with fibres isomorphic to $V^{\otimes N}$. Notice that in $(Y_N \times V^{\otimes N})/S_N$ we have

$$((z_1, \dots, z_N); \sigma.w) = ((z_{\sigma(1)}, \dots, z_{\sigma(N)}); w).$$

Furthermore, since by proposition 3.16 the values of a connection can be defined locally, the connection $\nabla^{(\Lambda)}$ on $(Y_N \times V^{\otimes N})$ induces a flat connection ∇^{KZ} on $(Y_N \times V^{\otimes N})/S_N$, and we obtain a well defined monodromy representation of the braid group B_N,

$$\rho_N^{KZ} : B_N \to Aut(V^{\otimes N}), \qquad \rho_N^{KZ} : \sigma_i \mapsto M_{\gamma_i},$$

where $\gamma_i : [0, 1] \to \mathcal{C}_N$ is the loop in $\mathcal{C}_N$ corresponding to the braid group generator σ_i (recall proposition 5.8).

In summary, local horizontal sections of the trivial bundle $(Y_N \times V^{\otimes N})$ are local solutions of KZ, and parallel transport along loops on Y_N with respect to the flat connection $\nabla^{(\Lambda)}$ defines the monodromy of KZ. It is a linear representation of the group $P_N \cong \pi_1(Y_N, z_0)$ with any base point $z_0 \in Y_N$, since the manifold Y_N is connected. However, we will first consider parallel transport of local horizontal sections of $(Y_N \times V^{\otimes N})/S_N$, which are nothing but horizontal sections of $(Y_N \times V^{\otimes N})$ modulo S_N, with respect to the flat connection ∇^{KZ} (which is $\nabla^{(\Lambda)}$ modulo S_N). This defines a linear representation of the fundamental group of the configuration space $\mathcal{C}_N$, namely $B_N \cong \pi_1(\mathcal{C}_N, [z_0])$, where $[z_0] \in \mathcal{C}_N$ denotes the equivalence class of the point $z_0 \in Y_N$ modulo S_N in $\mathcal{C}_N = Y_N/S_N$.

5.3 The two-point KZ-equations in $V_2 \otimes V_2$

Since KZ is a system of partial differential equations with solutions taking values in a representation of a semisimple Lie algebra, solving the equations is not easy in general. Next we consider the first nontrivial example of solving KZ($\mathfrak{sl}_2$) and computing the monodromy, namely $V_2 \otimes V_2$. By example 4.18

$$V_2 \otimes V_2 \cong V_1 \oplus V_3,$$

with the highest weight vectors $v^{(1)} \in V_1$ and $v^{(0)} \in V_3$. For convenience (and to keep in touch with physics, where s stands for *singlet* and t for *triplet*), denote by

$$\begin{aligned} s &:= v^{(1)} = v_0 \otimes v_1 - v_1 \otimes v_0, \\ t_+ &:= v^{(0)} = v_0 \otimes v_0, \\ t_0 &:= f.t_+ = v_1 \otimes v_0 + v_0 \otimes v_1, \\ t_- &:= f^2.t_+ = 2v_1 \otimes v_1, \end{aligned}$$

whence $\{s\}$ is a basis of $V_1 \subset V_2 \otimes V_2$ and $\{t_+, t_0, t_-\}$ is a basis of $V_3 \subset V_2 \otimes V_2$.

The action of Ω on $V_2 \otimes V_2$ is by direct computation

$$\Omega.s = -\frac{3}{2}s, \quad \Omega.t_+ = \frac{1}{2}t_+, \quad \Omega.t_0 = \frac{1}{2}t_0, \quad \Omega.t_- = \frac{1}{2}t_-.$$

Notice that Ω acts as a scalar on each irreducible component of the direct sum decomposition of $V_2 \otimes V_2$ separately, which is to be expected. Namely, Ω is a module map and hence by Schur's lemma 4.10 the maps $\Omega|_{V_1}$ and $\Omega|_{V_3}$ are either zero maps or isomorphisms of modules. Moreover, because every irreducible submodule in the Clebsch-Gordan decomposition is of different dimension, it follows that

$$\Omega|_{V_1} = \lambda_1 id_{V_1} \text{ and } \Omega|_{V_3} = \lambda_3 id_{V_3}$$

for some $\lambda_1, \lambda_3 \in \mathbb{C}$. From above we see that actually $\lambda_1 = -\frac{3}{2}$ and $\lambda_3 = \frac{1}{2}$.

Remark. Notice that by the above observation Ω acts as a scalar on each irreducible component of the direct sum decomposition of a general $\mathfrak{sl}_2$-module $V_{d_1} \otimes V_{d_2}$, since both Schur's lemma 4.10 and the Clebsch-Gordan decomposition hold in general dimensions. We will compute the eigenvalues of Ω on $V_{d_1} \otimes V_{d_2}$ in section 5.4.

5.3.1 Solutions of KZ in $V_2 \otimes V_2$

Let now $\{s, t_+, t_0, t_-\}$ be the ordered basis for $V_2 \otimes V_2$ given in the previous section. Then the Pfaffian system of equations KZ($\mathfrak{sl}_2$) can be written as

$$\begin{cases} \frac{\partial}{\partial z_1}\phi(z_1, z_2) = \frac{1}{\kappa}\frac{\Omega_{12}}{z_1 - z_2}\phi(z_1, z_2) \\ \frac{\partial}{\partial z_2}\phi(z_1, z_2) = \frac{1}{\kappa}\frac{\Omega_{21}}{z_2 - z_1}\phi(z_1, z_2), \end{cases}$$

where

$$\Omega_{12} = \Omega_{21} = \Omega = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

in the basis $\{s, t_+, t_0, t_-\}$.

Adding the two equations together we obtain

$$\left\{ \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right\} \phi(z_1, z_2) = 0,$$

which implies $\phi(z_1, z_2) = \psi(z_1 - z_2) = \psi(z)$, where $z = z_1 - z_2$. The derivative of ψ is

$$\frac{d}{dz}\psi(z) = \frac{d}{dz_1}\phi(z_1, z_2) = \frac{1}{\kappa}\frac{\Omega_{12}}{z_1 - z_2}\phi(z_1, z_2) = \frac{\Omega}{\kappa z}\psi(z).$$

Since the bundle $Y_2 \times (V_2^{\otimes 2})$ is of rank four, we want to find four (pointwise) linearly independent solutions, i.e. linearly independent horizontal sections, for the equation KZ($\mathfrak{sl}_2$). It suffices to find nonzero solutions proportional to the linearly independent basis vectors s, t_+, t_0 and t_-. Actually, by proposition 5.1 it is enough to find the two solutions proportional to s and t_+; the other two are obtained from the solution proportional to t_+ by repeated action of $f \in \mathfrak{sl}_2$. This is also true in general; in order to solve KZ($\mathfrak{sl}_2$) it suffices to find solutions proportional to highest weight vectors of the $\mathfrak{sl}_2$-modules.

Consider first the solution of the form

$$\phi_s(z_1, z_2) = \psi_s(z) = g_s(z)s,$$

where $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is analytic. By equation KZ($\mathfrak{sl}_2$)

$$\frac{d}{dz}\psi_s(z) = \frac{dg_s(z)}{dz}s = \frac{g_s(z)}{\kappa z}\Omega.s = -\frac{3g_s(z)}{2\kappa z}s$$

which has the fundamental solution

$$g_s(z) = z^{-\frac{3}{2\kappa}}.$$

Similarly we obtain the solutions

$$\phi_{t_+}(z_1, z_2) = g_t(z)t_+, \quad \phi_{t_0}(z_1, z_2) = g_t(z)t_0, \quad \phi_{t_-}(z_1, z_2) = g_t(z)t_-,$$

where

$$g_t(z) = z^{\frac{1}{2\kappa}}.$$

Hence a fundamental system of solutions of KZ($\mathfrak{sl}_2$) is

$$\begin{cases} \phi_s(z_1,z_2) = (z_1-z_2)^{-\frac{3}{2\kappa}} s \\ \phi_{t_+}(z_1,z_2) = (z_1-z_2)^{\frac{1}{2\kappa}} t_+ \\ \phi_{t_0}(z_1,z_2) = (z_1-z_2)^{\frac{1}{2\kappa}} t_0 \\ \phi_{t_-}(z_1,z_2) = (z_1-z_2)^{\frac{1}{2\kappa}} t_-, \end{cases}$$

and a general solution is

$$\phi(z_1,z_2) = C_s(z_1-z_2)^{-\frac{3}{2\kappa}} s + (z_1-z_2)^{\frac{1}{2\kappa}}(C_{t_+}t_+ + C_{t_0}t_0 + C_{t_-}t_-),$$

where $C_s, C_{t_+}, C_{t_0}, C_{t_-} \in \mathbb{C}$.

5.3.2 Monodromy of KZ in $V_2 \otimes V_2$

Next we would like to find out how the monodromy operator corresponding to the generator of the fundamental group of $\mathcal{C}_2$ acts on the solutions of KZ($\mathfrak{sl}_2$). Choose for base point of $\mathcal{C}_2$ the equivalence class of the point $(1,2) \in Y_2$, whence the generator of the fundamental group is the homotopy class of the path $\gamma_1 : [0,1] \to Y_2$ defined by

$$\gamma_1(r) = (z_1(r), z_2(r)) := \frac{1}{2}(3 - e^{\pi \mathrm{i} r}, 3 + e^{\pi \mathrm{i} r}).$$

Recall that the homotopy class of γ_1 corresponds to the *inverse* of the braid group generator $\sigma_1 \in B_2$. Moreover, notice that γ_1 defines a loop in $\mathcal{C}_2$, but on Y_2 it is not a closed path.

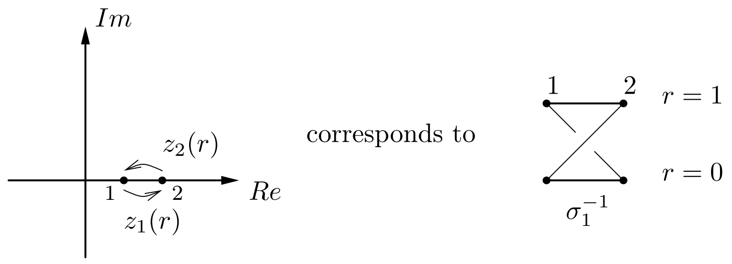


Figure 9: A loop in $\mathcal{C}_2$

Consider the effect of parallel transport along γ_1 on the fundamental system of solutions

$$\begin{cases} \phi_s(z_1,z_2) = g_s(z_1-z_2)s = (z_1-z_2)^{-\frac{3}{2\kappa}} s \\ \phi_t(z_1,z_2) = g_t(z_1-z_2)t = (z_1-z_2)^{\frac{1}{2\kappa}} t, \end{cases}$$

of KZ($\mathfrak{sl}_2$), $t \in \{t_+, t_0, t_-\}$. At $\gamma_1(0) = (1,2)$, fix the branches of the initial values

$$\begin{cases} g_s(1-2) = (-1)^{-\frac{3}{2\kappa}} \\ g_t(1-2) = (-1)^{\frac{1}{2\kappa}}. \end{cases}$$

At $\gamma_1(r)$, $r \in [0,1]$, we have

$$\begin{cases} g_s(z_1(r) - z_2(r)) = (-e^{\pi \mathrm{i} r})^{-\frac{3}{2\kappa}} \\ g_t(z_1(r) - z_2(r)) = (-e^{\pi \mathrm{i} r})^{\frac{1}{2\kappa}}, \end{cases}$$

and in particular, when $r = 1$,

$$\begin{cases} g_s(z_1(1) - z_2(1)) = g_s(2-1) = (-1)^{-\frac{3}{2\kappa}} e^{-\frac{3}{2\kappa}\pi \mathrm{i}} = g_s(1-2) e^{-\frac{3}{2\kappa}\pi \mathrm{i}} \\ g_t(z_1(1) - z_2(1)) = g_t(2-1) = (-1)^{\frac{1}{2\kappa}} e^{\frac{1}{2\kappa}\pi \mathrm{i}} = g_t(1-2) e^{\frac{1}{2\kappa}\pi \mathrm{i}}. \end{cases}$$

In view of examples 1.6 and 1.8 we can write the effect of parallel transport on Y_2 along γ_1, with respect to the flat connection ∇^Λ, on the horizontal sections with values $((z_1,z_2);\phi_s(z_1,z_2)), ((z_1,z_2);\phi_t(z_1,z_2)) \in Y_2 \times V_2^{\otimes 2}$ as

$$\begin{cases} ((1,2);\phi_s(1,2)) \mapsto e^{-\frac{3}{2\kappa}\pi \mathrm{i}}((2,1);\phi_s(1,2)) \\ ((1,2);\phi_t(1,2)) \mapsto e^{\frac{1}{2\kappa}\pi \mathrm{i}}((2,1);\phi_t(1,2)), \end{cases}$$

where $\phi_s(1,2) = (-1)^{-\frac{3}{2\kappa}} s$ and $\phi_t(1,2) = (-1)^{\frac{1}{2\kappa}} t$.

Notice that in $(Y_2 \times V_2^{\otimes 2})/S_2$

$$\begin{aligned}
((2,1);s) &= ((2,1); v_0 \otimes v_1 - v_1 \otimes v_0) \\
&= ((1,2); \tau_{V_2,V_2}(s)) = ((1,2); v_1 \otimes v_0 - v_0 \otimes v_1) = ((1,2); -s), \\
((2,1);t_+) &= ((2,1); v_0 \otimes v_0) = ((1,2); \tau_{V_2,V_2}(t_+)) = ((1,2); v_0 \otimes v_0) \\
&= ((1,2); t_+), \\
((2,1);t_0) &= ((2,1); v_1 \otimes v_0 + v_0 \otimes v_1) = ((1,2); \tau_{V_2,V_2}(t_0)) \\
&= ((1,2); v_0 \otimes v_1 + v_1 \otimes v_0) = ((1,2); t_0), \\
((2,1);t_-) &= ((2,1); 2v_1 \otimes v_1) = ((1,2); \tau_{V_2,V_2}(t_-)) \\
&= ((1,2); 2v_1 \otimes v_1) = ((1,2); t_-).
\end{aligned}$$

Hence the monodromy operator corresponding to the loop γ_1 on $\mathcal{C}_2$ is M_{γ_1}, acting on the space of solutions of KZ($\mathfrak{sl}_2$) by

$$\begin{cases} M_{\gamma_1} : \phi_s(z_1,z_2) \mapsto -e^{-\frac{3}{2\kappa}\pi \mathrm{i}} \phi_s(z_1,z_2) \\ M_{\gamma_1} : \phi_t(z_1,z_2) \mapsto e^{\frac{1}{2\kappa}\pi \mathrm{i}} \phi_t(z_1,z_2). \end{cases}$$

Letting $q := e^{\frac{\pi \mathrm{i}}{\kappa}}$ the inverse operator can be written in the form

$$M_{\gamma_1}^{-1}|_{V_1} = -q^{\frac{3}{2}} id_{V_1}, \quad M_{\gamma_1}^{-1}|_{V_3} = q^{-\frac{1}{2}} id_{V_3},$$

and the braid group representation associated to the solutions of KZ($\mathfrak{sl}_2$) is

$$\rho_2^{KZ} : B_2 \to Aut(V_2^{\otimes 2}), \quad \rho_2^{KZ} : \sigma_1 \mapsto M_{\gamma_1}^{-1}.$$

5.4 The two-point KZ-equations in $V_d \otimes V_d$

By proposition 4.16

$$V_d \otimes V_d \cong V_1 \oplus V_3 \oplus \cdots \oplus V_{2d-1},$$

and by Schur's lemma 4.10, because every submodule in this Clebsch-Gordan decomposition is of different dimension, Ω acts as a scalar on every component separately. Thus for every $i(p) = 2(d-p) - 1 \in \{1, 3, 5, \ldots, 2d-1\}$, $p = 0, \ldots, d-1$,

$$\Omega|_{V_{i(p)}} = \lambda_{i(p)} id_{V_{i(p)}}$$

for some $\lambda_{i(p)} \in \mathbb{C}$. In order to find out the monodromy representation of the braid group B_2, we need to compute the eigenvalues of Ω. After this it suffices to find for all highest weight vectors $v^{(p)}$, $p = 0, \ldots, d-1$, obtained from lemma 4.15, a nonzero solution of KZ($\mathfrak{sl}_2$) proportional to $v^{(p)}$. By proposition 5.1 the rest of the basis of the space of solutions can be constructed from these by repeated action of $f \in \mathfrak{sl}_2$.

In view of solving KZ in $V_2 \otimes V_2$, let

$$\phi_{v^{(p)}}(z_1, z_2) = g_{v^{(p)}}(z_1 - z_2) v^{(p)}$$

be such a solution, where $g_{v^{(p)}} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is analytic. As before, from KZ($\mathfrak{sl}_2$) it follows that we have a fundamental solution of the form

$$g_{v^{(p)}}(z) = z^{\frac{\lambda_{i(p)}}{\kappa}}$$

and a fundamental system of solutions of KZ($\mathfrak{sl}_2$) is obtained from

$$\phi_{v^{(p)}}(z_1, z_2) = (z_1 - z_2)^{\frac{\lambda_{i(p)}}{\kappa}} v^{(p)},$$

$p = 0, \ldots, d-1$, by repeated action of f.

Similarly as before, the monodromy representation

$$\rho_2^{KZ} : B_2 \to Aut(V_d^{\otimes 2}), \quad \rho_2^{KZ} : \sigma_1 \mapsto M_{\gamma_1}^{-1},$$

is found to be defined by

$$M_{\gamma_1}^{-1}|_{V_{i(p)}} = q^{-\lambda_{i(p)}} \tau_{V_d,V_d}^{-1}|_{V_{i(p)}},$$

where $q = e^{\frac{\pi i}{\kappa}}$.

Notice that

$$\tau_{V_d,V_d} \circ \tau_{V_d,V_d} = id_{V_d \otimes V_d},$$

whence τ_{V_d,V_d} has only the two possible eigenvalues ± 1. Moreover, since τ_{V_d,V_d} is a morphism of representations of $\mathfrak{sl}_2$, and every submodule in the Clebsch-Gordan decomposition of $V_d \otimes V_d$ is of different dimension, by Schur's lemma 4.10 τ_{V_d,V_d} acts as a scalar on each $V_{i(p)}$ separately, and it follows that

$$\tau_{V_d,V_d}^{-1}|_{V_{i(p)}} = \tau_{V_d,V_d}|_{V_{i(p)}} = \pm id_{V_{i(p)}}.$$

From the formula

$$v^{(p)} = \sum_{s=0}^{p} \frac{(-1)^s}{s!} \frac{p!}{(p-s)!} \frac{(d_1-1-s)!}{(d_1-1)!} \frac{(d_2-p-1+s)!}{(d_2-p-1)!} v_s \otimes v'_{p-s}$$

for the highest weight vectors generating the submodules $V_{i(p)} \subset V_d \otimes V_d$, where $d_1 = d_2 = d$, and $\{v_j\}_{j=0}^{d_1-1}, \{v'_j\}_{j=0}^{d_2-1}$ are the natural bases of V_{d_1} and V_{d_2}, respectively, we see that

$$\tau_{V_d,V_d}(v^{(p)}) = (-1)^p v^{(p)},$$

whence

$$\tau_{V_d,V_d}|_{V_{i(p)}} = (-1)^p id_{V_{i(p)}}.$$

Let us next compute the eigenvalues of $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e$ on $V_{d_1} \otimes V_{d_2}$. To shorten the notation, write

$$\alpha^{(p)}_{d_1,d_2}(s) := \frac{(-1)^s}{s!} \frac{p!}{(p-s)!} \frac{(d_1-1-s)!}{(d_1-1)!} \frac{(d_2-p-1+s)!}{(d_2-p-1)!},$$

$s = 0, \ldots, p$. By linearity, it is enough to compute

$$\begin{aligned}
&\Omega.(v_s \otimes v'_{p-s}) = \frac{1}{2} h.v_s \otimes h.v'_{p-s} + e.v_s \otimes f.v'_{p-s} + f.v_s \otimes e.v'_{p-s} \\
=&\frac{1}{2}(d_1-1-2s)(d_2-1-2(p-s))(v_s \otimes v'_{p-s}) \\
&+ s(d_1-s)(v_{s-1} \otimes v'_{p-s+1}) + (p-s)(d_2-(p-s))(v_{s+1} \otimes v'_{p-s-1}),
\end{aligned}$$

where we used the action of $\mathfrak{sl}_2$ on the irreducible module V_d given by theorem 4.13,

$$\begin{aligned}
h.v_j &= (d-1-2j)v_j, \\
f.v_j &= v_{j+1}, \\
e.v_j &= j(d-j)v_{j-1}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\alpha^{(p)}_{d_1,d_2}(s-1) = -\frac{(-1)^s}{(s-1)!} \frac{p!}{(p-s+1)!} \frac{(d_1-s)!}{(d_1-1)!} \frac{(d_2-p-2+s)!}{(d_2-p-1)!} \\
=& -\frac{s(d_1-s)}{(p-s+1)(d_2-p-1+s)} \frac{(-1)^s}{s!} \frac{p!}{(p-s)!} \frac{(d_1-1-s)!}{(d_1-1)!} \frac{(d_2-p-1+s)!}{(d_2-p-1)!} \\
=& -\frac{s(d_1-s)}{(p-s+1)(d_2-p-1+s)} \alpha^{(p)}_{d_1,d_2}(s)
\end{aligned}$$

and similarly

$$\begin{aligned}
&\alpha^{(p)}_{d_1,d_2}(s+1) = -\frac{(-1)^s}{(s+1)!} \frac{p!}{(p-s-1)!} \frac{(d_1-2-s)!}{(d_1-1)!} \frac{(d_2-p+s)!}{(d_2-p-1)!} \\
=& -\frac{(p-s)(d_2-(p-s))}{(s+1)(d_1-1-s)} \frac{(-1)^s}{s!} \frac{p!}{(p-s)!} \frac{(d_1-1-s)!}{(d_1-1)!} \frac{(d_2-p-1+s)!}{(d_2-p-1)!} \\
=& -\frac{(p-s)(d_2-(p-s))}{(s+1)(d_1-1-s)} \alpha^{(p)}_{d_1,d_2}(s).
\end{aligned}$$

We obtain

$$
\begin{aligned}
\Omega.v^{(p)} =& \sum_{s=0}^{p} \alpha^{(p)}_{d_1,d_2}(s)\Omega.(v_s \otimes v'_{p-s}) \\
=& \sum_{s=0}^{p} \alpha^{(p)}_{d_1,d_2}(s)\Big\{\frac{1}{2}(d_1-1-2s)(d_2-1-2(p-s))(v_s \otimes v'_{p-s}) \\
&+ s(d_1-s)(v_{s-1}\otimes v'_{p-s+1}) + (p-s)(d_2-(p-s))(v_{s+1}\otimes v'_{p-s-1})\Big\} \\
=& \sum_{s=0}^{p} \Big\{\frac{1}{2}\alpha^{(p)}_{d_1,d_2}(s)(d_1-1-2s)(d_2-1-2(p-s))(v_s \otimes v'_{p-s}) \\
&-(p-s+1)(d_2-p-1+s)\alpha^{(p)}_{d_1,d_2}(s-1)(v_{s-1}\otimes v'_{p-s+1}) \\
&-(s+1)(d_1-1-s)\alpha^{(p)}_{d_1,d_2}(s+1)(v_s \otimes v'_{p-s})\Big\} \\
=& \sum_{s=0}^{p} \alpha^{(p)}_{d_1,d_2}(s)\Big\{\frac{1}{2}(d_1-1-2s)(d_2-1-2(p-s)) \\
&-(p-s)(d_2-p+s) - s(d_1-s)\Big\}(v_s \otimes v'_{p-s}) \\
=& \left(\frac{1}{2}(d_1-1)(d_2-1)+p^2+p(1-d_1-d_2)\right)\sum_{s=0}^{p}\alpha^{(p)}_{d_1,d_2}(s)(v_s \otimes v'_{p-s}) \\
=& \lambda_{i(p)}v^{(p)}, \quad \text{and} \\
\lambda_{i(p)} =& \frac{1}{2}(d_1-1)(d_2-1)+p^2+p(1-d_1-d_2).
\end{aligned}
$$

Now the monodromy generator can be written as

$$M^{-1}_{\gamma_1}|_{V_{i(p)}} = q^{-\lambda_{i(p)}}\tau^{-1}_{V_d,V_d}|_{V_{i(p)}} = (-1)^p q^{-\frac{1}{2}(d_1-1)(d_2-1)-p^2-p(1-d_1-d_2)} id_{V_{i(p)}},$$

and the braid group representation associated to the solutions of KZ($\mathfrak{sl}_2$) is

$$\rho_2^{KZ} : B_2 \to Aut(V_d^{\otimes 2}), \quad \rho_2^{KZ} : \sigma_1 \mapsto M^{-1}_{\gamma_1}.$$

5.5 Solutions of the KZ-equations

When $N > 2$, solving the KZ-equations for $\mathfrak{sl}_2$ becomes more difficult. However, in analogy with the two-point case we have the solutions proportional to the highest weight vector $u_0 := v_0 \otimes \cdots \otimes v_0 \in V_{d_1} \otimes \cdots \otimes V_{d_N}$ of the form

$$\Psi_0(z_1,\ldots,z_N) = \prod_{1\le i<j\le N}(z_i-z_j)^{\frac{\mu_i\mu_j}{2\kappa}}u_0,$$

where we use the obvious notation for the basis vectors. Notice however that the vectors $v_0 \in V_{d_i}$ in different tensor components correspond to representations of possibly different dimensions. The proof that Ψ_0 defines a solution of KZ will be given in short; see proposition 5.9.

Denote by $W := V_{d_1} \otimes \cdots \otimes V_{d_N}$ and let $\mu_i \in \mathbb{C}$ be the highest weights in V_{d_i}, that is $d_i = \mu_i + 1$, $i = 1, \ldots, N$. We want to find solutions $\phi : Y_N \to W$ of KZ($\mathfrak{sl}_2$). Consider first the easiest case, namely the solution proportional to the highest weight vector $u_0 = v_0 \otimes \cdots \otimes v_0 \in W$ of weight $\sum_{i=1}^N \mu_i$. Notice that u_0 is indeed a highest weight vector in W since

$$h.u_0 = \Delta^{(N)}(h)u_0 = \sum_{i=1}^N v_0 \otimes \cdots \otimes h.v_0 \otimes \cdots \otimes v_0 = \sum_{i=1}^N \mu_i u_0 \quad \text{and}$$
$$e.u_0 = \Delta^{(N)}(e)u_0 = \sum_{i=1}^N v_0 \otimes \cdots \otimes e.v_0 \otimes \cdots \otimes v_0 = 0.$$

The action of the symmetric invariant tensor is

$$\Omega_{ij} u_0 = (\frac{1}{2} h \otimes h + e \otimes f + f \otimes e)_{ij} u_0 = (\frac{1}{2} h \otimes h + 0)_{ij} u_0 = \frac{\mu_i \mu_j}{2} u_0$$

for all $i, j = 1, \ldots, N$.

If $\phi : Y_N \to W$ is a solution of the form

$$\phi(z_1, \ldots, z_N) = \psi(z_1, \ldots, z_N) u_0,$$

where $\psi : Y_N \to \mathbb{C}$ is analytic, then the equation KZ($\mathfrak{sl}_2$) for ψ reads

$$\frac{\partial}{\partial z_i} \psi(z_1, \ldots, z_N) = \frac{1}{2\kappa} \sum_{j \neq i} \frac{\mu_i \mu_j}{z_i - z_j} \psi(z_1, \ldots, z_N).$$

Analogously to the two-point case we get the following solutions of KZ($\mathfrak{sl}_2$).

Proposition 5.9. *The multivalued function* $\Psi_0 : Y_N \to W$,

$$\Psi_0(z_1, \ldots, z_N) := \psi_0(z_1, \ldots, z_N) u_0,$$

where

$$\psi_0(z_1, \ldots, z_N) = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}},$$

is a solution of KZ($\mathfrak{sl}_2$) taking values in the one dimensional subspace of highest weight vectors of W of weight $\sum_{i=1}^N \mu_i$.

Proof. By direct computation, the function ψ_0 satisfies

$$\frac{\partial}{\partial z_i} \psi_0(z_1, \ldots, z_N) = \frac{1}{2\kappa} \sum_{j \neq i} \frac{\mu_i \mu_j}{z_i - z_j} \psi_0(z_1, \ldots, z_N).$$

□

Remark. We denote by $Sol(H_{u_0})$ the space of solutions of KZ($\mathfrak{sl}_2$) taking values in the one dimensional subspace $H_{u_0} \subset W$ of highest weight vectors of weight $\sum_{i=1}^N \mu_i$. Notice that $Sol(H_{u_0})$ is one dimensional, and spanned by Ψ_0.

We also denote by $Sol(W_{u_0})$ the space of solutions of KZ($\mathfrak{sl}_2$) taking values in the submodule $W_{u_0} \subset W$ generated by the vector u_0. By corollary 5.2 a basis

$$\{f^k.\Psi_0\}_{k=0}^{dim(W_{u_0})-1}$$

for $Sol(W_{u_0})$ is obtained by repeated action of $f \in \mathfrak{sl}_2$.

5.5.1 Solutions of KZ in $V_{d_1} \otimes V_{d_2} \otimes V_{d_3}$

Since for representations of Lie algebras the natural isomorphism of tensor products

$$(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$$

is an isomorphism of representations, we can use proposition 4.16 repeatedly to obtain a direct sum decomposition of $V_{d_1} \otimes V_{d_2} \otimes V_{d_3}$ into irreducible $\mathfrak{sl}_2$-modules.

Furthermore, by Schur's lemma 4.10 the operator Ω_{ij} shuffles the components of the same dimension of the direct sum. Recall that by the subscript ij, $i, j \in \{1, 2, 3\}$, we mean the operator acting on the i:th and j:th component of the tensor product $V_{d_1} \otimes V_{d_2} \otimes V_{d_3}$.

We consider first solutions of KZ($\mathfrak{sl}_2$) taking values in the general space $V_{d_1} \otimes V_{d_2} \otimes V_{d_3}$, where the dimensions d_1, d_2, d_3 might be different. For studying the monodromy representation of the braid group B_3 we need to take $d_1 = d_2 = d_3$ in order the action of S_3 on $V_{d_1} \otimes V_{d_2} \otimes V_{d_3}$ to be well defined.

The following construction is presented in [Var95] and [EFK98]. Fix numbers $\kappa \in \mathbb{C}$ and $m_1, m_2, m_3 \in \mathbb{C}$, and consider the multivalued holomorphic function

$$l_{z_1,z_2,z_3}(w) := \prod_{1 \le i < j \le 3} (z_i - z_j)^{\frac{m_i m_j}{2\kappa}} \prod_{i=1}^{3} (w - z_i)^{-\frac{m_i}{\kappa}},$$

defined on $\mathbb{C} \setminus \{z_1, z_2, z_3\}$. Define the multivalued holomorphic one-forms on $\mathbb{C} \setminus \{z_1, z_2, z_3\}$ by

$$\eta_i := l_{z_1,z_2,z_3}(w) \frac{dw}{w - z_i},$$

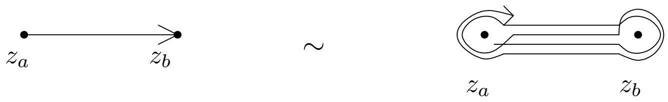
$i = 1, 2, 3$. We observe that

$$dl_{z_1,z_2,z_3}(w) = \sum_{i=1}^{3} -\frac{m_i}{\kappa} \frac{dw}{w - z_i} l_{z_1,z_2,z_3}(w) = -\frac{1}{\kappa} \sum_{i=1}^{3} m_i \eta_i,$$

which means that the form $\sum_{i=1}^3 m_i\eta_i \in \Omega^1(\mathbb{C}\setminus\{z_1,z_2,z_3\})$ is exact. Consider then the integrals

$$I_k := \int_{\Gamma_{z_1,z_2,z_3}} \eta_k,$$

where Γ_{z_1,z_2,z_3} is the line segment between z_a and z_b shown in the following figure and z_a, z_b are any two of the points z_1, z_2, z_3.



Changing the path of integration to the closed Pochammer contour, we see similarly as in chapter 2 that

$$\sum_{k=1}^3 m_k I_k = \int_{\Gamma_{z_1,z_2,z_3}} \sum_{k=1}^3 m_k\eta_k = -\kappa\int_{\Gamma_{z_1,z_2,z_3}} dl_{z_1,z_2,z_3}(w) = 0. \tag{8}$$

By Caychy's theorem of complex integrals (see [Rud87]) the value of I_k is homotopy invariant with respect to the path of integration. Now the integrals I_k are actually functions of the complex variables z_1, z_2, z_3, and we may consider their partial derivatives in these variables. Define the holomorphic function $I := (I_1, I_2, I_3) : Y_3 \to \mathbb{C}^3$, where

$$Y_3 = \{(z_1,z_2,z_3)\in\mathbb{C}^3\}\setminus\bigcup_{i<j}\{z_i = z_j\}.$$

By direct computation we get the following identity.

Proposition 5.10. $I(z_1,z_2,z_3)$ *satisfies the following system of linear partial differential equations*

$$\frac{\partial}{\partial z_i} I(z_1,z_2,z_3) = \frac{1}{\kappa}\sum_{j\neq i}\frac{A_{ij}}{z_i - z_j} I(z_1,z_2,z_3), \tag{9}$$

$i = 1,2,3,$ *where the matrix* $A_{ij} = ((A_{ij})_{kl})_{k,l=1,2,3},\ i,j = 1,2,3,$ *has the components*

$$(A_{ij})_{kk} = \frac{m_i m_j}{2} \ \ \textit{for}\ \ k\notin\{i,j\},$$
$$(A_{ij})_{ii} = \frac{m_i m_j}{2} - m_j, \quad (A_{ij})_{jj} = \frac{m_i m_j}{2} - m_i,$$
$$(A_{ij})_{ij} = m_j, \quad (A_{ij})_{ji} = m_i,$$

the other components being zero.

Proof. We compute one of the partial derivatives of I; the others are similar. First,

$$\begin{aligned}\frac{\partial}{\partial z_2}\eta_3 &= l_{z_1,z_2,z_3}(w)\left(\frac{-\frac{m_1m_2}{2\kappa}}{z_1-z_2}+\frac{\frac{m_2m_3}{2\kappa}}{z_2-z_3}+\frac{\frac{m_2}{\kappa}}{w-z_2}\right)\frac{dw}{w-z_3}\\ &= -\frac{m_1m_2}{2\kappa(z_1-z_2)}\eta_3+\frac{m_2m_3}{2\kappa(z_2-z_3)}\eta_3+\frac{m_2}{\kappa(z_2-z_3)}(\eta_2-\eta_3),\end{aligned}$$

which implies

$$\begin{aligned}\frac{\partial}{\partial z_2}I_3 &= \frac{m_2}{\kappa(z_2-z_3)}I_2+\left(-\frac{m_1m_2}{2\kappa(z_1-z_2)}+\frac{m_2(m_3-2)}{2\kappa(z_2-z_3)}\right)I_3\\ &= \frac{m_2}{\kappa(z_2-z_3)}I_2+\left(\frac{m_1m_2}{2\kappa(z_2-z_1)}+\left(\frac{m_2m_3}{2}-m_2\right)\frac{1}{\kappa(z_2-z_3)}\right)I_3.\end{aligned}$$

Above we used the relation

$$\frac{1}{(w-z_2)}\frac{1}{(w-z_3)}=\frac{1}{(z_2-z_3)}\frac{1}{(w-z_2)}+\frac{1}{(z_3-z_2)}\frac{1}{(w-z_3)}.$$

□

Next we will apply the previous results to obtain a new solution of KZ($\mathfrak{sl}_2$) in $W = V_{d_1}\otimes V_{d_2}\otimes V_{d_3}$. By proposition 5.1 it suffices to find solutions (pointwise) proportional to the highest weight vectors of W, which are the highest weight vectors corresponding to the components of the direct sum decomposition.

Recall that we denote by $\mu_1,\mu_2,\mu_3\in\mathbb{C}$ the highest weights of the $\mathfrak{sl}_2$-modules V_{d_1},V_{d_2},V_{d_3}, respectively. Recall also that in proposition 5.9 we found the solution $\Psi_0 : Y_3\to W$ of the form

$$\Psi_0(z_1,z_2,z_3)=\psi_0(z_1,z_2,z_3)u_0,$$

where

$$\psi_0(z_1,z_2,z_3)=\prod_{1\le i<j\le 3}(z_i-z_j)^{\frac{\mu_i\mu_j}{2\kappa}}.$$

This is the easiest case, namely a solution proportional to the highest weight vector $u_0 = v_0\otimes v_0\otimes v_0\in W$ of weight $\mu_1+\mu_2+\mu_3$.

Consider then solutions (pointwise) proportional to highest weight vectors in W of weight $\mu_1+\mu_2+\mu_3-2$. Denote by f_k the element $f\in\mathfrak{sl}_2$ acting on the k:th component of a vector $v\in W$.

Lemma 5.11. *Any highest weight vector of weight $\mu_1+\mu_2+\mu_3-2$ in W can be written as*

$$u_1^{(a_1,a_2,a_3)} := a_1v_1\otimes v_0\otimes v_0+a_2v_0\otimes v_1\otimes v_0+a_3v_0\otimes v_0\otimes v_1=\sum_{k=1}^{3}a_kf_ku_0,$$

where the coefficients $a_1, a_2, a_3 \in \mathbb{C}$ *satisfy*

$$\sum_{k=1}^{3} \mu_k a_k = 0.$$

In particular, the subspace of W *spanned by highest weight vectors of weight* $\mu_1 + \mu_2 + \mu_3 - 2$ *is two dimensional.*

Proof. Firstly,

$$\begin{aligned} h.u_1^{(a_1,a_2,a_3)} &= (\mu_1 + \mu_2 + \mu_3 - 2)a_1 v_1 \otimes v_0 \otimes v_0 \\ &+ (\mu_1 + \mu_2 + \mu_3 - 2)a_2 v_0 \otimes v_1 \otimes v_0 + (\mu_1 + \mu_2 + \mu_3 - 2)a_3 v_0 \otimes v_0 \otimes v_1 \\ &= (\mu_1 + \mu_2 + \mu_3 - 2)u_1^{(a_1,a_2,a_3)}, \end{aligned}$$

and secondly,

$$\begin{aligned} e.u_1^{(a_1,a_2,a_3)} &= a_1\mu_1 v_0 \otimes v_0 \otimes v_0 + a_2\mu_2 v_0 \otimes v_0 \otimes v_0 + a_3\mu_3 v_0 \otimes v_0 \otimes v_0 \\ &= \left(\sum_{k=1}^{3} \mu_k a_k\right) u_0 = 0 \quad \text{if and only if} \quad \sum_{k=1}^{3} \mu_k a_k = 0. \end{aligned}$$

Because two of the coefficients can be chosen freely, the space is two dimensional. □

Next we will show that if we let the coefficients depend on the complex variables z_1, z_2, z_3, we obtain a solution of KZ($\mathfrak{sl}_2$).

Lemma 5.12. *Let* A_{ij} *be the matrices of proposition 5.10. Then*

$$(A_{ij})_{kl} = (\Omega_{ij})_{kl}$$

for all $i, j, k, l \in \{1, 2, 3\}$, *where* $(\Omega_{ij})_{kl}$ *are the matrix elements of the operator* Ω_{ij} *acting on the basis* $\{f_1 u_0, f_2 u_0, f_3 u_0\}$.

Proof. The proof is a direct computation using the action of $\mathfrak{sl}_2$ stated in theorem 4.13. For instance, we compute the operator Ω_{23},

$$\begin{aligned} \Omega_{23}.(v_1 \otimes v_0 \otimes v_0) &= \frac{\mu_2\mu_3}{2} v_1 \otimes v_0 \otimes v_0, \\ \Omega_{23}.(v_0 \otimes v_1 \otimes v_0) &= \frac{(\mu_2 - 2)\mu_3}{2} v_0 \otimes v_1 \otimes v_0 + \mu_2 v_0 \otimes v_0 \otimes v_1, \\ \Omega_{23}.(v_0 \otimes v_0 \otimes v_1) &= \frac{\mu_2(\mu_3 - 2)}{2} v_0 \otimes v_0 \otimes v_1 + \mu_3 v_0 \otimes v_1 \otimes v_0, \end{aligned}$$

whence

$$\Omega_{23} = \begin{pmatrix} \frac{\mu_2\mu_3}{2} & 0 & 0 \\ 0 & \frac{(\mu_2-2)m_3}{2} & \mu_3 \\ 0 & \mu_2 & \frac{\mu_2(\mu_3-2)}{2} \end{pmatrix}$$

□

We have hence proved the following proposition. In this case the symbol $\mathcal{P}$ denotes the path Γ_{z_1,z_2,z_3} of integration. Recall that, as in (8), we may vary the complex variables z_1, z_2, z_3 infinitesimally, thinking Γ_{z_1,z_2,z_3} as the Pochammer contour. The notation $\mathcal{P}$ is motivated by the fact that there are also other solutions for KZ($\mathfrak{sl}_2$) of integral form, corresponding to highest weight vectors of lower weights. In treatment of these we will use similar notation.

Proposition 5.13. *The multivalued function* $\Psi_1^{(\mathcal{P})} : Y_3 \to W$,

$$\Psi_1^{(\mathcal{P})}(z_1, z_2, z_3) := \sum_{k=1}^{3} I_k^{(\mathcal{P})}(z_1, z_2, z_3) f_k u_0,$$

where

$$I_k^{(\mathcal{P})}(z_1, z_2, z_3) = \int_{\mathcal{P}} \prod_{1\le i<j\le 3} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}} \prod_{i=1}^{3} (w - z_i)^{-\frac{\mu_i}{\kappa}} \frac{dw}{w - z_k},$$

is a solution of KZ($\mathfrak{sl}_2$) taking values in the two dimensional subspace of highest weight vectors of W of weight $\mu_1 + \mu_2 + \mu_3 - 2$, given by lemma 5.11.

Remark. Notice that dimension of the space $Sol(H_{u_1})$ of solutions of the form $\Psi_1^{(\mathcal{P})}$ is at most two, which follows from lemma 5.11. Actually, if $\kappa \notin \mathbb{Q}$, a basis of $Sol(H_{u_1})$ is obtained by taking two homologically different paths of integration for the coefficients $I_k^{(\mathcal{P})}$, for example the paths $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ and $\Gamma^{(2,3)}_{z_1,z_2,z_3}$ between the points z_1, z_2, and z_2, z_3, respectively. The solutions corresponding to these paths of integration are linearly independent, which can be seen while computing the monodromy, similarly as in section 2.5 for the hypergeometric equation.

We also note that by corollary 5.2 a basis of the space $Sol(W_{u_1})$ of solutions taking values in the submodule $W_{u_1} \subset W$ generated by the highest weight vectors of W of weight $\mu_1 + \mu_2 + \mu_3 - 2$ is obtained by repeated action of $f \in \mathfrak{sl}_2$.

The solutions (pointwise) proportional to highest weight vectors of lower weights are considered in the next section, together with a general number of tensor components in the $\mathfrak{sl}_2$-module W.

5.5.2 Solutions of KZ in $V_{d_1} \otimes \cdots \otimes V_{d_N}$

Following the same lines as in the previous section, let $\mu_i = d_i - 1$ be the highest weights of the irreducible $\mathfrak{sl}_2$-modules V_{d_i}, $i = 1, \ldots, N$, $\kappa \in \mathbb{C}$, and let $u_0 = v_0 \otimes \cdots \otimes v_0$ be the highest weight vector of $W := V_{d_1} \otimes \cdots \otimes V_{d_N}$ of weight $\sum_{i=1}^{N} \mu_i$. Let $z_1, \ldots, z_N \in \mathbb{C}$ be fixed distinct points, and write

$z = (z_1, \ldots, z_N) \in Y_N$. Define $\psi_1 : Y_N \to \mathbb{C}$ by

$$\psi_1(z, w) = \prod_{i=1}^{N} (w - z_i)^{-\frac{\mu_i}{\kappa}}.$$

Let $\mathcal{P} : [0, 1] \to \mathbb{C} \setminus \{z_1, \ldots, z_N\}$ be a loop such that

$$t \mapsto \psi_1(z, \mathcal{P}(t))$$

is a well defined continuous complex function on $[0, 1]$ and takes the same values in the end points of the path $\mathcal{P}$, that is

$$\psi_1(z, \mathcal{P}(0)) = \psi_1(z, \mathcal{P}(1)).$$

An example of this kind of loop is the Pochammer contour, which is not homotopically trivial. These assumptions on $\mathcal{P}$ guarantee that the following integral is well defined, and that the boundary terms sum up to zero in partial integration, when considering the differential operator of the KZ-equations acting on the integral.

Define the multivalued function $\Psi_1^{(\mathcal{P})} : Y_N \to W$ by

$$\Psi_1^{(\mathcal{P})}(z) := \psi_0(z) \int_{\mathcal{P}} \psi_1(z, w) \sum_{k=1}^{N} \frac{dw}{w - z_k} f_k u_0,$$

where f_k denotes the element $f \in \mathfrak{sl}_2$ acting on the k:th component of a vector $v \in W$. Assume that a continuous branch of $t \mapsto \psi_1(z, \mathcal{P}(t))$ on $\mathcal{P}$ has been chosen so that the integral is well defined. It is important to notice that the value of the function $\Psi_1^{(\mathcal{P})}$ does not depend on this choice except up to a "phase factor".

We are ready to state the generalisation of proposition 5.13.

Proposition 5.14. *Let*

$$\psi_0(z) = \prod_{1 \le i < j \le N} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}}, \quad \psi_1(z, w) = \prod_{i=1}^{N} (w - z_i)^{-\frac{\mu_i}{\kappa}},$$

and let $\mathcal{P} : [0, 1] \to \mathbb{C} \setminus \{z_1, \ldots, z_N\}$ be a loop such that for a fixed point $z = (z_1, \ldots, z_N) \in Y_N$ the map $t \mapsto \psi_1(z, \mathcal{P}(t))$ is a well defined complex function on $[0, 1]$ satisfying $\psi_1(z, \mathcal{P}(0)) = \psi_1(z, \mathcal{P}(1))$. Then the multivalued function $\Psi^{(\mathcal{P})} : Y_N \to W$,

$$\Psi_1^{(\mathcal{P})}(z) := \psi_0(z) \int_{\mathcal{P}} \psi_1(z, w) \sum_{k=1}^{N} \frac{dw}{w - z_k} f_k u_0$$

is a solution of KZ($\mathfrak{sl}_2$) taking values in the subspace of highest weight vectors of W of weight $\sum_{i=1}^{N} \mu_i - 2$.

Proof. For convenience, introduce the following multivalued operator functions.

$$T_i(z,w) := \psi_1(z,w)\frac{f_i}{w-z_i},\ i = 1,\dots,N,$$
$$Y(z,w) := \sum_{i=1}^{N} T_i(z,w),$$
$$H(z,w) := \sum_{i=1}^{N} \frac{\mu_i - h_i}{w-z_i}.$$

Then by direct computation we obtain the following relation, where $[\cdot,\cdot]$ denotes the commutator of operators, that is $[A,B] = AB - BA$,

$$\frac{\partial}{\partial z_i}Y - \Big[\frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}, Y\Big] = \frac{1}{\kappa}Y\frac{\mu_i - h_i}{w-z_i} - \frac{1}{\kappa}T_iH - \frac{\partial}{\partial w}T_i.$$

We can now write

$$\Psi_1^{(\mathcal{P})}(z) = \psi_0(z)\int_{\mathcal{P}}\psi_1(z,w)\sum_{k=1}^{N}\frac{dw}{w-z_k}f_k u_0 = \psi_0(z)\int_{\mathcal{P}}Y(z,w)u_0 dw.$$

Consider the KZ-operator

$$\mathcal{D}_i := \frac{\partial}{\partial z_i} - \frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}$$

acting on $\Psi_1^{(\mathcal{P})}$. Since by proposition 5.9 the function $\psi_0 u_0$ is annihilated by the KZ-operator, we obtain

$$\mathcal{D}_i\Psi_1^{(\mathcal{P})}(z) = \mathcal{D}_i\Big(\psi_0(z)\int_{\mathcal{P}}Y(z,w)u_0dw\Big) = \int_{\mathcal{P}}\mathcal{D}_i\Big(\psi_0(z)Y(z,w)\Big)u_0dw$$
$$= \int_{\mathcal{P}}\Big(Y(z,w)\mathcal{D}_i\psi_0(z)u_0 + \psi_0(z)\mathcal{D}_iY(z,w)u_0\Big)dw = \int_{\mathcal{P}}\psi_0(z)\mathcal{D}_iY(z,w)u_0dw$$
$$= \int_{\mathcal{P}}\Big(\mathcal{D}_iY(z,w) - Y(z,w)\mathcal{D}_i\Big)\psi_0(z)u_0dw = \psi_0(z)\int_{\mathcal{P}}[\mathcal{D}_i, Y(z,w)]u_0dw$$
$$= \psi_0(z)\int_{\mathcal{P}}\Big[\frac{\partial}{\partial z_i} - \frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}, Y(z,w)\Big]u_0dw$$
$$= \psi_0(z)\int_{\mathcal{P}}\Big(\frac{\partial}{\partial z_i}Y(z,w) - \Big[\frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}, Y(z,w)\Big]\Big)u_0dw.$$

Using the above relation and the fact that $h_iu_0 = \mu_iu_0$ we obtain

$$\mathcal{D}_i\Psi_1^{(\mathcal{P})}(z) = \psi_0(z)\int_{\mathcal{P}} -\frac{\partial}{\partial w}T_i(z,w)u_0dw.$$

The assertion follows using the assumptions

$$\psi_1(z, \mathcal{P}(0)) = \psi_1(z, \mathcal{P}(1)), \quad \mathcal{P}(0) = \mathcal{P}(1)$$

on the path of integration $\mathcal{P}$, which imply

$$\begin{aligned}
&\psi_0(z) \int_{\mathcal{P}} -\frac{\partial}{\partial w} T_i(z, w) u_0 dw \\
&= -\psi_0(z)(T_i(z, \mathcal{P}(1)) - T_i(z, \mathcal{P}(0)))u_0 \\
&= -\psi_0(z)\Big(\psi_1(z, \mathcal{P}(1))\frac{f_i}{\mathcal{P}(1) - z_i} - \psi_1(z, \mathcal{P}(0))\frac{f_i}{\mathcal{P}(0) - z_i}\Big)u_0 = 0.
\end{aligned}$$

□

Next we will study the solutions (pointwise) proportional to highest weight vectors of lower weights. The weights are eigenvalues of $h \in \mathfrak{sl}_2$, and they are of the form

$$\sum_{i=1}^{N} \mu_i - 2l.$$

We call the number l the *level* of the solution; we have already found the solutions corresponding to levels zero and one. Write $w = (w_1, \ldots, w_l) \in \mathbb{C}^l$ and $dw = dw_1 \wedge \cdots \wedge dw_l$, and fix a point $z = (z_1, \ldots, z_N) \in Y_N$.

Define the complex manifold

$$Y_{z,l} := \mathbb{C}^l \setminus \left(\bigcup_{i<j} \{w_i = w_j\} \cup \bigcup_{i,j} \{w_i = z_j\} \right)$$

with holomorphic structure inherited from $\mathbb{C}^l$. Define also the multivalued holomorphic differential l-form $\eta_{z,l}$ by

$$\eta_{z,l} := \prod_{1 \le i < j \le l} (w_i - w_j)^{\frac{2}{\kappa}} \prod_{j,k} (w_j - z_k)^{-\frac{\mu_k}{\kappa}} \prod_{i=1}^{l} \sum_{k=1}^{N} \frac{f_k}{w_i - z_k} dw.$$

Notice that $\eta_{z,l}$ is operator valued, meaning that its values act on the vector space W. We can now express a solution of KZ($\mathfrak{sl}_2$) corresponding to arbitrary level l as follows.

Suppose $\mathcal{P}$ is an l-dimensional cohomologically closed surface in the sense that the boundary terms in the following integration add up to zero after applying the KZ-operator as in the proof of proposition 5.14. The reader may find in [EFK98] a rigorous inspection of the conditions that $\mathcal{P}$ needs to satisfy.

Proposition 5.15. *The multivalued function* $\Psi_l^{(\mathcal{P})} : Y_N \to W$,

$$\Psi_l^{(\mathcal{P})}(z) := \psi_0(z) \int_{\mathcal{P}} \eta_{z,l} u_0,$$

where

$$\psi_0(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}},$$

$$\eta_{z,l} = \prod_{1 \leq i < j \leq l} (w_i - w_j)^{\frac{2}{\kappa}} \prod_{j,k} (w_j - z_k)^{-\frac{\mu_k}{\kappa}} \prod_{i=1}^{l} \sum_{k=1}^{N} \frac{f_k}{w_i - z_k} dw,$$

is a solution of KZ($\mathfrak{sl}_2$) taking values in the subspace of highest weight vectors of W of weight $\sum_{i=1}^N \mu_i - 2l$.

Proof. The idea of the proof is similar to the proof of proposition 5.13, although this general case needs much more computations. We omit the proof and refer to [EFK98], section 4.4. □

5.6 Monodromy of the KZ-equations

Now that we have found solutions of KZ($\mathfrak{sl}_2$) we can finally compute their monodromy. We consider as examples the simplest cases, corresponding to the solutions Ψ_0 and $\Psi_1^{(\mathcal{P})}$. Moreover, we introduce a recursion which enables us to find a general formula for the monodromy of any solution of the form of proposition 5.15.

5.6.1 Monodromy of Ψ_0 in $V_d \otimes V_d \otimes V_d$

Let $q = e^{\frac{\pi \mathrm{i}}{\kappa}}$. Since

$$(\tau_{V_d,V_d})_{i,i+1}(u_0) = (\tau_{V_d,V_d})_{i,i+1}(v_0 \otimes v_0 \otimes v_0) = u_0$$

for $i = 1, 2$, in view of section 5.4 it is clear that the monodromy action on the solution $\Psi_0 : Y_3 \to W$,

$$\Psi_0(z) = \prod_{1 \leq i < j \leq 3} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}} u_0,$$

$z = (z_1, z_2, z_3) \in Y_3$, of proposition 5.9, is of the form

$$M^{-1}_{\gamma_{z_i,z_{i+1}}} : \Psi_0(z) \mapsto e^{-\pi \mathrm{i} \frac{\mu_i \mu_{i+1}}{2\kappa}} \Psi_0(z) = q^{-\frac{\mu_i \mu_{i+1}}{2}} \Psi_0(z),$$

where $\gamma_{z_i,z_{i+1}} : [0,1] \to Y_3$ is a half-loop exchanging the points z_i and z_{i+1} counterclockwise. Notice that $\gamma_{z_i,z_{i+1}}$ represents a loop in $\mathcal{C}_3$, and the homotopy classes of the loops γ_{z_1,z_2} and γ_{z_2,z_3} correspond to the inverses of the braid group generators $\sigma_1, \sigma_2 \in B_3$, respectively. Recall that the monodromy group is independent of the choice of a base point for the fundamental group of $\mathcal{C}_3$ because $\mathcal{C}_3$ is connected in the quotient topology.

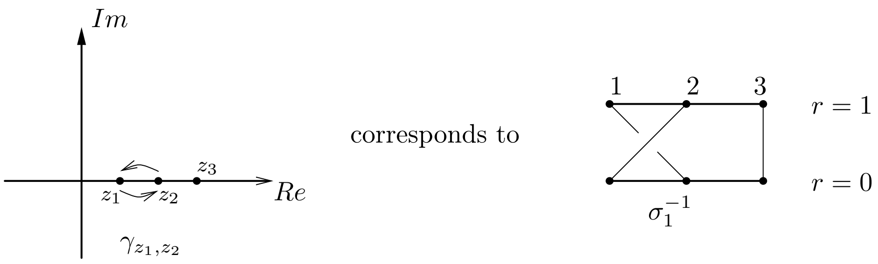


Figure 10: A loop in $\mathcal{C}_3$

Recall that $\Psi_0 \in Sol(H_{u_0})$ takes values in the subspace $H_{u_0} \subset W$ of highest weight vectors of weight $3(d-1)$, and that a basis for the space $Sol(W_{u_0})$ of solutions taking values in the submodule $W_{u_0} \subset W$ generated by u_0 can be obtained by repeated action of $f \in \mathfrak{sl}_2$. Similarly as in section 5.4, since the tensor flip $(\tau_{V_d,V_d})_{i,i+1}$ is a morphism of $\mathfrak{sl}_2$-modules for $i=1,2$, and the direct sum decomposition of $V_d \otimes V_d \otimes V_d$ contains only one component of dimension $dim(W_{u_0})$, by Schur's lemma 4.10 we have that

$$(\tau_{V_d,V_d})_{i,i+1}|_{W_{u_0}} = id_{W_{u_0}}.$$

The monodromy action corresponding to the solution Ψ_0 can now be written as

$$M^{-1}_{\gamma_{z_i,z_{i+1}}}|_{W_{u_0}} = q^{-\frac{\mu_i\mu_{i+1}}{2}} id_{W_{u_0}}.$$

Remark. The symmetric invariant tensor $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e$ acts on W_{u_0} as a scalar,

$$\Omega_{i,i+1}|_{W_{u_0}} = \frac{\mu_i\mu_{i+1}}{2} id_{W_{u_0}}$$

for $i=1,2$. Notice in particular that its eigenvalues appear in the exponent of the quantum deformation parameter q.

5.6.2 Monodromy of $\Psi_1^{(\mathcal{P})}$ in $V_2 \otimes V_2 \otimes V_2$

The monodromy action on the integral solutions of the form $\Psi_1^{(\mathcal{P})}$ of proposition 5.13 is more complicated to compute. We will use the "contour deformation" method as in section 2.5 for the hypergeometric equation HGE. Actually, the equation KZ($\mathfrak{sl}_2$) for level one solutions $\Psi_1^{(\mathcal{P})}$ (pointwise) proportional to highest weight vectors of weight $\mu_1+\mu_2+\mu_3-2$ can be reduced to HGE for which the monodromy is already computed. However, as we want to study the relationship of the monodromy of the KZ-equations and the braid group representation defined by $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, we shall compute the monodromy explicitly in the simplest nontrivial case $d_1 = d_2 = d_3 = d = 2$.

By repeated use of proposition 4.16 (Clebsch-Gordan) we obtain the direct sum decomposition

$$W = V_2 \otimes V_2 \otimes V_2 \cong (V_1 \oplus V_3) \otimes V_2 \cong V_4 \oplus V_2 \oplus V_2,$$

where the highest weight vectors are

$$u_0 = v_0 \otimes v_0 \otimes v_0$$

of weight 3, and two linearly independent highest weight vectors of weight 1, which by lemma 5.11 are of the form

$$u_1 = a_1 v_1 \otimes v_0 \otimes v_0 + a_2 v_0 \otimes v_1 \otimes v_0 + a_3 v_0 \otimes v_0 \otimes v_1 = \sum_{k=1}^{3} a_k f_k u_0,$$

where the coefficients satisfy $\sum_{k=1}^{3} a_k = 0$.

Recall that unless $\kappa \in \mathbb{Q}$, two linearly independent solutions of the form suggested by proposition 5.13 are obtained for example by taking two homologically different paths of integration for the coefficients I_k, namely $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ between the points z_1, z_2, and $\Gamma^{(2,3)}_{z_1,z_2,z_3}$ between the points z_2, z_3. The linear independence of the solutions of $\mathrm{KZ}(\mathfrak{sl}_2)$ so obtained can be seen after computing the monodromy.

It turns out that the integration surfaces $\mathcal{P}$ of solutions of $\mathrm{KZ}(\mathfrak{sl}_2)$ of the form of proposition 5.15 associated to level l correspond to linear combinations of non-intersecting families of loops with l members [FW90]. Hence it is convenient to first consider a suitable complex vector space consisting of non-intersecting families of loops on $\mathbb{C} \setminus \{z_1, \ldots, z_N\}$, with a basis whose monodromy is straightforward to compute. We shall next introduce this kind of families of loops, and see that if the integration surface $\mathcal{P}$ can be written as a linear combination of elements of this type then we have an explicit formula for the monodromy. Moreover, it can actually be shown that any admissible integration surface $\mathcal{P}$ in the sense of proposition 5.15 is this kind of a linear combination. The proof and more details about this method can be found in [EFK98] and [FW90].

Before proving the general result, we consider the case of the one dimensional surface, that is the loop $\mathcal{P} : [0,1] \to \mathbb{C} \setminus \{z_1, z_2, z_3\}$, in the solution $\Psi_1^{(\mathcal{P})} : Y_3 \to V_2 \otimes V_2 \otimes V_2$,

$$\Psi_1^{(\mathcal{P})}(z_1, z_2, z_3) = \sum_{k=1}^{3} \int_{\mathcal{P}} \prod_{1 \le i < j \le 3} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}} \prod_{i=1}^{3} (w - z_i)^{-\frac{\mu_i}{\kappa}} \frac{dw}{w - z_k} f_k u_0,$$

where $\mathcal{P}$ is either $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ or $\Gamma^{(2,3)}_{z_1,z_2,z_3}$. However, we need first to make the general definition of the suitable non-intersecting families of loops.

Definition 5.16. Fix a point $z = (z_1, \dots, z_N) \in Y_N$ so that

$$Re(z_1) < \dots < Re(z_N).$$

Define $\mathcal{P}_{j_1,\dots,j_N}$ as a family of non-intersecting loops with a fixed base point $b \in \mathbb{R}$, such that the following conditions hold.

(i) The base point satisfies $b < Re(z_1)$.

(ii) For all $i = 1, \dots, N$ there are j_i nested loops around the point z_i counterclockwise, and these loops do not enclose any other point z_j.

(iii) The loops intersect only at the base point b.

(iv) If $\gamma_i, \gamma_k \in \mathcal{P}_{j_1,\dots,j_N}$, $\gamma_i, \gamma_k : [0,1] \to \mathbb{C} \setminus \{z_1, \dots, z_N\}$, are two different loops around different points z_i, z_k, $1 \le i < k \le N$, and $t, s \in [0,1]$ such that $Re(\gamma_i(t)) = Re(\gamma_k(s))$, then

$$Im(\gamma_i(t)) > Im(\gamma_k(s)).$$

We call the number $l = \sum_{i=1}^N j_i$ the *level* of $\mathcal{P}_{j_1,\dots,j_N}$. The families $\mathcal{P}_{j_1,\dots,j_N}$ can be thought of as l-surfaces on $Y_{z,l}$, and loops around different points z_i correspond to different integration variables w_j when integrating the l-forms $\eta_{z,l}$ of proposition 5.15 over $\mathcal{P}_{j_1,\dots,j_N}$. Figure 11 shows how the families $\mathcal{P}_{j_1,\dots,j_N}$ look like for $N = 5$. The star in the figure indicates the point where the branch of the integrand is chosen, such that the integrand is positive and real at that point.

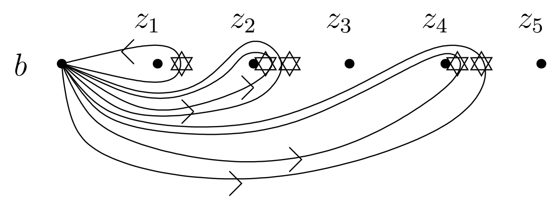


Figure 11: $\mathcal{P}_{1,2,0,2,0}$

Since we are interested in representations of the braid group B_3, we will consider action of the half-loop

$$\gamma_{z_1,z_2} : s \mapsto \left(\left(\frac{z_1 + z_2}{2} \right) + e^{\pi i s} \left(\frac{z_1 - z_2}{2} \right), \left(\frac{z_1 + z_2}{2} \right) - e^{\pi i s} \left(\frac{z_1 - z_2}{2} \right), z_3 \right)$$

exchanging the points z_1 and z_2 counterclockwise in Y_3. Recall that γ_{z_1,z_2} represents a loop in $\mathcal{C}_3$, and corresponds to the inverse of the braid group generator $\sigma_1 \in B_3$. We will keep the highest weights μ_1, μ_2, μ_3 in the computations; for the case $d_1 = d_2 = d_3 = d = 2$ they are $\mu_1 = \mu_2 = \mu_3 = 1$.

First we want to express the contours $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ and $\Gamma^{(2,3)}_{z_1,z_2,z_3}$ in terms of the families $\mathcal{P}_{j_1,j_2}$, where $j_1, j_2 \geq 0$. Motivated by the claim that level one solutions correspond to linear combinations of non-intersecting families of loops with only one member, we consider the elements $\mathcal{P}_{1,0,0}$, $\mathcal{P}_{0,1,0}$, and $\mathcal{P}_{0,0,1}$. We consider the integrals

$$I_k^{(\mathcal{P})}(z) = \int_{\mathcal{P}} \prod_{1\leq i<j\leq 3} (z_i - z_j)^{\frac{\mu_i\mu_j}{2\kappa}} \prod_{i=1}^{3} (w - z_i)^{-\frac{\mu_i}{\kappa}} \frac{dw}{w - z_k}$$

$z = (z_1, z_2, z_3) \in Y_3$, $\mathcal{P} \in \{\Gamma^{(1,2)}_{z_1,z_2,z_3}, \Gamma^{(2,3)}_{z_1,z_2,z_3}\}$, of proposition 5.13 for any index $k = 1, 2, 3$, written as linear combinations of integrals over the families $\mathcal{P}_{1,0,0}$, $\mathcal{P}_{0,1,0}$, and $\mathcal{P}_{0,0,1}$. The idea is similar to the "contour deformation" method in chapter 2 for integral solutions of the hypergeometric equation. Write optimistically

$$\begin{cases} I_k^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(z) = \alpha I_k^{(\mathcal{P}_{1,0,0})}(z) + \beta I_k^{(\mathcal{P}_{0,1,0})}(z) \\ I_k^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}(z) = \alpha' I_k^{(\mathcal{P}_{0,1,0})}(z) + \beta' I_k^{(\mathcal{P}_{0,0,1})}(z), \end{cases}$$

where $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$. With a slight abuse of notation, we shall write the above equations only for the paths of integration, that is

$$\begin{cases} \Gamma^{(1,2)}_{z_1,z_2,z_3} = \alpha\mathcal{P}_{1,0,0} + \beta\mathcal{P}_{0,1,0} \\ \Gamma^{(2,3)}_{z_1,z_2,z_3} = \alpha'\mathcal{P}_{0,1,0} + \beta'\mathcal{P}_{0,0,1}, \end{cases}$$

and consider all the indices $k = 1, 2, 3$ at the same time. The coefficients $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$ will be the same for any k. Figures 12 and 13 illustrate the above equations. Notice that the integrals $I_k^{(\mathcal{P})}(z)$ are nothing but coefficient functions of the solution

$$\Psi_1^{(\mathcal{P})}(z) = \sum_{k=1}^{3} I_k^{(\mathcal{P})}(z) f_k u_0$$

of proposition 5.13 written as a linear combination of $f_k u_0 \in V_{d_1} \otimes V_{d_2} \otimes V_{d_3}$, $k = 1, 2, 3$.

We will use pictures to compute the coefficients. The star in the figures indicates the point where the branch of the integrand is chosen, such that the integrand is positive and real at that point. Notice that the figures are *sketches* only, and in particular the points b, z_1, z_2 and z_3 appear to be in slightly different positions in different pictures.

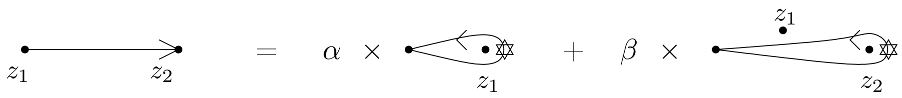


Figure 12: $\Gamma^{(1,2)}_{z_1,z_2,z_3} = \alpha\mathcal{P}_{1,0,0} + \beta\mathcal{P}_{0,1,0}$

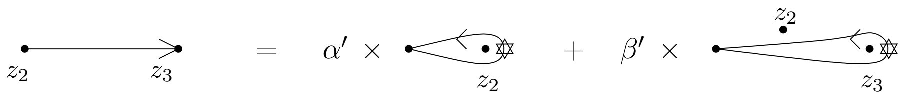


Figure 13: $\Gamma^{(2,3)}_{z_1,z_2,z_3} = \alpha'\mathcal{P}_{0,1,0} + \beta'\mathcal{P}_{0,0,1}$

Denote $q = e^{\frac{\pi i}{\kappa}}$. Let us write the elements $\mathcal{P}_{1,0,0}$ and $\mathcal{P}_{0,1,0}$ in a different manner. They can be drawn as in figures 14 and 15.

From these we see that, if $\kappa \notin \mathbb{Q}$, which implies that q is not a root of unity, the coefficients $\alpha, \beta, \alpha', \beta'$ must satisfy

$$\begin{cases} \beta = \frac{q^{-2\mu_1}-1}{q^{\mu_2}-q^{-\mu_2}}\alpha \\ \beta' = \frac{q^{-2\mu_2}-1}{q^{\mu_3}-q^{-\mu_3}}\alpha', \end{cases}$$

and the choice of branch of the integrand along $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ fixes the value of α, and choice of branch of the integrand along $\Gamma^{(2,3)}_{z_1,z_2,z_3}$ fixes the value of α'.

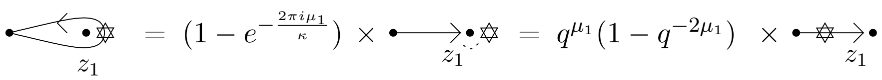


Figure 14: $\mathcal{P}_{1,0,0}$

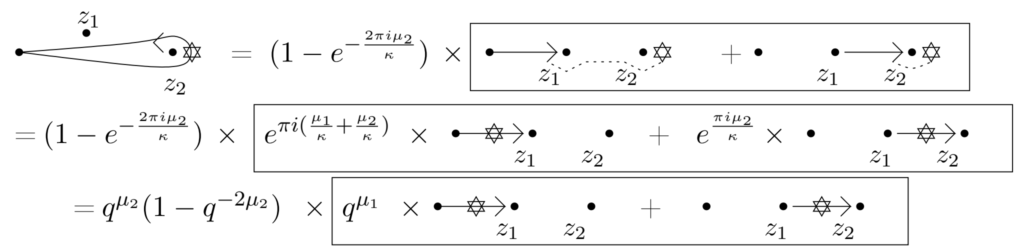


Figure 15: $\mathcal{P}_{0,1,0}$

Next we will compute the monodromy along γ_{z_1,z_2}. Since we know how to express the paths of integration $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ and $\Gamma^{(2,3)}_{z_1,z_2,z_3}$ as linear combinations of the elements $\mathcal{P}_{1,0,0}$, $\mathcal{P}_{0,1,0}$ and $\mathcal{P}_{0,0,1}$, we first compute the monodromy of the integrals $I_k^{(\mathcal{P})}$, $k = 1, 2, 3$, over these elements along γ_{z_1,z_2}. Notice that the integals $I_k^{(\mathcal{P})}$ with $\mathcal{P} \in \{\mathcal{P}_{1,0,0}, \mathcal{P}_{0,1,0}, \mathcal{P}_{0,0,1}\}$ *do not* necessarily produce solutions of $\mathrm{KZ}(\mathfrak{sl}_2)$ of the form of proposition 5.13. However, we obtain solutions using the expressions for $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ and $\Gamma^{(2,3)}_{z_1,z_2,z_3}$ as linear combinations of the elements $\mathcal{P}_{1,0,0}$, $\mathcal{P}_{0,1,0}$ and $\mathcal{P}_{0,0,1}$, computed above.

Recall that the action of γ_{z_1,z_2} corresponds to the action of the inverse of the braid group generator $\sigma_1 \in B_3$. Notice also that we are considering parallel transport in the bundle over the configuration space $\mathcal{C}_3$, where the points

$$(z_1, z_2) = \gamma_{z_1,z_2}(0) \;\text{ and }\; (z_2, z_1) = \gamma_{z_1,z_2}(1)$$

are identified. Another crucial observation is that the rational function $\frac{1}{w-z_k}$ appearing in the integral $I_k^{(\mathcal{P})}$ is *single-valued*, and hence does not affect the monodromy. In particular, the monodromy of $I_k^{(\mathcal{P})}$ is *independent of the index* $k = 1, 2, 3$.

The monodromy of $I_k^{(\mathcal{P}_{1,0,0})}$ is easy (figure 16), resulting

$$I_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(1)) = q^{\frac{\mu_1\mu_2}{2}-\mu_2} I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0)).$$

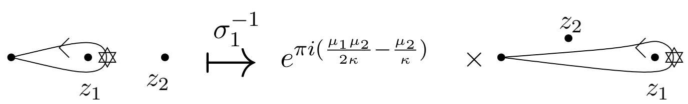


Figure 16: The monodromy of $I_k^{(\mathcal{P}_{1,0,0})}$

On the other hand, the monodromy of $I_k^{(\mathcal{P}_{0,1,0})}$ is slightly more complicated. In figure 17 below we have divided the loop after the monodromy action into two separate loops, and the loop around both of the points z_1 and z_2 into two loops enclosing only one of the points z_1, z_2.

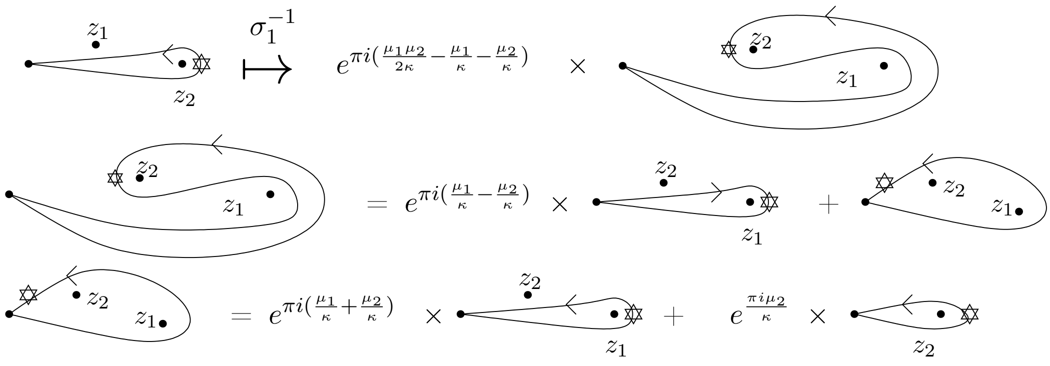


Figure 17: The monodromy of $I_k^{(\mathcal{P}_{0,1,0})}$

Combining the above pictures we obtain the result

$$
\begin{aligned}
&I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(1)) \\
&\quad = q^{\frac{\mu_1\mu_2}{2}-\mu_1-\mu_2}\Big\{-q^{\mu_1-\mu_2}I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0)) \\
&\qquad + q^{\mu_1+\mu_2}I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0)) + q^{\mu_2}I_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0))\Big\} \\
&\quad = \Big(q^{\frac{\mu_1\mu_2}{2}} - q^{\frac{\mu_1\mu_2}{2}-2\mu_2}\Big) I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0)) + q^{\frac{\mu_1\mu_2}{2}-\mu_1}I_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0)) \\
&\quad = q^{\frac{\mu_1\mu_2}{2}-\mu_2}(q^{\mu_2}-q^{-\mu_2})I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0)) + q^{\frac{\mu_1\mu_2}{2}-\mu_1}I_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0)).
\end{aligned}
$$

Finally, the monodromy of $I_k^{(\mathcal{P}_{0,0,1})}$ is also easy (figure 18), resulting

$$I_k^{(\mathcal{P}_{0,0,1})}(\gamma_{z_1,z_2}(1)) = q^{\frac{\mu_1\mu_2}{2}}I_k^{(\mathcal{P}_{0,0,1})}(\gamma_{z_1,z_2}(0)).$$

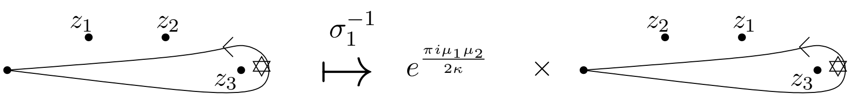


Figure 18: The monodromy of $I^{(\mathcal{P}_{0,0,1})}$

Using the monodromy of the elements $I_k^{(\mathcal{P}_{1,0,0})}$, $I_k^{(\mathcal{P}_{0,1,0})}$ and $I_k^{(\mathcal{P}_{0,0,1})}$ we obtain

$$\begin{aligned}
&I_k^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(1))\\
&=\alpha q^{\frac{\mu_1\mu_2}{2}-\mu_2} I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0))+\beta\Big\{q^{\frac{\mu_1\mu_2}{2}-\mu_2}\left(q^{\mu_2}-q^{-\mu_2}\right) I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0))\\
&\quad+q^{\frac{\mu_1\mu_2}{2}-\mu_1} I_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0))\Big\}\\
&=\alpha q^{\frac{\mu_1\mu_2}{2}-2\mu_1}\Big\{q^{-\mu_2} I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0))+\frac{q^{-\mu_1}-q^{\mu_1}}{q^{\mu_2}-q^{-\mu_2}} I_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0))\Big\},
\end{aligned}$$

where we used the relation between the coefficients α and β, and

$$\begin{aligned}
&I_k^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(1))\\
&=\alpha'\Big\{q^{\frac{\mu_1\mu_2}{2}-\mu_2}\left(q^{\mu_2}-q^{-\mu_2}\right) I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0))+q^{\frac{\mu_1\mu_2}{2}-\mu_1} I_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0))\Big\}\\
&\quad+\beta' q^{\frac{\mu_1\mu_2}{2}} I_k^{(\mathcal{P}_{0,0,1})}(\gamma_{z_1,z_2}(0))\\
&=\alpha'\Big\{q^{\frac{\mu_1\mu_2}{2}}\Big\{q^{-\mu_2}\left(q^{\mu_2}-q^{-\mu_2}\right) I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0))+q^{-\mu_1} I_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0))\Big\}\\
&\quad+\frac{q^{-2\mu_2}-1}{q^{\mu_3}-q^{-\mu_3}} I_k^{(\mathcal{P}_{0,0,1})}(\gamma_{z_1,z_2}(0))\Big\},
\end{aligned}$$

where we used the relation between the coefficients α' and β'.

Recall that our aim is to compute the monodromy of the solution $\Psi_1^{(\mathcal{P})}$ of $\mathrm{KZ}(\mathfrak{sl}_2)$ in $V_2\otimes V_2\otimes V_2$. Hence assume next that $\mu_1=\mu_2=\mu_3=1$, and choose the branch of the integrand along $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ so that $\beta=1$ and along $\Gamma^{(2,3)}_{z_1,z_2,z_3}$ so that $\beta'=1$. Then the monodromy action of the half-loop γ_{z_1,z_2} can be written as

$$\begin{aligned}
I_k^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(1)) &= -q^{-\frac{3}{2}}\Big\{(I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0))-qI_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0))\Big\}\\
&=-\,q^{-\frac{3}{2}} I_k^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0)),\\
I_k^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(1)) &= q^{\frac{1}{2}}\Big\{I_k^{(\mathcal{P}_{0,0,1})}(\gamma_{z_1,z_2}(0))-qI_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0))\Big\}\\
&\quad+q^{-\frac{1}{2}}\Big\{I_k^{(\mathcal{P}_{0,1,0})}(\gamma_{z_1,z_2}(0))-qI_k^{(\mathcal{P}_{1,0,0})}(\gamma_{z_1,z_2}(0))\Big\}\\
&=q^{\frac{1}{2}} I_k^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0))+q^{-\frac{1}{2}} I_k^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0)).
\end{aligned}$$

Finally, the monodromy of the solutions

$$\Psi_1^{(\mathcal{P})}(z)=\sum_{k=1}^{3} I_k^{(\mathcal{P})}(z) f_k u_0,$$

with $\mathcal{P} \in \{\Gamma^{(1,2)}_{z_1,z_2,z_3}, \Gamma^{(2,3)}_{z_1,z_2,z_3}\}$ reads

$$\begin{aligned}
\Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(1)) &= \sum_{k=1}^{3} I_k^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(1)) f_k u_0 \\
&= \sum_{k=1}^{3} -q^{-\frac{3}{2}} I_k^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0)) f_k u_0 = -q^{-\frac{3}{2}} \Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0)), \\
\Psi_1^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(1)) &= \sum_{k=1}^{3} I_k^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(1)) f_k u_0 \\
&= \sum_{k=1}^{3} \Big(q^{\frac{1}{2}} I_k^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0)) + q^{-\frac{1}{2}} I_k^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0)) \Big) f_k u_0 \\
&= q^{\frac{1}{2}} \Psi_1^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0)) + q^{-\frac{1}{2}} \Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}(\gamma_{z_1,z_2}(0)),
\end{aligned}$$

and we have the monodromy operator

$$\begin{cases} M_{\gamma_{z_1,z_2}} : \Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})} \mapsto -q^{-\frac{3}{2}} \Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})} \\ M_{\gamma_{z_1,z_2}} : \Psi_1^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})} \mapsto q^{\frac{1}{2}} \Psi_1^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})} + q^{-\frac{1}{2}} \Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}. \end{cases}$$

Notice that $\Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}$ is an eigenvector of the monodromy operator $M_{\gamma_{z_1,z_2}}$ but $\Psi_1^{(\Gamma^{(2,1)}_{z_1,z_2,z_3})}$ is not. Similarly as in section 2.5 for solutions of HGE we deduce that the solutions associated to $\Gamma^{(1,2)}_{z_1,z_2,z_3}$ and $\Gamma^{(2,3)}_{z_1,z_2,z_3}$ are linearly independent.

We can now write the monodromy operator $M_{\gamma_{z_1,z_2}}$ acting on the (sub)space of solutions of KZ($\mathfrak{sl}_2$) in $V_2 \otimes V_2 \otimes V_2$ as a matrix with respect to the basis

$$\{\Psi_0, f.\Psi_0, f^2.\Psi_0, f^3.\Psi_0; \Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}, f.\Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}; \Psi_1^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}, f.\Psi_1^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}\}$$

where

$$\begin{aligned}
\Psi_0(z_1,z_2,z_3) &= \prod_{1\le i<j\le 3} (z_i - z_j)^{\frac{1}{2\kappa}} u_0, \\
\Psi_1^{(\Gamma^{(1,2)}_{z_1,z_2,z_3})}&(z_1,z_2,z_3) \\
&= \sum_{k=1}^{3} \int_{\Gamma^{(1,2)}_{z_1,z_2,z_3}} \prod_{1\le i<j\le 3} (z_i - z_j)^{\frac{\mu_i\mu_j}{2\kappa}} \prod_{i=1}^{3} (w - z_i)^{-\frac{\mu_i}{\kappa}} \frac{dw}{w - z_k} f_k u_0, \\
\Psi_1^{(\Gamma^{(2,3)}_{z_1,z_2,z_3})}&(z_1,z_2,z_3) \\
&= \sum_{k=1}^{3} \int_{\Gamma^{(2,3)}_{z_1,z_2,z_3}} \prod_{1\le i<j\le 3} (z_i - z_j)^{\frac{\mu_i\mu_j}{2\kappa}} \prod_{i=1}^{3} (w - z_i)^{-\frac{\mu_i}{\kappa}} \frac{dw}{w - z_k} f_k u_0
\end{aligned}$$

with some fixed choices of branches of the initial values.
The matrix $M_{\gamma_{z_1,z_2}}$ in the above basis reads

$$M_{\gamma_{z_1,z_2}} = \begin{pmatrix} q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q^{-\frac{3}{2}} & 0 & q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-\frac{3}{2}} & 0 & q^{-\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix}$$

Recall that this is the monodromy operator corresponding to the inverse of the braid group generator $\sigma_1 \in B_3$. The matrix corresponding to σ_1 is the inverse of $M_{\gamma_{z_1,z_2}}$, that is

$$M^{-1}_{\gamma_{z_1,z_2}} = \begin{pmatrix} q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q^{\frac{3}{2}} & 0 & q^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{\frac{3}{2}} & 0 & q^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} \end{pmatrix}$$

The action of the monodromy operator $M_{\gamma_{z_2,z_3}}$ exchanging the points z_2 and z_3 is found by a similar computation. The monodromy representation of the braid group B_3 associated to the solutions of KZ($\mathfrak{sl}_2$) in $W = V_2 \otimes V_2 \otimes V_2$ is determined by the inverses of these operators, that is,

$$\rho_3^{KZ} : B_3 \to Aut(V_2^{\otimes 3}), \quad \sigma_1 \mapsto M^{-1}_{\gamma_{z_1,z_2}} \quad \sigma_2 \mapsto M^{-1}_{\gamma_{z_2,z_3}}.$$

5.6.3 The general “contour deformation“ method

Next we introduce the general “contour deformation“ method for solutions of KZ($\mathfrak{sl}_2$) of arbitrary level. Recall the formula for the solutions,

$$\Psi_l^{(\mathcal{P})}(z) = \psi_0(z) \int_{\mathcal{P}} \eta_{z,l} u_0,$$

where

$$\psi_0(z) = \prod_{1\le i<j\le N} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}},$$

$$\eta_{z,l} = \prod_{1\le i<j\le l} (w_i - w_j)^{\frac{2}{\kappa}} \prod_{j,k} (w_j - z_k)^{-\frac{\mu_k}{\kappa}} \prod_{i=1}^{l} \sum_{k=1}^{N} \frac{f_k}{w_i - z_k} dw,$$

and $\mathcal{P}$ is a suitable l-surface on the manifold

$$Y_{z,l} = \mathbb{C}^l \setminus \left(\bigcup_{i<j} \{w_i = w_j\} \cup \bigcup_{i,j} \{w_i = z_j\} \right)$$

We notice that the part

$$\prod_{i=1}^{l} \sum_{k=1}^{N} \frac{f_k}{w_i - z_k} u_0$$

can be written as a linear combination of terms proportional to vectors of the form

$$f_1^{m_1} \cdots f_N^{m_N} u_0 = v_{m_1} \otimes \cdots \otimes v_{m_N} \in W,$$

containing a *single-valued* rational function $r_{m_1,\ldots,m_N}(z,w)$ having as a function of $w = (w_1, \ldots, w_l)$ no poles in $Y_{z,l}$. Indeed,

$$\begin{aligned}\prod_{i=1}^{l} \sum_{k=1}^{N} \frac{f_k}{w_i - z_k} u_0 &= \sum_{m_1,\ldots,m_N \ge 0,\ \sum_i m_i = l} r_{m_1,\ldots,m_N}(z,w) f_1^{m_1} \cdots f_N^{m_N} u_0 \\ &= \sum_{m_1,\ldots,m_N \ge 0,\ \sum_i m_i = l} r_{m_1,\ldots,m_N}(z,w) v_{m_1} \otimes \cdots \otimes v_{m_N}.\end{aligned}$$

Define the complex valued functions $F_{j_1,\ldots,j_N}^{m_1,\ldots,m_N} : Y_N \to \mathbb{C}$ by

$$\begin{aligned}&F_{j_1,\ldots,j_N}^{m_1,\ldots,m_N}(z) \\ &:= \psi_0(z) \int_{\mathcal{P}_{j_1,\ldots,j_N}} \prod_{1\le i<j\le l} (w_i - w_j)^{\frac{2}{\kappa}} \prod_{j,k} (w_j - z_k)^{-\frac{\mu_k}{\kappa}} r_{m_1,\ldots,m_N}(z,w) dw,\end{aligned}$$

where $m_1, \ldots, m_N, j_1, \ldots, j_N \ge 0$ satisfy $\sum_{i=1}^{N} j_i = l = \sum_{i=1}^{N} m_i$.

We consider the monodromy action of the half-loop exchanging the points z_i and z_{i+1} counterclockwise, denoted by $\gamma_{z_i,z_{i+1}} : [0,1] \to Y_N$, on the functions $F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}$. The homotopy class of $\gamma_{z_i,z_{i+1}}$ corresponds to the inverse of the braid group generator $\sigma_i \in B_N$. Notice that since the complex function $r_{m_1,\ldots,m_N}(w)$ is single-valued, the monodromy affects only the part

$$\psi_0(z) \int_{\mathcal{P}_{j_1,\ldots,j_N}} \prod_{1\leq i<j\leq l} (w_i - w_j)^{\frac{2}{\kappa}} \prod_{j,k} (w_j - z_k)^{-\frac{\mu_k}{\kappa}} dw$$

of the function $F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}$. In particular, the monodromy is *independent on the indices* $m_1, \ldots, m_N \geq 0$.

We will compute the monodromy of the functions $F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}$ using the contour deformation method on the families $\mathcal{P}_{j_1,\ldots,j_N}$ as in the previous section. We will use a slight abuse of notation so that while computing the monodromy, we denote by $\mathcal{P}_{j_1,\ldots,j_N}$ also the function $F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}$ for any indices $m_1, \ldots, m_N \geq 0$ satisfying $\sum_{i=1}^{N} m_i = l = \sum_{i=1}^{N} j_i$, with the path of integration $\mathcal{P}_{j_1,\ldots,j_N}$. Actually, since we are only interested in the exchange of two adjacent points, it suffices to consider the action of γ_{z_1,z_2} on $\mathcal{P}_{j_1,j_2}$. For this, we shall define some auxiliary families of loops, using the following figures. All the loops are assumed to be non-intersecting and to have the same fixed base point $b < Re(z_1)$ as the family $\mathcal{P}_{j_1,j_2}$.

Firstly, the action of γ_{z_1,z_2} on $\mathcal{P}_{j_1,j_2}$ can be pictured as in the below figure 19. Secondly, define the auxiliary families of loops called $\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2}$ and $\hat{\mathcal{G}}^{(p)}_{j_1,j_2}$ by figures 20 and 21.

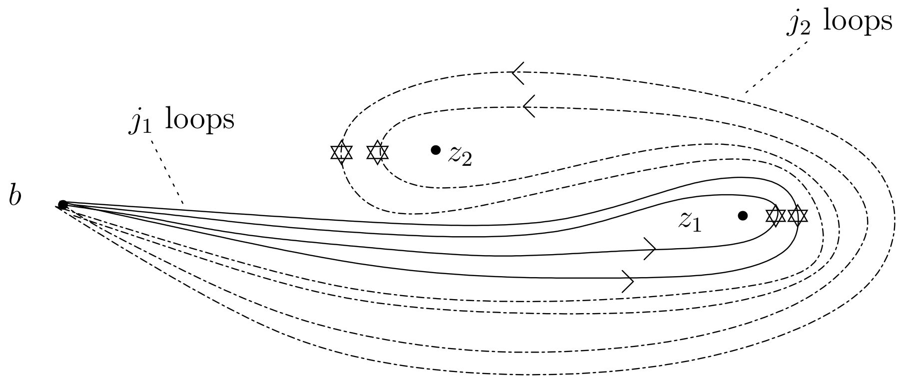


Figure 19: $M_{\gamma_{z_1,z_2}} \mathcal{P}_{j_1,j_2}$

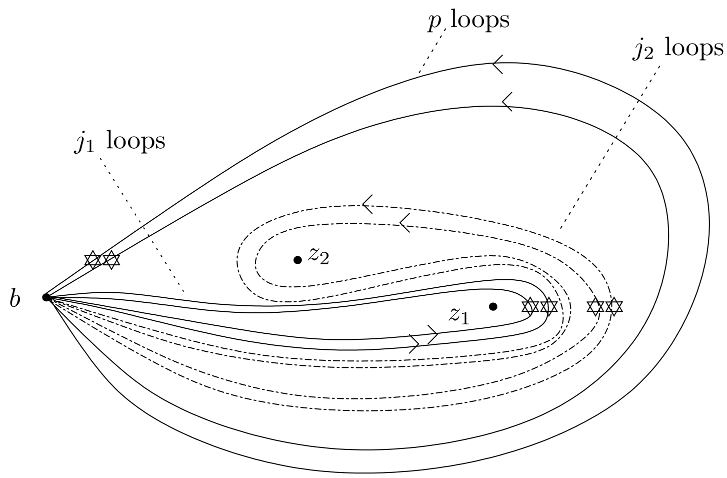


Figure 20: $\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2}$

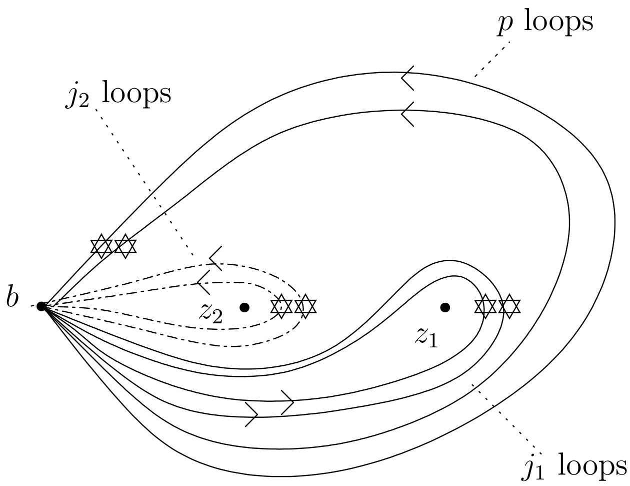


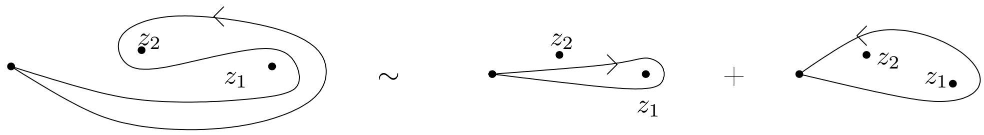
Figure 21: $\hat{\mathcal{G}}^{(p)}_{j_1,j_2}$

Taking into account the resulting "phase factors" from the function $F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}$ we can write the action of γ_{z_1,z_2} on $\mathcal{P}_{j_1,j_2}$ as

$$M_{\gamma_{z_1,z_2}}\mathcal{P}_{j_1,j_2} = q^{\frac{\mu_1\mu_2}{2}-j_1\mu_2}\widetilde{\mathcal{G}}^{(0)}_{j_1,j_2},$$

where we denote $q = e^{\frac{\pi i}{\kappa}}$. In order to express $M_{\gamma_{z_1,z_2}}\mathcal{P}_{j_1,j_2}$ as a linear combination of families of non-intersecting loops of the form $\mathcal{P}_{k_1,k_2}$ we will compute the monodromy of the auxiliary families $\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2}$ and $\hat{\mathcal{G}}^{(p)}_{j_1,j_2}$.

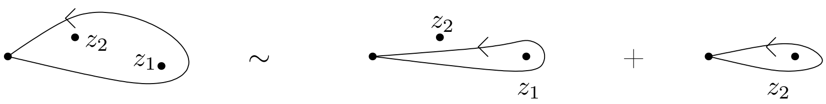
Divide the outermost loop in $\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2}$ around z_2 into two loops of the following type.



Taking into account the resulting "phase factors" from the function $F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}$ we obtain the following recursion for $\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2}$,

$$\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2} = q^{2(j_1+j_2-1)-\mu_1-\mu_2}\widetilde{\mathcal{G}}^{(p+1)}_{j_1,j_2-1} - q^{2(j_2-1-\mu_2)}\widetilde{\mathcal{G}}^{(p)}_{j_1+1,j_2-1}.$$

Similarly, divide the innermost loop in $\hat{\mathcal{G}}^{(p)}_{j_1,j_2}$ around both z_1 and z_2 into two loops of the following type.



Taking into account the resulting "phase factors" from the function $F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}$ we obtain the following recursion for $\hat{\mathcal{G}}^{(p)}_{j_1,j_2}$,

$$\hat{\mathcal{G}}^{(p)}_{j_1,j_2} = q^{-2j_2+\mu_2}\hat{\mathcal{G}}^{(p-1)}_{j_1,j_2+1} + q^{-2(j_1+j_2)+\mu_1+\mu_2}\hat{\mathcal{G}}^{(p-1)}_{j_1+1,j_2}.$$

Moreover, notice also that if there are no loops around both z_1 and z_2 in $\hat{\mathcal{G}}^{(p)}_{j_1,j_2}$, we obtain the family $\mathcal{P}_{j_2,j_1}$,

$$\hat{\mathcal{G}}^{(0)}_{j_1,j_2} = \mathcal{P}_{j_2,j_1}.$$

Similarly, if there are no loops around z_2 then we have

$$\hat{\mathcal{G}}^{(p)}_{j_1,0} = \widetilde{\mathcal{G}}^{(p)}_{j_1,0}.$$

From the recursions it follows that decreasing the indices j_2 and p of the families $\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2}$ and $\hat{\mathcal{G}}^{(p)}_{j_1,j_2}$, respectively, we can write

$$\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2} = \sum_{k=0}^{m} \alpha_{m,k}\widetilde{\mathcal{G}}^{(p+m-k)}_{j_1+k,j_2-m}$$
$$\hat{\mathcal{G}}^{(p)}_{j_1,j_2} = \sum_{k=0}^{m} \beta_{m,k}\hat{\mathcal{G}}^{(p-m)}_{j_1+k,j_2+m-k}.$$

Furthermore, using the recursions we obtain corresponding recursions for the coefficients $\alpha_{m,k}$ and $\beta_{m,k}$. From the equations

$$\begin{aligned}\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2} &= \sum_{k=0}^{m-1} \alpha_{m-1,k} \widetilde{\mathcal{G}}^{(p+m-1-k)}_{j_1+k,j_2-m+1} \\ &= \sum_{k=0}^{m-1} \alpha_{m-1,k} \Big(q^{2(j_1+k+j_2-m)-\mu_1-\mu_2} \widetilde{\mathcal{G}}^{(p+m-k)}_{j_1+k,j_2-m} \\ &\quad - q^{2(j_2-m-\mu_2)} \widetilde{\mathcal{G}}^{(p+m-1-k)}_{j_1+k+1,j_2-m} \Big)\end{aligned}$$

and

$$\begin{aligned}\hat{\mathcal{G}}^{(p)}_{j_1,j_2} &= \sum_{k=0}^{m-1} \beta_{m-1,k} \hat{\mathcal{G}}^{(p-m+1)}_{j_1+k,j_2+m-1-k} \\ &= \sum_{k=0}^{m-1} \beta_{m-1,k} \Big(q^{-2(j_2+m-k-1)+\mu_2} \hat{\mathcal{G}}^{(p-m)}_{j_1+k,j_2+m-k} \\ &\quad + q^{-2(j_1+j_2+m-1)+\mu_1+\mu_2} \hat{\mathcal{G}}^{(p-m)}_{j_1+k+1,j_2+m-1-k} \Big)\end{aligned}$$

we see that

$$\begin{cases} \alpha_{m,k} = q^{2(j_1+k+j_2-m)-\mu_1-\mu_2} \alpha_{m-1,k} - q^{2(j_2-m-\mu_2)} \alpha_{m-1,k-1} \\ \beta_{m,k} = q^{-2(j_2+m-k-1)+\mu_2} \beta_{m-1,k} + q^{-2(j_1+j_2+m-1)+\mu_1+\mu_2} \beta_{m-1,k-1}. \end{cases}$$

Solutions of the recursions. In order to solve the recursions determining the monodromy action of γ_{z_1,z_2} on $\mathcal{P}_{j_1,j_2}$ we present some useful results concerning the quantum binomial coefficients. Recall that the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where

$$\begin{aligned}[n]_q &:= \frac{q^n - q^{-n}}{q - q^{-1}}, \\ [n]_q! &:= [n]_q[n-1]_q \cdots [1]_q,\end{aligned}$$

and $q \in \mathbb{C} \setminus \{0, \pm 1\}$.

Proposition 5.17. *The q-binomial coefficients satisfy the recursion*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{k-n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Proof.

$$\begin{aligned}
&\begin{bmatrix} n \\ k \end{bmatrix}_q - q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q - q^{k-n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\
&= \frac{[n]_q!}{[k]_q![n-k]_q!} - q^k \frac{[n-1]_q!}{[k]_q![n-1-k]_q!} - q^{k-n} \frac{[n-1]_q!}{[k-1]_q![n-1-(k-1)]_q!} \\
&= \frac{[n-1]_q!}{[k]_q![n-k]_q!} \left([n]_q - q^k [n-k]_q - q^{k-n} [k]_q \right) \\
&= \frac{[n-1]_q!}{[k]_q![n-k]_q!} \left(\frac{q^n - q^{-n} - q^k(q^{n-k} - q^{k-n}) - q^{k-n}(q^k - q^{-k})}{q - q^{-1}} \right) = 0.
\end{aligned}$$

□

Lemma 5.18. *Suppose the complex numbers* $(a_{n,k})_{n \in \mathbb{N}, k=0,\ldots,n}$ *satisfy the recursion*

$$a_{n,k} = (A_+^n B_+^k C_+) a_{n-1,k} + (A_-^n B_-^k C_-) a_{n-1,k-1},$$

where $a_{n,k} = 0$ *for* $k \notin \{0, \ldots, n\}$*. Then*

$$a_{n,k} = \left(\begin{bmatrix} n \\ k \end{bmatrix}_q A_+^{\frac{(n+1)(n-k)}{2}} A_-^{\frac{(n+1)k}{2}} B_+^{\frac{(n-k)k}{2}} B_-^{\frac{(k+1)k}{2}} C_+^{n-k} C_-^k \right) a_{0,0},$$

where $q = \sqrt{\frac{A_+ B_+}{A_-}}$.

Proof. Using proposition 5.17 with $q = \sqrt{\frac{A_+ B_+}{A_-}}$ we obtain

$$\begin{aligned}
&\frac{a_{n,k}}{a_{0,0}} = \begin{bmatrix} n \\ k \end{bmatrix}_q A_+^{\frac{(n+1)(n-k)}{2}} A_-^{\frac{(n+1)k}{2}} B_+^{\frac{(n-k)k}{2}} B_-^{\frac{(k+1)k}{2}} C_+^{n-k} C_-^k \\
&= \left(q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{k-n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \right) \cdot A_+^{\frac{(n+1)(n-k)}{2}} A_-^{\frac{(n+1)k}{2}} B_+^{\frac{(n-k)k}{2}} B_-^{\frac{(k+1)k}{2}} C_+^{n-k} C_-^k \\
&= \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(\frac{A_+ B_+}{A_-} \right)^{\frac{k}{2}} \cdot A_+^{\frac{(n+1)(n-k)}{2}} A_-^{\frac{(n+1)k}{2}} B_+^{\frac{(n-k)k}{2}} B_-^{\frac{(k+1)k}{2}} C_+^{n-k} C_-^k \\
&\quad + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \left(\frac{A_+ B_+}{A_-} \right)^{\frac{k-n}{2}} \cdot A_+^{\frac{(n+1)(n-k)}{2}} A_-^{\frac{(n+1)k}{2}} B_+^{\frac{(n-k)k}{2}} B_-^{\frac{(k+1)k}{2}} C_+^{n-k} C_-^k \\
&= \frac{a_{n-1,k}}{a_{0,0}} A_+^n A_-^0 B_+^k B_-^0 C_+^1 C_-^0 + \frac{a_{n-1,k-1}}{a_{0,0}} A_+^0 A_-^n B_+^0 B_-^k C_+^0 C_-^1 \\
&= \frac{a_{n-1,k}}{a_{0,0}} A_+^n B_+^k C_+ + \frac{a_{n-1,k-1}}{a_{0,0}} A_-^n B_-^k C_-.
\end{aligned}$$

Hence the proposed formula satisfies the recursion, and it is clear that the solution is unique with a given initial condition $a_{0,0}$. □

From this we get as a special case the following useful result.

Lemma 5.19.

$$\prod_{s=0}^{r-1}(q^{t+s}-q^{-(t+s)}) = \sum_{l=0}^{r}(-1)^l \begin{bmatrix} r \\ l \end{bmatrix}_q q^{rt-2lt+2l-r+\frac{1}{2}(r+1)(r-2l)},$$

and conversely,

$$\sum_{l=0}^{r}(-1)^l \begin{bmatrix} r \\ l \end{bmatrix}_q q^{l(\beta-r)} = q^{\frac{1}{2}r(\beta-r)} \prod_{s=0}^{r-1}(q^{\frac{1}{2}(1-\beta)+s} - q^{\frac{1}{2}(\beta-1)-s}).$$

Proof. By collecting fixed powers of q we can write

$$\prod_{s=0}^{r-1}(q^{t+s}-q^{-(t+s)}) = (q^t-q^{-t})(q^{t+1}-q^{-(t+1)})\cdots(q^{t+r-1}-q^{-(t+r-1)})$$

$$= q^{\sum_{s=0}^{r-1} t+s} + \cdots + q^{-\sum_{s=0}^{r-1} t+s} = q^{rt}\sum_{l=0}^{r} c_{r,l}q^{-2lt}$$

with suitable coefficients $c_{r,l}$. We see that $c_{r,l}$ satisfy the recursion

$$c_{r,l} = q^{r-1}c_{r-1,l} - q^{1-r}c_{r-1,l-1},$$

which is of the form of lemma 5.18 with $A_+ = q$, $A_- = q^{-1}$, $B_+ = 1 = B_-$, $C_+ = q^{-1}$, $C_- = -q$. □

We will now use lemma 5.18 to solve the recursions

$$\begin{cases} \alpha_{m,k} = q^{2(j_1+k+j_2-m)-\mu_1-\mu_2}\alpha_{m-1,k} - q^{2(j_2-m-\mu_2)}\alpha_{m-1,k-1} \\ \beta_{m,k} = q^{-2(j_2+m-k-1)+\mu_2}\beta_{m-1,k} + q^{-2(j_1+j_2+m-1)+\mu_1+\mu_2}\beta_{m-1,k-1} \end{cases}$$

for the coefficients of the auxiliary families $\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2}$ and $\hat{\mathcal{G}}^{(p)}_{j_1,j_2}$.

Proposition 5.20. *The recursions*

$$\begin{cases} \alpha_{m,k} = q^{2(j_1+k+j_2-m)-\mu_1-\mu_2}\alpha_{m-1,k} - q^{2(j_2-m-\mu_2)}\alpha_{m-1,k-1} \\ \beta_{m,k} = q^{-2(j_2+m-k-1)+\mu_2}\beta_{m-1,k} + q^{-2(j_1+j_2+m-1)+\mu_1+\mu_2}\beta_{m-1,k-1} \end{cases}$$

have the unique solutions

$$\begin{cases} \alpha_{m,k} = (-1)^k \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_q q^{-k^2-2j_1(k-m)+k(m+\mu_1-\mu_2)-m(1-2j_2+m+\mu_1+\mu_2)} \\ \beta_{m,k} = \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_q q^{-2j_1k-k^2+k(m+\mu_1)+m(1-2j_2-m+\mu_2)} \end{cases}$$

Proof. For $\alpha_{m,k}$, put in lemma 5.18

$$A_+ = A_- = q^{-2},\ B_+ = q^2,\ B_- = 1,\ C_+ = q^{2(j_1+j_2)-\mu_1-\mu_2},\ C_- = -q^{2(j_2-\mu_2)}.$$

Then

$$\sqrt{\frac{q^{-2}q^{2}}{q^{-2}}}=q, \quad \alpha_{0,0}=1$$

and

$$\begin{aligned}\alpha_{m,k} &= \begin{bmatrix} m \\ k \end{bmatrix}_q q^{-(m+1)(m-k)} q^{-(m+1)k} q^{(m-k)k} q^{(2(j_1+j_2)-\mu_1-\mu_2)(m-k)} (-q)^{2(j_2-\mu_2)k} \\ &= (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{-k^2-2j_1(k-m)+k(m+\mu_1-\mu_2)-m(1-2j_2+m+\mu_1+\mu_2)}.\end{aligned}$$

For $\beta_{m,k}$, put in lemma 5.18

$$\begin{aligned}&A_+ = A_- = q^{-2},\ B_+ = q^2,\ B_- = 1,\ C_+ = q^{-2(j_2-1)+\mu_2}, \\ &C_- = q^{-2(j_1+j_2-1)+\mu_1+\mu_2}.\end{aligned}$$

Then

$$\sqrt{\frac{q^{-2}q^{2}}{q^{-2}}}=q, \quad \beta_{0,0}=1$$

and

$$\begin{aligned}\beta_{m,k} = &\begin{bmatrix} n \\ k \end{bmatrix}_q q^{-(m+1)(m-k)} q^{-(m+1)k} q^{2(m-k)k} q^{(-2(j_2-1)+\mu_2)(m-k)} \\ &\cdot q^{(-2(j_1+j_2-1)+\mu_1+\mu_2)k} \\ = &\begin{bmatrix} m \\ k \end{bmatrix}_q q^{-2j_1k-k^2+k(m+\mu_1)+m(1-2j_2-m+\mu_2)}.\end{aligned}$$

□

Using the previous proposition we obtain from the expressions

$$\begin{aligned}\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2} &= \sum_{k=0}^{m} \alpha_{m,k} \widetilde{\mathcal{G}}^{(p+m-k)}_{j_1+k,j_2-m} \\ \hat{\mathcal{G}}^{(p)}_{j_1,j_2} &= \sum_{k=0}^{m} \beta_{m,k} \hat{\mathcal{G}}^{(p-m)}_{j_1+k,j_2+m-k}\end{aligned}$$

by taking $m = j_2$ in the first equation and $m = p$ in the second equation the formulas

$$\begin{aligned}\widetilde{\mathcal{G}}^{(p)}_{j_1,j_2} &= \sum_{k=0}^{j_2} \alpha_{j_2,k} \widetilde{\mathcal{G}}^{(p+j_2-k)}_{j_1+k,0} \\ &= \sum_{k=0}^{j_2} (-1)^k \begin{bmatrix} j_2 \\ k \end{bmatrix}_q q^{-k^2-2j_1(k-j_2)+k(j_2+\mu_1-\mu_2)-j_2(1-j_2+\mu_1+\mu_2)} \widetilde{\mathcal{G}}^{(p+j_2-k)}_{j_1+k,0}\end{aligned}$$

and

$$\begin{aligned}\hat{\mathcal{G}}^{(p)}_{j_1,j_2} &= \sum_{k=0}^{p} \beta_{p,k} \hat{\mathcal{G}}^{(0)}_{j_1+k,j_2+p-k} \\ &= \sum_{k=0}^{p} \begin{bmatrix} p \\ k \end{bmatrix}_q q^{-2j_1k-k^2+k(p+\mu_1)+p(1-2j_2-p+\mu_2)} \hat{\mathcal{G}}^{(0)}_{j_1+k,j_2+p-k}.\end{aligned}$$

5.6.4 Monodromy of $\Psi_l^{(\mathcal{P})}$ in $V_d^{\otimes N}$

Combining the results obtained in the previous sections we conclude that

$$\begin{aligned}\widetilde{\mathcal{G}}^{(0)}_{j_1,j_2} &= \sum_{k=0}^{j_2} \alpha_{j_2,k} \widetilde{\mathcal{G}}^{(j_2-k)}_{j_1+k,0} \\ &= \sum_{k=0}^{j_2} (-1)^k \begin{bmatrix} j_2 \\ k \end{bmatrix}_q q^{-k^2-2j_1(k-j_2)+k(j_2+\mu_1-\mu_2)-j_2(1-j_2+\mu_1+\mu_2)} \widetilde{\mathcal{G}}^{(j_2-k)}_{j_1+k,0} \\ &= \sum_{k=0}^{j_2} (-1)^k \begin{bmatrix} j_2 \\ k \end{bmatrix}_q q^{-k^2-2j_1(k-j_2)+k(j_2+\mu_1-\mu_2)-j_2(1-j_2+\mu_1+\mu_2)} \hat{\mathcal{G}}^{(j_2-k)}_{j_1+k,0} \\ &= \sum_{k=0}^{j_2} (-1)^k \begin{bmatrix} j_2 \\ k \end{bmatrix}_q q^{-k^2-2j_1(k-j_2)+k(j_2+\mu_1-\mu_2)-j_2(1-j_2+\mu_1+\mu_2)} \\ &\quad \cdot \sum_{l=0}^{j_2-k} \begin{bmatrix} j_2-k \\ l \end{bmatrix}_q q^{-2(j_1+k)l-l^2+l(j_2-k+\mu_1)+(j_2-k)(1-(j_2-k)+\mu_2)} \\ &\quad \cdot \hat{\mathcal{G}}^{(0)}_{j_1+k+l,j_2-k-l} \\ &= \sum_{k=0}^{j_2} \sum_{l=0}^{j_2-k} (-1)^k \begin{bmatrix} j_2 \\ k \end{bmatrix}_q \begin{bmatrix} j_2-k \\ l \end{bmatrix}_q \\ &\quad \cdot q^{j_2(2j_1-\mu_1)+k(\mu_1-2(k+j_1+\mu_2)+3j_2-1)+l(\mu_1-l-2j_1+j_2-3k)} \mathcal{P}_{j_2-k-l,j_1+k+l} \\ &= \sum_{m=0}^{j_2} q^{j_2(2j_1-\mu_1)+m(\mu_1-m-2j_1+j_2)} \begin{bmatrix} j_2 \\ m \end{bmatrix}_q \\ &\quad \cdot \left(\sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{k(-1+2j_2-m-2\mu_2)} \right) \mathcal{P}_{j_2-m,j_1+m},\end{aligned}$$

where we denoted $m = k + l$ and used the relations

$$\hat{\mathcal{G}}^{(p)}_{j_1,0} = \widetilde{\mathcal{G}}^{(p)}_{j_1,0}, \quad \hat{\mathcal{G}}^{(0)}_{j_1,j_2} = \mathcal{P}_{j_2,j_1},$$

and

$$\begin{bmatrix} j_2 \\ k \end{bmatrix}_q \begin{bmatrix} j_2-k \\ l \end{bmatrix}_q = \begin{bmatrix} j_2 \\ k \end{bmatrix}_q \begin{bmatrix} j_2-k \\ m-k \end{bmatrix}_q = \begin{bmatrix} j_2 \\ m \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q$$

Using lemma 5.19 with $\beta = 2j_2 - 2\mu_2 - 1$ we further compute the sum over k

$$\begin{aligned}
&\sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{k(-1+2j_2-m-2\mu_2)} \\
&= q^{\frac{m}{2}(2j_2-2\mu_2-1-m)} \prod_{s=0}^{m-1}(q^{1+\mu_2-j_2+s} - q^{-1-\mu_2+j_2-s}) \\
&= q^{\frac{m}{2}(2j_2-2\mu_2-1-m)} \frac{[\mu_2 - j_2 + m]_q!}{[\mu_2 - j_2]_q!}(q - q^{-1})^m.
\end{aligned}$$

Hence

$$\begin{aligned}
\widetilde{\mathcal{G}}^{(0)}_{j_1,j_2} = \sum_{m=0}^{j_2} & q^{j_2(2j_1-\mu_1)+m(-2j_1+2j_2-\frac{3}{2}m+\mu_1-\mu_2-\frac{1}{2})}(q-q^{-1})^m \\
& \cdot \frac{[j_2]_q![\mu_2 - j_2 + m]_q!}{[m]_q![j_2 - m]_q![\mu_2 - j_2]_q!}\mathcal{P}_{j_2-m,j_1+m}
\end{aligned}$$

and

$$\begin{aligned}
M_{\gamma_{z_1,z_2}}\mathcal{P}_{j_1,j_2} &= q^{\frac{\mu_1\mu_2}{2}-j_1\mu_2}\widetilde{\mathcal{G}}^{(0)}_{j_1,j_2} \\
&= q^{\frac{\mu_1\mu_2}{2}-j_1\mu_2} \sum_{m=0}^{j_2} q^{j_2(2j_1-\mu_1)+m(-2j_1+2j_2-\frac{3}{2}m+\mu_1-\mu_2-\frac{1}{2})}(q-q^{-1})^m \\
&\quad \cdot \frac{[j_2]_q![\mu_2 - j_2 + m]_q!}{[m]_q![j_2 - m]_q![\mu_2 - j_2]_q!}\mathcal{P}_{j_2-m,j_1+m} \\
&= \sum_{m=0}^{j_2} q^{2(\frac{\mu_1}{2}-(j_1+m))(\frac{\mu_2}{2}-(j_2-m))+\frac{1}{2}m(m-1)}(q-q^{-1})^m \\
&\quad \cdot \frac{[j_2]_q![\mu_2 - j_2 + m]_q!}{[m]_q![j_2 - m]_q![\mu_2 - j_2]_q!}\mathcal{P}_{j_2-m,j_1+m}
\end{aligned}$$

Recall that the monodromy of $\mathcal{P}_{j_1,j_2}$ determines the monodromy of the functions

$$\begin{aligned}
&F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}(z) \\
&\quad = \psi_0(z)\int_{\mathcal{P}_{j_1,\ldots,j_N}} \prod_{1\le i<j\le l}(w_i - w_j)^{\frac{2}{\kappa}} \prod_{j,k}(w_j - z_k)^{-\frac{\mu_k}{\kappa}} r_{m_1,\ldots,m_N}(z,w)dw
\end{aligned}$$

for $m_1,\ldots,m_N \ge 0$, $\sum_{i=1}^N m_i = l$, that is

$$\begin{aligned}
F^{m_1,\ldots,m_N}_{j_1,\ldots,j_N}(\gamma_{z_i,z_{i+1}}(1)) &= \sum_{m=0}^{j_{i+1}} q^{2(\frac{\mu_i}{2}-(j_i+m))(\frac{\mu_{i+1}}{2}-(j_{i+1}-m))+\frac{1}{2}m(m-1)}(q-q^{-1})^m \\
\cdot \frac{[j_{i+1}]_q![\mu_{i+1} - j_{i+1} + m]_q!}{[m]_q![j_{i+1} - m]_q![\mu_{i+1} - j_{i+1}]_q!} & F^{m_1,\ldots,m_{i-1},m_{i+1},m_i,m_{i+2},\ldots,m_N}_{j_1,\ldots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\ldots,j_N}(\gamma_{z_i,z_{i+1}}(0))
\end{aligned}$$

Suppose $\mathcal{P}$ is an admissible l-surface in the sense of proposition 5.15 such that it can be written as a linear combination of families of loops of the form $\mathcal{P}_{j_1,\dots,j_N}$ of level $\sum_{i=1}^N j_i = l$. Then by linearity the monodromy of the solution

$$\Psi_l^{(\mathcal{P})}(z) = \psi_0(z) \int_{\mathcal{P}} \eta_{z,l} u_0$$

of $\mathrm{KZ}(\mathfrak{sl}_2)$ is determined by the monodromy of the components of the form

$$\begin{aligned}\Psi_l^{(\mathcal{P}_{j_1,\dots,j_N})}(z) &= \psi_0(z) \int_{\mathcal{P}_{j_1,\dots,j_N}} \eta_{z,l} u_0 \\ &= \sum_{m_1,\dots,m_N \geq 0,\ \sum_i m_i = l} F_{j_1,\dots,j_N}^{m_1,\dots,m_N}(z) f_1^{m_1} \cdots f_N^{m_N} u_0 \\ &= \sum_{m_1,\dots,m_N \geq 0,\ \sum_i m_i = l} F_{j_1,\dots,j_N}^{m_1,\dots,m_N}(z) v_{m_1} \otimes \cdots \otimes v_{m_N},\end{aligned}$$

that is

$$\begin{aligned}&\Psi_l^{(\mathcal{P}_{j_1,\dots,j_N})}(\gamma_{z_i,z_{i+1}}(1)) \\ &= \psi_0(\gamma_{z_i,z_{i+1}}(1)) \int_{\mathcal{P}_{j_1,\dots,j_N}} \eta_{\gamma_{z_i,z_{i+1}}(1),l} u_0 \\ &= \sum_{m_1,\dots,m_N \geq 0,\ \sum_i m_i = l} F_{j_1,\dots,j_N}^{m_1,\dots,m_N}(\gamma_{z_i,z_{i+1}}(1)) v_{m_1} \otimes \cdots \otimes v_{m_N} \\ &= \sum_{m_1,\dots,m_N \geq 0,\ \sum_i m_i = l} \Bigg(\sum_{m=0}^{j_{i+1}} q^{2(\frac{\mu_i}{2}-(j_i+m))(\frac{\mu_{i+1}}{2}-(j_{i+1}-m))+\frac{1}{2}m(m-1)} (q-q^{-1})^m \\ &\cdot \frac{[j_{i+1}]_q! [\mu_{i+1}-j_{i+1}+m]_q!}{[m]_q! [j_{i+1}-m]_q! [\mu_{i+1}-j_{i+1}]_q!} F_{j_1,\dots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\dots,j_N}^{m_1,\dots,m_{i-1},m_{i+1},m_i,m_{i+2},\dots,m_N}(\gamma_{z_i,z_{i+1}}(0)) \Bigg) \\ &\cdot v_{m_1} \otimes \cdots \otimes v_{m_N}.\end{aligned}$$

Notice that these are not in general solutions of $\mathrm{KZ}(\mathfrak{sl}_2)$.

Recall that we wanted to obtain a representation of the braid group B_N. Hence, let $d_1 = \cdots = d_N = d$, whence the highest weights satisfy $\mu_1 = \cdots = \mu_N = \mu = d-1$. Taking into account the action of the symmetric group on Y_N and writing

$$\begin{aligned}\gamma_{z_i,z_{i+1}}(1) &= (z_1, \dots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \dots, z_N), \\ \gamma_{z_i,z_{i+1}}(0) &= (z_1, \dots, z_N)\end{aligned}$$

we obtain in $(Y_N \times V_d^{\otimes N})/S_N$ the monodromy

$$
\begin{aligned}
&\Psi_l^{(\mathcal{P}_{j_1,\dots,j_N})}(\gamma_{z_i,z_{i+1}}(1)) \\
&= \psi_0(\gamma_{z_i,z_{i+1}}(1)) \int_{\mathcal{P}_{j_1,\dots,j_N}} \eta_{\gamma_{z_i,z_{i+1}}(1),l} u_0 \\
&= \psi_0(z_1,\dots,z_{i-1},z_{i+1},z_i,z_{i+2},\dots,z_N) \int_{\mathcal{P}_{j_1,\dots,j_N}} \eta_{\gamma_{z_i,z_{i+1}}(1),l} u_0 \\
&= \sum_{m_1,\dots,m_N\geq 0,\ \sum_i m_i = l} \Big(\sum_{m=0}^{j_{i+1}} q^{2(\frac{\mu_i}{2}-(j_i+m))(\frac{\mu_{i+1}}{2}-(j_{i+1}-m))+\frac{1}{2}m(m-1)} (q-q^{-1})^m \\
&\quad \cdot \frac{[j_{i+1}]_q! [\mu - j_{i+1}+m]_q!}{[m]_q! [j_{i+1}-m]_q! [\mu - j_{i+1}]_q!} F^{m_1,\dots,m_{i-1},m_{i+1},m_i,m_{i+2},\dots,m_N}_{j_1,\dots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\dots,j_N}(z_1,\dots,z_N) \Big) \\
&\quad \cdot v_{m_1}\otimes\cdots\otimes v_{m_N} \\
&= \sum_{m_1,\dots,m_N\geq 0,\ \sum_i m_i = l} \Big(\sum_{m=0}^{j_{i+1}} q^{2(\frac{\mu_i}{2}-(j_i+m))(\frac{\mu_{i+1}}{2}-(j_{i+1}-m))+\frac{1}{2}m(m-1)} (q-q^{-1})^m \\
&\quad \cdot \frac{[j_{i+1}]_q! [\mu - j_{i+1}+m]_q!}{[m]_q! [j_{i+1}-m]_q! [\mu - j_{i+1}]_q!} \\
&\quad \cdot F^{m_1,\dots,m_{i-1},m_{i+1},m_i,m_{i+2},\dots,m_N}_{j_1,\dots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\dots,j_N}(z_1,\dots,z_{i-1},z_{i+1},z_i,z_{i+2},\dots,z_N) \Big) \\
&\quad \cdot v_{m_1}\otimes\cdots\otimes v_{m_{i-1}}\otimes v_{m_{i+1}}\otimes v_{m_i}\otimes v_{m_{i+2}}\otimes\cdots\otimes v_{m_N} \\
&= \sum_{m_1,\dots,m_N\geq 0,\ \sum_i m_i = l} \Big(\sum_{m=0}^{j_{i+1}} q^{2(\frac{\mu_i}{2}-(j_i+m))(\frac{\mu_{i+1}}{2}-(j_{i+1}-m))+\frac{1}{2}m(m-1)} (q-q^{-1})^m \\
&\quad \cdot \frac{[j_{i+1}]_q! [\mu - j_{i+1}+m]_q!}{[m]_q! [j_{i+1}-m]_q! [\mu - j_{i+1}]_q!} F^{m_1,\dots,m_{i-1},m_{i+1},m_i,m_{i+2},\dots,m_N}_{j_1,\dots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\dots,j_N}(\gamma_{z_i,z_{i+1}}(1)) \Big) \\
&\quad \cdot (\tau_{V_d,V_d})_{i,i+1}(v_{m_1}\otimes\cdots\otimes v_{m_N}) \\
&= \sum_{m=0}^{j_{i+1}} q^{2(\frac{\mu_i}{2}-(j_i+m))(\frac{\mu_{i+1}}{2}-(j_{i+1}-m))+\frac{1}{2}m(m-1)} (q-q^{-1})^m \\
&\quad \cdot \frac{[j_{i+1}]_q! [\mu - j_{i+1}+m]_q!}{[m]_q! [j_{i+1}-m]_q! [\mu - j_{i+1}]_q!} \\
&\quad \cdot \psi_0(\gamma_{z_i,z_{i+1}}(1)) \int_{\mathcal{P}_{j_1,\dots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\dots,j_N}} \eta_{\gamma_{z_i,z_{i+1}}(1),l} u_0 \\
&= \sum_{m=0}^{j_{i+1}} q^{2(\frac{\mu_i}{2}-(j_i+m))(\frac{\mu_{i+1}}{2}-(j_{i+1}-m))+\frac{1}{2}m(m-1)} (q-q^{-1})^m \\
&\quad \cdot \frac{[j_{i+1}]_q! [\mu - j_{i+1}+m]_q!}{[m]_q! [j_{i+1}-m]_q! [\mu - j_{i+1}]_q!} \\
&\quad \cdot \Psi_l^{(\mathcal{P}_{j_1,\dots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\dots,j_N})}(\gamma_{z_i,z_{i+1}}(1)).
\end{aligned}
$$

We formulate the result as follows.

Theorem 5.21. *Suppose $\mathcal{P}$ is an admissible l-surface in the sense of proposition 5.15 such that it can be written as a linear combination of families of loops of the form $\mathcal{P}_{j_1,\ldots,j_N}$ of level $\sum_{i=1}^N j_i = l$. Let the associated solution of KZ($\mathfrak{sl}_2$) be $\Psi_l^{(\mathcal{P})} : Y_N \to V_d^{\otimes N}$,*

$$\Psi_l^{(\mathcal{P})}(z) = \psi_0(z) \int_{\mathcal{P}} \eta_{z,l} u_0 = \sum_{j_1,\ldots,j_N \geq 0,\ \sum_i j_i = l} c_{j_1,\ldots,j_N} \Psi_l^{(\mathcal{P}_{j_1,\ldots,j_N})}(z),$$

where $c_{j_1,\ldots,j_N} \in \mathbb{C}$. Let $\gamma_{z_i,z_{i+1}} : [0,1] \to Y_N$ be a half-loop on Y_N exchanging the points z_i and z_{i+1}, and let $M_{\gamma_{z_i,z_{i+1}}}$ be the corresponding monodromy operator. Then the action of $M_{\gamma_{z_i,z_{i+1}}}$ on $\Psi_l^{(\mathcal{P})}$ is defined by the formula

$$M_{\gamma_{z_i,z_{i+1}}} \Psi_l^{(\mathcal{P})} = \sum_{j_1,\ldots,j_N \geq 0,\ \sum_i j_i = l} c_{j_1,\ldots,j_N} M_{\gamma_{z_i,z_{i+1}}} \Psi_l^{(\mathcal{P}_{j_1,\ldots,j_N})},$$

where

$$M_{\gamma_{z_i,z_{i+1}}} \Psi_l^{(\mathcal{P}_{j_1,\ldots,j_N})} = \sum_{m=0}^{j_{i+1}} \widetilde{r}^{\,m}_{j_i,j_{i+1}} \Psi_l^{(\mathcal{P}_{j_1,\ldots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\ldots,j_N})}$$

and

$$\widetilde{r}^{\,m}_{j_i,j_{i+1}} = \frac{[j_{i+1}]_q! [\mu - j_{i+1} + m]_q!}{[m]_q! [j_{i+1} - m]_q! [\mu - j_{i+1}]_q!} (q - q^{-1})^m \cdot q^{2(\frac{\mu}{2} - (j_i + m))(\frac{\mu}{2} - (j_{i+1} - m)) + \frac{1}{2} m(m-1)},$$

$q = e^{\frac{\pi \mathrm{i}}{\kappa}}$*, and $\mu = d - 1$ is the highest weight of the $\mathfrak{sl}_2$-module V_d.*

6 The braid group representation of quantum $\mathfrak{sl}_2$

In this section we will go deeper in the theory of quantum groups and construct a representation of the braid group B_n from the quantum $\mathfrak{sl}_2$. We will see that this representation of B_n is actually equivalent to the monodromy representation of B_n arising from the monodromy of solutions of KZ($\mathfrak{sl}_2$). This is a most suprising result, relating two quite different braches of mathematics. The equivalence (for any semisimple Lie algebra) was first stated by Toshitake Kohno in 1987 [Koh87], who gave a description of the monodromy representation of the braid group arising from the KZ-equations in terms of quantum groups. In 1990 Vladimir Drinfeld [Dri90] established the relation between the monodromy of KZ and the braid group representation defined by the universal R-matrix of the associated quantum group in a more general framework. Drinfeld accepted the prestigious Fields medal in 1990. The equivalence of these two representations of B_n carries the name Drinfeld-Kohno theorem, and it is proved using topological algebra e.g. in [Kas95]. We will not consider the original proof in the case of a general Lie algebra in this thesis, but rather give an *explicit relation* between the monodromy of KZ($\mathfrak{sl}_2$) and the braid group representation of quantum $\mathfrak{sl}_2$.

We will first introduce the notion of R-matrices, which are one of the tools used in constructing representations of B_n. Indeed, we will see that any R-matrix defines a linear representation of B_n, and that special kind of R-matrices acting on tensor products of representations of bialgebras produce representations of B_n commuting with the action of the bialgebra. For example, the so called braided bialgebras have these kind of R-matrices.

Motivated by braided bialgebras, we will use the technique of R-matrices to construct representations of B_n on tensor products of representations of the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. For this we will also introduce the notion of the Drinfeld double (quantum double), which is a construction originally invented by Drinfeld in 1986 [Dri86] in order to supply many non-commutative, non-cocommutative Hopf algebras. The Drinfeld double is a Hopf algebra which as a vector space is a tensor product of a Hopf algebra and a certain subspace of its algebraic dual. For finite dimensional Hopf algebras the Drinfeld double construction yields braided Hopf algebras. Motivated by this, we will introduce the quantum Borel algebra $H_{q^2} \subset U_q(\mathfrak{sl}_2)[\sqrt{K}]$ and find that the Drinfeld double associated to this Hopf algebra produces as a quotient structure the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. This enables us to construct an R-matrix which in turn defines representations of the braid group on tensor products of repesentations of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, found later to be equivalent to the monodromy of KZ($\mathfrak{sl}_2$). We state most of the results concerning Hopf algebras and quantum groups without proofs, which can be found in [Kas95] and [Kyt11].

6.1 R-matrices

We will now introduce the notion of R-matrices, which define linear representations of the braid group. Let A be a bialgebra. If $R = \sum_i s_i \otimes t_i \in A \otimes A$, we write

$$R_{12} := \sum_i s_i \otimes t_i \otimes 1_A, \quad R_{23} := \sum_i 1_A \otimes s_i \otimes t_i, \quad R_{13} := \sum_i s_i \otimes 1_A \otimes t_i.$$

Recall that for a bialgebra $(A, \mu, \Delta, \eta, \epsilon)$ we denote the *co-opposite coproduct* by $\Delta^{cop} : A \to A \otimes A$,

$$\Delta^{cop} = \tau_{A,A} \circ \Delta.$$

Definition 6.1. Let A be a bialgebra. An invertible element $R \in A \otimes A$ is a *universal R-matrix* if the following conditions hold.

$$\begin{aligned} &\Delta^{cop}(a) = R\Delta(a)R^{-1} \text{ for all } a \in A, \\ &(\Delta \otimes id_A)(R) = R_{13}R_{23}, \\ &(id_A \otimes \Delta)(R) = R_{13}R_{12}. \end{aligned}$$

If A admits a universal R-matrix it is said to be *braided.*

If A is cocommutative, that is $\Delta^{cop} = \Delta$, then $1_A \otimes 1_A$ is a universal R-matrix. Hence braided bialgebras generalise cocommutative bialgebras.

Definition 6.2. A linear automorphism $\check{R} \in End(V \otimes V)$ of the second tensor power of a vector space V is called an *R-matrix* if it satisfies the *Yang-Baxter equation*

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}, \tag{YBE}$$

where the notation $\check{R}_{ij}$ denotes the operator $\check{R}$ acting on the i:th and j:th tensor component of $V \otimes V \otimes V$.

The following lemma is easy to verify.

Lemma 6.3. *The equation YBE is equivalent to*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{YBE'}$$

where $R = \tau_{V,V} \circ \check{R}$.

In representations, universal R-matrices give solutions to YBE. Solutions of YBE induce representations of the braid group.

Proposition 6.4. *Let V be a vector space and $\check{R}$ an R-matrix. Then for any integer $n > 0$ on $V^{\otimes n}$ there is a representation*

$$\rho_n^{\check{R}} : B_n \to Aut(V^{\otimes n})$$

of the braid group, such that $\rho_n^{\check{R}}(\sigma_i) = \check{R}_{i,i+1}$, $i = 1, \ldots, n-1$.

Proof. By invertibility, $\check{R}_{i,i+1} \in Aut(V^{\otimes n})$ for every $i = 1, \ldots, n-1$. The braid group relation (6) is trivial, and (7) follows from YBE. $\square$

Let A be a braided bialgebra with universal R-matrix R. Let ρ_V, ρ_W be two representations of A on the vector spaces V and W.

The following theorem is proved in detail in [Kyt11]. The proof is a direct computation.

Theorem 6.5. *Let A be a braided bialgebra with universal R-matrix R. Let ρ_V, ρ_W be two representations of A on the vector spaces V and W. Then*

(i) the linear map

$$c_{V,W} := \tau_{V,W} \circ (\rho_V \otimes \rho_W)(R) : V \otimes W \to W \otimes V$$

is an isomorphism of representations, and

(ii) the linear map $\check{R} := c_{V,V} \in End(V \otimes V)$ is an R-matrix.

Moreover, the braid group action on $V^{\otimes n}$ defined by $\check{R}$ commutes with the action of A.

6.2 The restricted dual

For a linear map $f : V \to W$ the *transpose* of f is defined as the linear map $f^* : W^* \to V^*$ between the algebraic duals, given by

$$f^*(\varphi)(v) := \varphi(f(v)).$$

The tensor product of duals has the natural inclusion

$$V^* \otimes W^* \subset (V \otimes W)^*,$$

with the identification

$$(\psi \otimes \varphi)(v \otimes w) = \psi(v)\varphi(w).$$

We also identify $\mathbb{C}$ and its dual by $f(1) \leftrightarrow f \in \mathbb{C}^*$.

It can easily be shown (see [Kyt11]) that the dual A^* of a coalgebra (A, Δ, ϵ) is a unital associative algebra equipped with the product

$$\mu = \Delta^*|_{A^* \otimes A^*} : A^* \otimes A^* \to A^*$$

and the unit

$$\eta = \epsilon^* : \mathbb{C} \to A^*.$$

However, in order the dual of an algebra (A, μ, η) to be a coalgebra, one needs to ensure that the transpose μ^* of the product takes values not only in the space $(A \otimes A)^*$ but in $A^* \otimes A^*$. Since this is not always the case, we define the restricted dual as the inverse image of μ^*.

Definition 6.6. Let (A, μ, η) be a unital associative algebra. The *restricted dual* of A is the vector space

$$A^o := (\mu^*)^{-1}(A^* \otimes A^*).$$

In [Kyt11] it is shown that in this way a coalgebra structure for A^o is naturally defined. Moreover, for Hopf algebras we have the following; see also [Kyt11].

Theorem 6.7. *Let $(A, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra. Then the restricted dual*

$$(A^o, \Delta^*|_{A^o \otimes A^o}, \mu^*|_{A^o}, \epsilon^*, \eta^*|_{A^o}, \gamma^*|_{A^o})$$

is a Hopf algebra.

In [Kyt11] it is also shown that the restricted dual of an algebra A is spanned by the *representative forms* of finite dimensional A-modules, which are linear maps $\lambda_{i,j} : A \to \mathbb{C}$ defined by the rule

$$x.v_j = \sum_{i=1}^{n} \lambda_{i,j}(x) v_i$$

for all $x \in A$, where $\{v_i\}_{i=1}^n$ is a basis of the associated A-module. Notice that the representative forms satisfy

$$\lambda_{i,j}(xy) = \sum_{k=1}^{n} \lambda_{i,k}(x) \lambda_{k,j}(y),$$

$i, j = 1, \ldots, n$, which follows from the fact that the action of A on an A-module respects the product of the algebra.

6.3 The quantum Borel algebra H_{q^2}

The quantum Borel algebra H_{q^2} is another tool needed for constructing R-matrices for quantum groups. In sections 6.4 and 6.5 we present a construction of a Drinfeld double associated to H_{q^2} which enables us to interpret the quantum Borel algebra H_{q^2} as a Hopf subalgebra of the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, defined in section 6.5.

Let $q \in \mathbb{C}\backslash\{0\}$. Let H_{q^2} be the algebra generated by the elements a, a^{-1}, b satisfying the relations

$$aa^{-1} = 1 = a^{-1}a, \quad ab = q^2 ba.$$

The collection $\{b^m a^n\}_{m \in \mathbb{N}, n \in \mathbb{Z}}$ is a vector space basis for H_{q^2}, and the product in this basis reads

$$b^{m_1} a^{n_1} b^{m_2} a^{n_2} \equiv \mu(b^{m_1} a^{n_1} \otimes b^{m_2} a^{n_2}) = q^{2 n_1 m_2} b^{m_1 + m_2} a^{n_1 + n_2}.$$

It is a direct computation, using the defining relations and the definition of a Hopf algebra, to see that there is a unique Hopf algebra structure on H_{q^2} such that the coproduct is defined by

$$\Delta(a) = a \otimes a, \qquad \Delta(b) = a \otimes b + b \otimes 1,$$

which implies the following formulas:

$$\begin{aligned}
\Delta(b^m a^n) &= \sum_{k=0}^{m} q^{k(m-k)} \begin{bmatrix} m \\ k \end{bmatrix}_q b^k a^{m-k+n} \otimes b^{m-k} a^n, \\
\epsilon(b^m a^n) &= \delta_{m,0}, \\
\gamma(b^m a^n) &= (-1)^m q^{-m(m+1)-2nm} b^m a^{-(n+m)}.
\end{aligned}$$

We will assume from now on that q is not a root of unity. Then H_{q^2} is neither commutative nor cocommutative, and the antipode γ is not involutive, since $\gamma(\gamma(b)) = q^{-2}b \neq b$.

In [Kyt11] representative forms associated to one and two dimensional representations of H_{q^2} are computed. Some of them span a Hopf subalgebra H'_{q^2} of the restricted dual $H^o_{q^2}$, isomorphic to H_{q^2}. It is convenient to define for $k \geq 0$ the elements $h_z^{(k)} \in H^o_{q^2}$ by

$$h_z^{(k)}(b^m a^n) := \delta_{m,k} z^n,$$

and denote by $h_z^{(0)} = g_z$. The elements $h_z^{(k)}$ for $k = 0, 1$ arise from one and two dimensional representations of H_{q^2}, and those can be generalised to all $k \geq 0$.

Using the Hopf algebra relations of H_{q^2}, the products and coproducts of these elements in $H^o_{q^2}$ can be computed. The computations are straightforward, and the reader may consult [Kyt11] for more details. We will only state the result.

Lemma 6.8. *Define for $k \geq 0$ the elements $h_z^{(k)} \in H^o_{q^2}$ by*

$$h_z^{(k)}(b^m a^n) := \delta_{m,k} z^n, \quad h_z^{(0)} = g_z.$$

Then for any $k, l \geq 0$

$$\begin{aligned}
h_z^{(k)} h_w^{(l)} &\equiv \Delta^*(h_z^{(k)} \otimes h_w^{(l)}) = z^l q^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix}_q h_{zw}^{(k+l)}, \\
\mu^*(g_z) &= g_z \otimes g_z, \\
\mu^*(h_z^{(1)}) &= g_{q^2 z} \otimes h_z^{(1)} + h_z^{(1)} \otimes g_z.
\end{aligned}$$

Definition 6.9. Let $(A, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra. If a nonzero element $a \in A$ satisfies $\Delta(a) = a \otimes a$, it is said to be *grouplike*. Notice that the elements g_z are grouplike in $H^o_{q^2}$.

Lemma 6.10. *Grouplike elements in A are invertible, and they form a linearly independent set in A. Moreover, $\epsilon(a) = 1$ for any grouplike $a \in A \setminus \{0\}$.*

Proof. Let $a \in A \setminus \{0\}$ be grouplike. By (H2')

$$0 \neq a = (\epsilon \otimes id_A) \circ \Delta(a) = (\epsilon \otimes id_A)(a \otimes a) = \epsilon(a)a,$$

which implies $\epsilon(a) = 1$. Similarly, by (H3)

$$\begin{aligned}\mu(\gamma(a) \otimes a) &= \mu \circ (\gamma \otimes id_A) \circ \Delta(a) = \eta \circ \epsilon(a) = 1_A \\ &= \eta \circ \epsilon(a) = \mu \circ (id_A \otimes \gamma) \circ \Delta(a) = \mu(a \otimes \gamma(a)),\end{aligned}$$

which implies $\gamma(a) = a^{-1}$, and in particular a is invertible.

Let then $a_1, \ldots, a_n$ be grouplike elements in A having a nontrivial relation

$$\sum_{i=1}^{n} c_i a_i = 0,$$

where $c_i \neq 0$ for all $i = 1, \ldots, n$, and suppose $n \in \mathbb{N}$ is the smallest number such that a relation of this kind exists. By renormalising the coefficients c_i we may assume that $c_1 = 1$ and write the relation in the form

$$\sum_{i=2}^{n} c_i a_i = a_1.$$

Now

$$\begin{aligned}&1 = \epsilon(a_1) = \sum_{i=2}^{n} c_i \epsilon(a_i) = \sum_{i=2}^{n} c_i, \text{ and} \\ &\sum_{i,j=2}^{n} c_i c_j (a_i \otimes a_j) = a_1 \otimes a_1 = \Delta(a_1) = \sum_{i=2}^{n} c_i \Delta(a_i) = \sum_{i=2}^{n} c_i (a_i \otimes a_i).\end{aligned}$$

Hence, there exists an index $k \in \{2, \ldots, n\}$ such that $a = a_k$, contradicting the minimality of n. □

Proposition 6.11. *Let $q \in \mathbb{C} \setminus \{0\}$, and suppose q is not a root of unity. Then the algebra H_{q^2} can be embedded to its restricted dual by an injective morphism of Hopf algebras $b^m a^n \mapsto \widetilde{b}^m \widetilde{a}^n$, where*

$$\widetilde{a} = g_{q^2}, \qquad \widetilde{b} = h_1^{(1)}.$$

We denote the image of this embedding by $H'_{q^2} \subset H^o_{q^2}$.

Proof. Since $\widetilde{a}$ is a grouplike element, by lemma 6.10 it is invertible. The defining relations of H_{q^2} as an algebra hold for $\widetilde{a}$ and $\widetilde{b}$, since

$$\widetilde{a}\widetilde{b} = g_{q^2} h_1^{(1)} = q^2 h_{q^2}^{(1)} \quad \text{and}$$
$$\widetilde{b}\widetilde{a} = h_1^{(1)} g_{q^2} = h_{q^2}^{(1)},$$

whence

$$\widetilde{a}\widetilde{b} = q^2 \widetilde{b}\widetilde{a}.$$

The Hopf algebra structure of H'_{q^2} is uniquely determined by the relations

$$\mu^*(\widetilde{a}) = \widetilde{a} \otimes \widetilde{a},$$
$$\mu^*(\widetilde{b}) = \widetilde{a} \otimes \widetilde{b} + \widetilde{b} \otimes \widetilde{1},$$

where $\widetilde{1} = \epsilon^*(1) = g_1$ is the unit element in $H^o_{q^2}$. Finally, the images of the basis elements are

$$\widetilde{b}^r \widetilde{a}^s = q^{\frac{1}{2}r(r-1)} [r]_q! h_{q^{2s}}^{(r)},$$

which are nonzero and linearly independent in $H^o_{q^2}$ when q is not a root of unity. Namely,

$$\widetilde{a}^s = (g_{q^2})^s = (g_{q^2})^{s-2} g_{q^2 q^2} = g_{q^{2s}} \text{ and}$$
$$\widetilde{b}^r = (h_1^{(1)})^r = (h_1^{(1)})^{r-2}(h_1^{(1)})^2 = (h_1^{(1)})^{r-2} q^1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q h_1^{(2)}$$
$$=(h_1^{(1)})^{r-3} q^1 [2]_q q^2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q h_1^{(3)} = (h_1^{(1)})^{r-3} q^1 [2]_q q^2 [3]_q h_1^{(3)}$$
$$= \cdots = q^{\sum_{i=1}^{r-1} i} [r]_q! h_1^{(r)} = q^{\frac{1}{2}r(r-1)} [r]_q! h_1^{(r)},$$

whence the action of $\widetilde{b}^r \widetilde{a}^s$ on the basis elements of H_{q^2} is

$$\widetilde{b}^r \widetilde{a}^s (b^m a^n) = ((h_1^{(1)})^r \otimes (g_{q^2})^s)(\Delta(b^m a^n))$$
$$= \sum_{k=0}^{m} q^{k(m-k)} \begin{bmatrix} m \\ k \end{bmatrix}_q ((h_1^{(1)})^r \otimes (g_{q^2})^s)(b^k a^{m-k+n} \otimes b^{m-k} a^n)$$
$$= \sum_{k=0}^{m} q^{k(m-k)} \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\frac{1}{2}r(r-1)} [r]_q! h_1^{(r)} (b^k a^{m-k+n}) g_{q^{2s}} (b^{m-k} a^n)$$
$$= \sum_{k=0}^{m} q^{k(m-k)+\frac{1}{2}r(r-1)} \begin{bmatrix} m \\ k \end{bmatrix}_q [r]_q! \delta_{k,r} \delta_{m-k,0} q^{2sn}$$
$$= q^{\frac{1}{2}r(r-1)} [r]_q! \delta_{m,r} q^{2sn} = q^{\frac{1}{2}r(r-1)} [r]_q! h_{q^{2s}}^{(r)} (b^m a^n).$$

Thus the embedding $b^m a^n \mapsto \widetilde{b}^m \widetilde{a}^n$ is injective. □

6.4 The Drinfeld double

Let $(A, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra such that γ has an inverse γ^{-1}. Let $B \subset A^o$ be a Hopf subalgebra, and denote the unit of A^o by $\widetilde{1}$. In the sequel we will use use the Sweedler's sigma notation for the coproduct, defined as follows. By properties of the tensor product the coproduct $\Delta : A \to A \otimes A$ can for every $a \in A$ be written as a linear combination of simple tensors, that is

$$\Delta(a) = \sum_{i=1}^{k} a_i' \otimes a_i'', \tag{10}$$

where $a_i', a_i'' \in A$, and by coassociativity the double coproduct can be written as

$$\sum_{i=1}^{k}\sum_{j=1}^{m} a_i' \otimes \left((a_i'')_j' \otimes (a_i'')_j''\right) = \sum_{i=1}^{k} a_i' \otimes \left(\sum_{j=1}^{m} (a_i'')_j' \otimes (a_i'')_j''\right)$$

$$= \sum_{i=1}^{k} a_i' \otimes \Delta(a_i'') = (id_A \otimes \Delta)(\Delta(a)) = (\Delta \otimes id_A)(\Delta(a)) = \sum_{i=1}^{k} \Delta(a_i') \otimes a_i''$$

$$= \sum_{i=1}^{k} \left(\sum_{j=1}^{n} (a_i')_j' \otimes (a_i')_j''\right) \otimes a_i'' = \sum_{i=1}^{k}\sum_{j=1}^{n} \left((a_i')_j' \otimes (a_i')_j''\right) \otimes a_i''.$$

In such expressions the choices of the simple tensors are not unique. Hence it is convenient to denote the coproduct of an element $a \in A$ by

$$\Delta(a) := \sum_{(a)} a_{(1)} \otimes a_{(2)},$$

where the notation represents any of the possible expressions of the form (10). This is called the *Sweedler's sigma notation.* By coassociativity the double coproduct reads then

$$\sum_{(a)}\sum_{(a_{(2)})} a_{(1)} \otimes \left((a_{(2)})_{(1)} \otimes (a_{(2)})_{(2)}\right) = (id_A \otimes \Delta)(\Delta(a))$$

$$= (\Delta \otimes id_A)(\Delta(a)) = \sum_{(a)}\sum_{(a_{(1)})} \left((a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)}\right) \otimes a_{(2)}.$$

By slight abuse of notation we write the above as

$$(id_A \otimes \Delta)(\Delta(a)) = (\Delta \otimes id_A)(\Delta(a)) := \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

Notice also that in A^o the coproduct of an element $\varphi \in A^o$ is

$$\mu^*(\varphi) := \sum_{(\varphi)} \varphi_{(1)} \otimes \varphi_{(2)}.$$

The following construction was originally established by Drinfeld in [Dri86]. For details the reader may also consult [Kyt11].

Theorem 6.12. *Let $(A, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra such that γ has an inverse γ^{-1}. Let $B \subset A^o$ be a Hopf subalgebra. Then the space $A \otimes B$ admits a unique Hopf algebra structure such that*

(i) The map $\iota_A : A \to A \otimes B$,

$$\iota_A(a) = a \otimes \widetilde{1}$$

is a morphism of Hopf algebras.

(ii) The map $\iota_B : B^{cop} \to A \otimes B$,

$$\iota_B(\varphi) = 1 \otimes \varphi$$

is a morphism of Hopf algebras.

(iii) For all $a \in A$, $\varphi \in B$

$$(a \otimes \widetilde{1})(1 \otimes \varphi) = a \otimes \varphi.$$

(iv) For all $a \in A$, $\varphi \in B$

$$(1 \otimes \varphi)(a \otimes \widetilde{1}) = \sum_{(a)} \sum_{(\varphi)} \{\varphi_{(1)}(a_{(3)})\}\{\varphi_{(3)}(\gamma^{-1}(a_{(1)}))\}(a_{(2)} \otimes \varphi_{(2)}).$$

This Hopf algebra is denoted by $\mathcal{D}(A, B)$ and called the Drinfeld double *associated to A and B.*

The proof of this theorem relies on the definition of a Hopf algebra and the properties (i)-(iv). Most of the details can be found in [Kyt11]. We list the Hopf algebra structure of $\mathcal{D}(A, B)$ following from the properties (i)-(iv). The product reads for all $a, b \in A$, $\varphi, \psi \in B$

$$(a \otimes \varphi)(b \otimes \psi) = \sum_{(b)} \sum_{(\varphi)} \{\varphi_{(1)}(a_{(3)})\}\{\varphi_{(3)}(\gamma^{-1}(a_{(1)}))\}(ab_{(2)} \otimes \varphi_{(2)}\psi),$$

and the unit is

$$\eta_{\mathcal{D}}(1) = 1 \otimes \widetilde{1};$$

the coproduct reads for all $a \in A$, $\varphi \in B$

$$\Delta_{\mathcal{D}}(a \otimes \varphi) = \sum_{(a)} \sum_{(\varphi)} (a_{(1)} \otimes \varphi_{(2)}) \otimes (a_{(2)} \otimes \varphi_{(1)}),$$

and the counit is

$$\epsilon_{\mathcal{D}}(a \otimes \varphi) = \epsilon(a)\varphi(1).$$

The antipode of $\mathcal{D}(A, B)$ has the following expression, for $a \in A$, $\varphi \in B$,

$$\gamma_{\mathcal{D}}(a \otimes \varphi) = \sum_{(a)} \sum_{(\varphi)} \{\varphi_{(1)}(\gamma^{-1}(a_{(3)}))\}\{\varphi_{(3)}(a_{(1)})\}(\gamma(a_{(2)}) \otimes (\gamma^*)^{-1}(\varphi_{(2)})).$$

Example 6.13. When A is finite dimensional, it can be shown that the antipode is invertible. The restricted dual equals the algebraic dual of A, that is, $A^o = A^*$. The Drinfeld double associated to A and A^* is usually denoted simply by $\mathcal{D}(A)$.

Example 6.14. When $q \in \mathbb{C} \setminus \{0\}$ is not a root of unity, the antipode of the Hopf algebra H_{q^2} is invertible, with the inverse given by

$$\gamma^{-1}(b^m a^n) = (-1)^m q^{-m(m-1)-2mn} b^m a^{-(m+n)}.$$

This is seen by direct computation. Hence, the Drinfeld double $\mathcal{D}(H_{q^2}, B)$ associated to H_{q^2} and any Hopf subalgebra $B \subset H^o_{q^2}$ is well defined.

For finite dimensional Hopf algebras, the Drinfeld doubles are always braided. The proof can be found in [Kyt11].

Theorem 6.15. *Let A be a finite dimensional Hopf algebra and $\{e_i\}_{i=1}^n$ and $\{\delta^i\}_{i=1}^n$ the basis and dual basis of A and A^*, respectively. Then $\mathcal{D}(A)$ is braided with universal R-matrix*

$$R = \sum_{i=1}^{n} (e_i \otimes \widetilde{1}) \otimes (1 \otimes \delta^i).$$

However, in the infinite dimensional case we cannot use the bases to define a universal R-matrix as an infinite sum. Hence finding a universal R-matrix is quite a problematic task. Nevertheless, in some special cases R-matrices, that is solutions of YBE, associated to finite dimensional representations of the Drinfeld double can be found - and these yield representations of the braid group, commuting with the action of the Drinfeld double.

6.5 The extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$

We want to construct an R-matrix associated to representations of the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, which can be interpreted as a quotient of a Drinfeld double. In this section we assume that q is a nonzero complex number which is *not a root of unity*. Then by example 6.14 the quantum Borel Hopf algebra H_{q^2} has a well defined Drinfeld double associated to the Hopf subalgebra

$$H''_{q^2} := H'_{q^2}[\sqrt{\widetilde{a}}] \subset H^o_{q^2}$$

of $H^o_{q^2}$ generated by the elements

$$\sqrt{\widetilde{a}} := g_q, \quad \widetilde{b} = h_1^{(1)}.$$

Denote the Drinfeld double by

$$\mathcal{D}_{q^2} := \mathcal{D}(H_{q^2}, H''_{q^2}).$$

By definition both H_{q^2} and H''_{q^2} are embedded to $\mathcal{D}_{q^2}$. Denote the embedded generators by

$$\alpha := a \otimes \widetilde{1}, \quad \beta := b \otimes \widetilde{1}, \quad \sqrt{\widetilde{\alpha}} := 1 \otimes \sqrt{\widetilde{a}}, \quad \widetilde{\beta} := 1 \otimes \widetilde{b}.$$

We also denote naturally $\widetilde{\alpha} = (\sqrt{\widetilde{\alpha}})^2$. By the properties $(i) - (iii)$ of the Drinfeld double $\mathcal{D}_{q^2}$ has a vector space basis

$$\{\beta^m \alpha^n \widetilde{\beta}^{m'} (\sqrt{\widetilde{\alpha}})^{n'}\}_{m,m' \in \mathbb{N} \cup \{0\}, n,n' \in \mathbb{Z}}.$$

By direct computation one can convince oneself that the following relations hold in $\mathcal{D}_{q^2}$. For details one may also consult [Kyt11], where a similar Drinfeld double construction is made for the pair H_{q^2}, H'_{q^2}.

Proposition 6.16. *The Hopf algebra $\mathcal{D}_{q^2}$ is as an algebra generated by the elements $\alpha, \alpha^{-1}, \beta, \sqrt{\widetilde{\alpha}}, (\sqrt{\widetilde{\alpha}})^{-1}, \widetilde{\beta}$ with the relations*

$$\begin{aligned} \alpha\alpha^{-1} &= 1 = \alpha^{-1}\alpha, & \sqrt{\widetilde{\alpha}}(\sqrt{\widetilde{\alpha}})^{-1} &= 1 = (\sqrt{\widetilde{\alpha}})^{-1}\sqrt{\widetilde{\alpha}}, \\ \alpha\beta &= q^2\beta\alpha, & \sqrt{\widetilde{\alpha}}\widetilde{\beta} &= q\widetilde{\beta}\sqrt{\widetilde{\alpha}}, \\ \alpha\widetilde{\beta} &= q^{-2}\widetilde{\beta}\alpha, & \sqrt{\widetilde{\alpha}}\beta &= q^{-1}\beta\sqrt{\widetilde{\alpha}}, \\ \alpha\sqrt{\widetilde{\alpha}} &= \sqrt{\widetilde{\alpha}}\alpha, & \widetilde{\beta}\beta - \beta\widetilde{\beta} &= \alpha - \widetilde{\alpha}. \end{aligned}$$

The Hopf algebra structure on $\mathcal{D}_{q^2}$ is uniquely determined by the values of the following coproduct.

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha, & \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes 1 \\ \Delta(\sqrt{\widetilde{\alpha}}) &= \sqrt{\widetilde{\alpha}} \otimes \sqrt{\widetilde{\alpha}}, & \Delta(\widetilde{\beta}) &= \widetilde{\beta} \otimes \widetilde{\alpha} + 1 \otimes \widetilde{\beta}. \end{aligned}$$

We can define the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ analogously to the definition 4.8 of $U_q(\mathfrak{sl}_2)$.

Definition 6.17. The *extended quantum group* $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ is the algebra generated by the elements $E, F, \sqrt{K}, (\sqrt{K})^{-1}$ satisfying the relations

$$\begin{aligned} \sqrt{K}(\sqrt{K})^{-1} &= 1 = (\sqrt{K})^{-1}\sqrt{K}, & \sqrt{K}E(\sqrt{K})^{-1} &= qE, \\ EF - FE &= \frac{1}{q - q^{-1}}(K - K^{-1}), & \sqrt{K}F\sqrt{K}^{-1} &= q^{-1}F, \end{aligned}$$

where we denote $K = (\sqrt{K})^2$.

Moreover, similarly as for the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ we obtain a Hopf algebra structure for $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. The proof of the following lemma is analogous to the result for $U_q(\mathfrak{sl}_2)$.

Lemma 6.18. *$U_q(\mathfrak{sl}_2)[\sqrt{K}]$ admits a unique Hopf algebra structure determined by the coproduct*

$$\Delta(\sqrt{K}) = \sqrt{K}\otimes\sqrt{K}, \quad \Delta(E) = E\otimes K+1\otimes E, \quad \Delta(F) = K^{-1}\otimes F+F\otimes 1.$$

The Hopf algebra structure is the following.

$$\epsilon(\sqrt{K}) = 1, \qquad \epsilon(E) = 0 = \epsilon(F),$$
$$\gamma(\sqrt{K}) = (\sqrt{K})^{-1}, \quad \gamma(E) = -EK^{-1}, \quad \gamma(F) = -KF.$$

It turns out that the Hopf algebra $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ can be interpreted as a quotient structure of the Drinfeld double $\mathcal{D}_{q^2}$. Hence, by property (i) of the Drinfeld double, the quantum Borel algebra H_{q^2} can be interpreted as a subalgebra of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. The following lemma can be proved by direct computation similarly as in [Kyt11].

Lemma 6.19. *The element $\alpha\widetilde{\alpha}$ is a grouplike central element in $\mathcal{D}_{q^2}$, and the two sided ideal generated by $\alpha\widetilde{\alpha} - 1$ is a Hopf ideal.*

Thus the quotient Hopf algebra $\mathcal{D}_{q^2}/(\alpha\widetilde{\alpha} - 1)$ is well defined. Using the notations $\sqrt{K}, E$ and F for the equivalence classes of $\sqrt{\widetilde{\alpha}}$, $\frac{1}{q^{-1}-q}\widetilde{\beta}$ and β in $\mathcal{D}_{q^2}/(\alpha\widetilde{\alpha} - 1)$, respectively, the defining relations of $\mathcal{D}_{q^2}/(\alpha\widetilde{\alpha} - 1)$ become the same as the relations of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$.

Proposition 6.20. *When $q \in \mathbb{C}\setminus\{0\}$ is not a root of unity the Hopf algebras $U_q(\mathfrak{sl}_2)$ and $\mathcal{D}_{q^2}/(\alpha\widetilde{\alpha} - 1)$ are isomorphic, with*

$$\sqrt{K} \leftrightarrow \left[\sqrt{\widetilde{\alpha}}\right] \qquad\qquad (\sqrt{K})^{-1} \leftrightarrow \left[\sqrt{\widetilde{\alpha}}\alpha\right]$$
$$E \leftrightarrow \left[\frac{1}{q^{-1}-q}\widetilde{\beta}\right] \qquad\qquad F \leftrightarrow [\beta]\,.$$

Notice that the irreducible representations of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ can be found by considering the irreducible representations of $\mathcal{D}_{q^2}$, letting the element $\alpha\widetilde{\alpha}$ act as the identity. We state the following result without proof, which has the same idea as the case of $\mathfrak{sl}_2$, with somewhat more computations. A proof of a similar result for the pair H_{q^2}, H'_{q^2} is presented in [Kyt11], and it applies to the pair H_{q^2}, H''_{q^2} following the same lines. Fix a branch of $\sqrt{q} \in \mathbb{C}\setminus\{0\}$.

Theorem 6.21. *For any integer $d > 0$ and for any $\lambda \in \mathbb{C}\setminus\{0\}$ and a choice of the branch of the fourth root $\lambda^{\frac{1}{4}}$ there exists a d-dimensional irreducible representation $W_d^{\lambda^{\frac{1}{4}}}$ of $\mathcal{D}_{q^2}$ with basis $\{w_j\}_{j=0}^{d-1}$ such that*

$$\begin{aligned}
\alpha.w_j &= \lambda^{\frac{1}{2}} q^{1-d+2j} w_j,\\
\beta.w_j &= w_{j+1},\\
\sqrt{\widetilde{\alpha}}.w_j &= \lambda^{\frac{1}{4}} q^{\frac{d-1}{2}-j} w_j,\\
\widetilde{\beta}.w_j &= \lambda^{\frac{1}{2}} [j]_q [d-j]_q (q^{-1}-q) w_{j-1}.
\end{aligned}$$

There are no other finite dimensional irreducible $\mathcal{D}_{q^2}$-modules.

For the irreducible $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-modules we obtain the following result, identifying the element $\alpha\widetilde{\alpha}$ with the identity $1\otimes\widetilde{1}$ in $\mathcal{D}_{q^2}$. For convenience we will use the same notation for the representations of $\mathcal{D}_{q^2}$ and $U_q(\mathfrak{sl}_2)[\sqrt{K}]$.

Theorem 6.22. *For any integer $d>0$ and $\nu\in\{\pm1,\pm i\}$ there exists a d-dimensional irreducible representation W_d^ν of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ with basis $\{w_j\}_{j=0}^{d-1}$ such that*

$$\begin{aligned}\sqrt{K}.w_j &= \nu q^{\frac{d-1}{2}-j}w_j,\\ F.w_j &= w_{j+1},\\ E.w_j &= \nu^2[j]_q[d-j]_q w_{j-1}.\end{aligned}$$

There are no other finite dimensional irreducible $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-modules.

The reader is invited to compare this result with the irreducible representations of $U_q(\mathfrak{sl}_2)$ stated in theorem 4.14. Notice in particular that the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ is a Hopf subalgebra of the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, whence representations of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ yield representations of $U_q(\mathfrak{sl}_2)$ by restriction.

6.6 Heuristics concerning the R-matrix

Suppose $(A,\mu,\Delta,\eta,\epsilon,\gamma)$ is an infinite dimensional Hopf algebra such that the antipode γ is invertible. Then by theorem 6.12 the Drinfeld double $\mathcal{D}(A,A^o)$ is uniquely determined. As a vector space, $\mathcal{D}(A,A^o)=A\otimes A^o$, with the embedded Hopf subalgebras A and A^o by

$$\iota_A : a\mapsto a\otimes\widetilde{1} \text{ and } \iota_{A^o}:\varphi\mapsto 1\otimes\varphi,$$

respectively. We would like $\mathcal{D}(A,A^o)$ to have a universal R-matrix with a similar expression as in the finite dimensional case,

$$R=\sum_{i\in I}\iota_A(e_i)\otimes\iota_{A^o}(\delta^i),$$

where $\{e_i\}_{i\in I}$ is a basis of A and $\{\delta^i\}_{i\in I}$ the "dual basis" of A^o. However, when A is infinite dimensional the index set I is infinite, and we have no reasonable sum or dual basis above. The problem of infinite sums can be avoided defining R in terms of representative forms of $\mathcal{D}(A,A^o)$.

Since A is an embedded Hopf subalgebra of $\mathcal{D}(A,A^o)$ we can consider restrictions of elements $\varphi\in\mathcal{D}(A,A^o)^o\subset\mathcal{D}(A,A^o)^*$ on A, defined by

$$\varphi|_A(a):=\varphi(\iota_a(a))$$

for every $a\in A$. On the other hand, A^o is an embedded Hopf subalgebra of $\mathcal{D}(A,A^o)$, whence the element $\varphi|_A\in A^o\subset A^*$ can be taken to be an element of $\mathcal{D}(A,A^o)$, defining

$$\varphi' := \iota_{A^o}(\varphi|_A)\in\mathcal{D}(A,A^o).$$

Recall that in the finite dimensional case, if $\{e_i\}_{i\in I}$ is a basis of A and $\{\delta^i\}_{i\in I}$ the dual basis of $A^o = A^*$ then for any $\psi \in A^o$ the formula

$$\sum_{i\in I} \psi(e_i)\delta^i = \psi$$

holds. If we are able to define a concept of a "dual basis" for $A^o \subset A^*$ in the infinite dimensional case, we would expect a similar formula to hold. In particular,

$$\varphi|_A = \sum_{i\in I} \varphi|_A(e_i)\delta^i,$$

or in $\mathcal{D}(A, A^o)$,

$$\varphi' = \iota_{A^o}(\varphi|_A) = \sum_{i\in I} \varphi|_A(e_i)\iota_{A^o}(\delta^i).$$

How would then the universal R-matrix act on representations of $\mathcal{D}(A, A^o)$? Let V be a $\mathcal{D}(A, A^o)$-module with basis $\{v_i\}_{i=0}^{d-1}$ and representative forms

$$\lambda_{i,j} : \mathcal{D}(A, A^o) \to \mathbb{C}, \;\; \lambda_{i,j} \in \mathcal{D}(A, A^o)^o \subset \mathcal{D}(A, A^o)^*,$$

such that for every $x \in \mathcal{D}(A, A^o)$

$$x.v_j = \sum_{i=0}^{d-1} \lambda_{i,j}(x)v_i.$$

If $\mathcal{D}(A, A^o)$ has a universal R-matrix of the form

$$R = \sum_{m\in I} \iota_A(e_m) \otimes \iota_{A^o}(\delta^m)$$

the action of R on $V \otimes V$ would be of the form

$$\begin{aligned}
R(v_i \otimes v_j) &= \sum_{m\in I} \left(\iota_A(e_m) \otimes \iota_{A^o}(\delta^m)\right)(v_i \otimes v_j) \\
&= \sum_{m\in I} \sum_{k,l=0}^{d-1} \left(\lambda_{l,i}(\iota_A(e_m))\lambda_{k,j}(\iota_{A^o}(\delta^m))\right)(v_l \otimes v_k) \\
&= \sum_{k,l=0}^{d-1} \lambda_{k,j}\left(\sum_{m\in I} \lambda_{l,i}|_A(e_m)\iota_{A^o}(\delta^m)\right)(v_l \otimes v_k) \\
&= \sum_{k,l=0}^{d-1} \lambda_{k,j}(\lambda'_{l,i})(v_l \otimes v_k).
\end{aligned}$$

Notice that the last expression contains only representative forms of $\mathcal{D}(A, A^o)$ and does not depend on the dimension of A. Moreover, the sum is finite and can thus be computed. In view of theorem 6.5 consider the operator $\check{R} = \tau_{V,V} \circ R : V \otimes V \to V \otimes V$,

$$\check{R}(v_i \otimes v_j) = \sum_{k,l=0}^{d-1} r_{i,j}^{k,l}(v_k \otimes v_l),$$

where

$$r_{i,j}^{k,l} = \lambda_{k,j}(\lambda'_{l,i}).$$

The following theorem is for most parts proved in [Kyt11].

Theorem 6.23. *Let $(A, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra with invertible antipode, and $B \subset A^o$ a Hopf subalgebra. Let V be a $\mathcal{D}(A, B)$-module with basis $\{v_i\}_{i=0}^{d-1}$ such that the representative forms $\lambda_{i,j} \in \mathcal{D}(A, B)^o$ satisfy*

$$\lambda_{i,j}|_A \in B.$$

Then the linear map $\check{R} : V \otimes V \to V \otimes V$,

$$\check{R}(v_i \otimes v_j) = \sum_{k,l=0}^{d-1} r_{i,j}^{k,l}(v_k \otimes v_l) = \sum_{k,l=0}^{d-1} \lambda_{k,j}(\lambda'_{l,i})(v_k \otimes v_l),$$

is an R-matrix, and the braid group representation

$$\rho_n^{\check{R}} : B_n \to Aut(V^{\otimes n}), \quad \rho_n^{\check{R}}(\sigma_i) = R_{i,i+1},$$

$i = 1, \ldots, n-1$, $n > 0$, *commutes with the action of $\mathcal{D}(A, B)$ on $V^{\otimes n}$.*

Remark. Recall that $\mathcal{D}(A, B)$ acts on $V^{\otimes n}$ by the $(n-1)$-fold coproduct

$$\Delta^{(n)} := (\Delta \otimes id_{\mathcal{D}(A,B)} \otimes id_{\mathcal{D}(A,B)} \otimes \cdots \otimes id_{\mathcal{D}(A,B)}) \circ \cdots \circ (\Delta \otimes id_{\mathcal{D}(A,B)}) \circ \Delta,$$

with the representation

$$\rho_{V^{\otimes n}} := (\rho_V \otimes \cdots \otimes \rho_V) \circ \Delta^{(n)} : \mathcal{D}(A, B) \to End(V^{\otimes n}).$$

6.7 An R-matrix for $U_q(\mathfrak{sl}_2)[\sqrt{K}]$

We shall construct an R-matrix for the algebra $\mathcal{D}_{q^2}$ using representative forms of finite dimensional irreducible representations of $\mathcal{D}_{q^2}$ obtained from theorem 6.21. We denote the unit element of $H_{q^2}^o$ by $\widetilde{1} := g_1$, defined by

$$\widetilde{1}(b^m a^n) = g_1(b^m a^n) = \delta_{m,0}.$$

Fix an irreducible representation $W_d^{\lambda^{\frac{1}{4}}}$ of $\mathcal{D}_{q^2}$, with basis $\{w_i\}_{i=0}^{d-1}$ of the form of theorem 6.21. We want to find the numbers $\lambda \in \mathbb{C}$, $d \in \mathbb{N}$, so that the representative forms $\lambda_{i,j} \in (\mathcal{D}_{q^2})^o$ associated to $W_d^{\lambda^{\frac{1}{4}}}$ satisfy

$$\lambda_{i,j}|_{H_{q^2}} \in H''_{q^2}.$$

Then by theorem 6.23 we obtain an R-matrix

$$\check{R} : W_d^{\lambda^{\frac{1}{4}}} \otimes W_d^{\lambda^{\frac{1}{4}}} \to W_d^{\lambda^{\frac{1}{4}}} \otimes W_d^{\lambda^{\frac{1}{4}}}$$

yielding a braid group representation on $(W_d^{\lambda^{\frac{1}{4}}})^{\otimes n}$, $n > 0$, which moreover commutes with the action of $\mathcal{D}_{q^2}$.

Recall first that the representative forms $\lambda_{i,j} \in (\mathcal{D}_{q^2})^o$ are defined by the relation

$$x.w_j = \sum_{i=0}^{d-1} \lambda_{i,j}(x) w_i$$

for every $x \in \mathcal{D}_{q^2}$. Consider the action of the restricted map $\lambda_{i,j}|_{H_{q^2}}$ on the basis vector $b^m a^n \in H_{q^2}$,

$$\lambda_{i,j}|_{H_{q^2}}(b^m a^n) = \lambda_{i,j}(b^m a^n \otimes \widetilde{1}) = \lambda_{i,j}(\beta^m \alpha^n).$$

By definition,

$$\sum_{i=0}^{d-1} \lambda_{i,j}(b^m a^n \otimes \widetilde{1}) w_i = (b^m a^n \otimes \widetilde{1}).w_j = \beta^m \alpha^n.w_j = (\sqrt{\lambda})^n q^{n(1-d+2j)} \beta^m.w_j$$

$$= \begin{cases} (\sqrt{\lambda})^n q^{n(1-d+2j)} w_{j+m} & \text{for } 0 \leq j+m < d \\ 0 \ \text{ otherwise.} \end{cases}$$

Hence

$$\lambda_{i,j}|_{H_{q^2}}(b^m a^n) = \lambda_{i,j}(b^m a^n \otimes \widetilde{1}) = \begin{cases} (\sqrt{\lambda})^n q^{n(1-d+2j)} \delta_{i-j,m} & \text{for } 0 \leq j+m < d \\ 0 \ \text{ otherwise.} \end{cases}$$

Using the formulas in section 6.3 we compute the action of the basis vector $\widetilde{b}^r \sqrt{\widetilde{a}}^s \in H''_{q^2}$ on the basis vector $b^m a^n \in H_{q^2}$,

$$\widetilde{b}^r \sqrt{\widetilde{a}}^s (b^m a^n) = q^{\frac{1}{2}r(r-1)} [r]_q! h_{q^s}^{(r)} (b^m a^n) = q^{\frac{1}{2}r(r-1)} [r]_q! \delta_{m,r} q^{ns}.$$

Now if $0 \leq i, j < d-1$ then

$$\lambda_{i,j}|_{H_{q^2}} \in H''_{q^2}$$

if and only if there exist constants $C_{rs} \in \mathbb{C}$ such that

$$\begin{aligned} \lambda_{i,j}|_{H_{q^2}}(b^m a^n) &= (\sqrt{\lambda})^n q^{n(1-d+2j)} \delta_{i-j,m} \\ &= \sum_{r \geq 0, s \in \mathbb{Z}} C_{rs} \widetilde{b}^r \sqrt{\widetilde{a}}^s (b^m a^n) = \sum_{r \geq 0, s \in \mathbb{Z}} C_{rs} q^{\frac{1}{2}r(r-1)} [r]_q! \delta_{m,r} q^{ns}. \end{aligned}$$

for all values of $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{Z}$.

Suppose $\lambda_{i,j}|_{H_{q^2}} \in H''_{q^2}$. Since the coefficients C_{rs} do not depend on n, we have $\lambda = 1$. From the above formula we also see that for $r = i - j$

$$q^{n(1-d+2j)} = q^{ns},$$

whence $s = 1 - d + 2j$ and

$$C_{rs} = \frac{1}{q^{\frac{1}{2}r(r-1)}[r]_q!} = \frac{1}{q^{\frac{1}{2}(i-j)(i-j-1)}[i-j]_q!}.$$

We have proved the following.

Lemma 6.24. *For any integer $d > 0$ the representative forms $\lambda_{i,j} \in (\mathcal{D}_{q^2})^o$ associated to the irreducible representation W^1_d satisfy*

$$\lambda_{i,j}|_{H_{q^2}} = \frac{\widetilde{b}^{i-j}\sqrt{\widetilde{a}}^{1-d+2j}}{q^{\frac{1}{2}(i-j)(i-j-1)}[i-j]_q!}\mathbb{I}_{j\leq i} \in H''_{q^2},$$

where

$$\mathbb{I}_{j\leq i} = \begin{cases} 1, & \text{for } 0 \leq i-j < d \\ 0 & \text{otherwise.}\end{cases}$$

From theorem 6.23 it follows that the formula

$$\check{R}(v_i \otimes v_j) = \sum_{k,l=0}^{d-1} r^{k,l}_{i,j}(v_k \otimes v_l) = \sum_{k,l=0}^{d-1} \lambda_{k,j}(\lambda'_{l,i})(v_k \otimes v_l)$$

defines an R-matrix $\check{R} : W^1_d \otimes W^1_d \to W^1_d \otimes W^1_d$ yielding for every $n > 0$ a braid group representation on $(W^1_d)^{\otimes n}$ which commutes with the action of $\mathcal{D}_{q^2}$. Moreover, if we identify the element $\alpha\widetilde{\alpha}$ with the identity $1 \otimes \widetilde{1}$ in $\mathcal{D}_{q^2}$ we obtain such a R-matrix for the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, acting on the $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-modules denoted also by W^1_d.

Let us compute the matrix elements of $\check{R}$. Since for $0 \leq j - m < d$

$$\begin{aligned}
&\sum_{k=0}^{d-1} \lambda_{k,j}(1 \otimes \widetilde{b}^m\sqrt{\widetilde{a}}^n)w_k = (1 \otimes \widetilde{b}^m\sqrt{\widetilde{a}}^n).w_j = \widetilde{\beta}^m\sqrt{\widetilde{\alpha}}^n.w_j \\
=&[j]_q \cdots [j-m+1]_q[d-j]_q \cdots [d-j+m-1]_q(q^{-1}-q)^m q^{n(\frac{d-1}{2}-j)}w_{j-m} \\
=&\frac{[j]_q![d-j+m-1]_q!}{[j-m]_q![d-j-1]_q!}(q^{-1}-q)^m q^{n(\frac{d-1}{2}-j)}w_{j-m}
\end{aligned}$$

and

$$\sum_{k=0}^{d-1} \lambda_{k,j}(1 \otimes \widetilde{b}^m\sqrt{\widetilde{a}}^n)w_k = (1 \otimes \widetilde{b}^m\sqrt{\widetilde{a}}^n).w_j = \widetilde{\beta}^m\sqrt{\widetilde{\alpha}}^n.w_j = 0$$

otherwise, we have by definition

$$\lambda_{k,j}(1\otimes \widetilde{b}^m\sqrt{\widetilde{a}}^n) = \mathbb{I}_{m\leq j}\frac{[j]_q![d-j+m-1]_q!}{[j-m]_q![d-j-1]_q!}(q^{-1}-q)^m q^{n(\frac{d-1}{2}-j)}\delta_{j-m,k}$$
$$= \frac{[j]_q![d-k-1]_q!}{[k]_q![d-j-1]_q!}(q^{-1}-q)^m q^{n(\frac{d-1}{2}-j)}\delta_{j-m,k},$$

and the action of the representative forms of W_d^1 on H''_{q^2} is

$$\lambda_{k,j}(\lambda'_{l,i}) = \lambda_{k,j}(\iota_{H''_{q^2}}(\lambda_{l,i}|_{H_{q^2}})) = \lambda_{k,j}\left(\frac{\widetilde{\beta}^{l-i}\sqrt{\widetilde{\alpha}}^{1-d+2i}}{q^{\frac{1}{2}(l-i)(l-i-1)}[l-i]_q!}\mathbb{I}_{i\leq l}\right)$$
$$= \frac{\mathbb{I}_{i\leq l}[j]_q![d-k-1]_q!}{[k]_q![d-j-1]_q!q^{\frac{1}{2}(l-i)(l-i-1)}[l-i]_q!}(q^{-1}-q)^{l-i}q^{(1-d+2i)(\frac{d-1}{2}-j)}\delta_{i+j,k+l}.$$

By theorem 6.23 the above expression defines an R-matrix

$$\check{R}(w_i\otimes w_j) = \sum_{k,l=0}^{d-1} r_{i,j}^{k,l}(w_k\otimes w_l) \text{ with}$$
$$r_{i,j}^{k,l} = \frac{\mathbb{I}_{i\leq l}\delta_{i+j,k+l}[j]_q![d-k-1]_q!}{[k]_q![d-j-1]_q![l-i]_q!}(q^{-1}-q)^{l-i}q^{2(i-\frac{d-1}{2})(\frac{d-1}{2}-j)-\frac{1}{2}(l-i)(l-i-1)}$$

for $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. The expression can be slightly simplified. Write $m := l-i$. Then we have the relations

$$i\leq l \iff m = l-i\geq 0$$
$$i,l\in\{1,\dots,d-1\} \Rightarrow m = l-i\leq d-1$$
$$i+j = k+l \text{ and } k\geq 0 \Rightarrow j-m = k\geq 0 \Rightarrow m\leq j.$$

It follows that the R-matrix can be written as

$$\check{R}(w_i\otimes w_j) = \sum_{m=0}^{j}\check{r}_{i,j}^m(w_{j-m}\otimes w_{i+m}) \text{ with}$$
$$\check{r}_{i,j}^m = \frac{[j]_q![d+m-j-1]_q!}{[j-m]_q![d-j-1]_q![m]_q!}(q^{-1}-q)^m q^{2(i-\frac{d-1}{2})(\frac{d-1}{2}-j)-\frac{1}{2}m(m-1)}.$$

To summarise, we state the second main result of this thesis, proved above.

Theorem 6.25. *The formula*

$$\check{R}(w_i\otimes w_j) = \sum_{m=0}^{j}\check{r}_{i,j}^m(w_{j-m}\otimes w_{i+m}) \text{ with}$$
$$\check{r}_{i,j}^m = \frac{[j]_q![d+m-j-1]_q!}{[j-m]_q![d-j-1]_q![m]_q!}(q^{-1}-q)^m q^{2(i-\frac{d-1}{2})(\frac{d-1}{2}-j)-\frac{1}{2}m(m-1)}$$

defines an R-matrix $\check{R} : W_d^1 \otimes W_d^1 \to W_d^1 \otimes W_d^1$ *acting on tensor products of irreducible* $U_q(\mathfrak{sl}_2)[\sqrt{K}]$*-modules, and the braid group representation*

$$\rho_n^{\check{R}} : B_n \to Aut((W_d^1)^{\otimes n}), \quad \rho_n^{\check{R}}(\sigma_i) = \check{R}_{i,i+1},$$

$i = 1, \dots, n-1$, $n > 0$, *commutes with the action of* $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ *on* $(W_d^1)^{\otimes n}$.

6.8 The equivalence of monodromy of KZ($\mathfrak{sl}_2$) and braiding of quantum $\mathfrak{sl}_2$

As already mentioned, the braid group representation defined by the R-matrix of theorem 6.25 is equivalent to the braid group representation arising from the monodromy of KZ($\mathfrak{sl}_2$). In this section we prove this fact. Notice that the proof gives an explicit equivalence between these two representations, although we have not shown in detail that all solutions of KZ($\mathfrak{sl}_2$) can be written using the non-intersecting families of loops for which the monodromy action was computed in section 5.6. Recall from theorem 5.21 that the monodromy operator corresponding to the action of the inverse of the braid group generator $\sigma_i \in B_N$ is defined by the formula

$$M_{\gamma_{z_i,z_{i+1}}} \Psi_l^{(\mathcal{P}_{j_1,\dots,j_N})} = \sum_{m=0}^{j_{i+1}} \widetilde{r}_{j_i,j_{i+1}}^m \Psi_l^{(\mathcal{P}_{j_1,\dots,j_{i-1},j_{i+1}-m,j_i+m,j_{i+2}\dots j_N})}$$

with

$$\widetilde{r}_{j_i,j_{i+1}}^m = \frac{[j_{i+1}]_q![\mu - j_{i+1} + m]_q!}{[m]_q![j_{i+1} - m]_q![\mu - j_{i+1}]_q!}(q - q^{-1})^m \cdot q^{2(\frac{\mu}{2} - (j_i+m))(\frac{\mu}{2} - (j_{i+1}-m)) + \frac{1}{2}m(m-1)},$$

where $\mathcal{P}_{j_1,\dots,j_N}$, with $j_1, \dots, j_N \geq 0$, $\sum_{i=1}^N j_i = l$, is the family of non-intersecting loops with a fixed base point, as in definition 5.16, consisting of j_k loops around the point z_k, $k = 1, \dots, N$, and $\Psi_l^{(\mathcal{P}_{j_1,\dots,j_N})} : Y_N \to V_d^{\otimes N}$ is a component of the solution $\Psi_l^{(\mathcal{P})} : Y_N \to V_d^{\otimes N}$,

$$\Psi_l^{(\mathcal{P})}(z) = \psi_0(z) \int_{\mathcal{P}} \eta_{z,l} u_0 = \sum_{j_1,\dots,j_N \geq 0,\ \sum_i j_i = l} c_{j_1,\dots,j_N} \Psi_l^{(\mathcal{P}_{j_1,\dots,j_N})}(z),$$

of KZ($\mathfrak{sl}_2$) of level l, whose integration surface $\mathcal{P}$ is admissible in the sense of proposition 5.15 and can be written as a linear combination of families of loops of the form $\mathcal{P}_{j_1,\dots,j_N}$. The constant $\mu = d - 1 \in \mathbb{C}$ is the highest weight of the irreducible $\mathfrak{sl}_2$-module V_d.

It turns out that we have a natural one-to-one correspondence between the families $\mathcal{P}_{j_1,\dots,j_N}$ and the vectors $w_{j_N} \otimes \dots \otimes w_{j_1} \in (W_d^1)^{\otimes N}$. This is actually an isomorphism of $U_q(\mathfrak{sl}_2)$-modules, as proved in [FW90], where an action of the quantum group on the vector space of the families $\mathcal{P}_{j_1,\ldots,j_N}$ is defined. We will now identify the elements

$$\mathcal{P}_{j_1,\ldots,j_N} \leftrightarrow w_{j_N} \otimes \cdots \otimes w_{j_1},$$

or more precisely, the elements

$$\Psi_l^{(\mathcal{P}_{j_1,\ldots,j_N})} \leftrightarrow w_{j_N} \otimes \cdots \otimes w_{j_1}.$$

Then the monodromy action of σ_i^{-1} reads on $w_{j_1} \otimes \cdots \otimes w_{j_N}$

$$\begin{aligned}
&M_{\gamma_{z_i,z_{i+1}}}(w_{j_1} \otimes \cdots \otimes w_{j_N}) = M_{\gamma_{z_i,z_{i+1}}} \Psi_l^{(\mathcal{P}_{j_1,\ldots,j_{i-1},j_{i+1},j_i,j_{i+2}\ldots,j_N})} \\
&= \sum_{m=0}^{j_i} \widetilde{r}^m_{j_{i+1},j_i} \Psi_l^{(\mathcal{P}_{j_1,\ldots,j_{i-1},j_i-m,j_{i+1}+m,j_{i+2}\ldots,j_N})} \\
&= \sum_{m=0}^{j_i} \widetilde{r}^m_{j_{i+1},j_i} (w_{j_1} \otimes \cdots \otimes w_{j_{i-1}} \otimes w_{j_{i+1}+m} \otimes w_{j_i-m} \otimes w_{j_{i+2}} \otimes \cdots \otimes w_{j_N}).
\end{aligned}$$

Next we show that the operator $(M_{\gamma_{z_i,z_{i+1}}})_{i,i+1} : W^1_d \otimes W^1_d \to W^1_d \otimes W^1_d$ is the inverse of the operator $\check{R} : W^1_d \otimes W^1_d \to W^1_d \otimes W^1_d$ defined by the formula of theorem 6.25

$$\begin{aligned}
\check{R}(w_{j_i} \otimes w_{j_{i+1}}) &= \sum_{m=0}^{j_{i+1}} \check{r}^m_{j_i,j_{i+1}} (w_{j_{i+1}-m} \otimes w_{j_i+m}) \quad \text{with} \\
\check{r}^m_{j_i,j_{i+1}} &= \frac{[j_{i+1}]_q! [d+m-j_{i+1}-1]_q!}{[j_{i+1}-m]_q! [d-j_{i+1}-1]_q! [m]_q!} (q^{-1}-q)^m \\
&\quad \cdot q^{2(j_i - \frac{d-1}{2})(\frac{d-1}{2} - j_{i+1}) - \frac{1}{2} m(m-1)}.
\end{aligned}$$

We compute

$$\begin{aligned}
(M_{\gamma_{z_i,z_{i+1}}})_{i,i+1}(\check{R}(w_{j_i} \otimes w_{j_{i+1}})) &= \sum_{m=0}^{j_{i+1}} \check{r}^m_{j_i,j_{i+1}} (M_{\gamma_{z_i,z_{i+1}}})_{i,i+1} (w_{j_{i+1}-m} \otimes w_{j_i+m}) \\
&= \sum_{m=0}^{j_{i+1}} \check{r}^m_{j_i,j_{i+1}} \sum_{n=0}^{j_{i+1}-m} \widetilde{r}^n_{j_i+m,j_{i+1}-m} (w_{j_i+m+n} \otimes w_{j_{i+1}-m-n}) \\
&= \sum_{k=0}^{j_{i+1}} \sum_{m=0}^{k} (-1)^m (q-q^{-1})^k q^{m(1-k)+2k(j_{i+1}-j_i-k)+\frac{1}{2}k(k-1)} \\
&\quad \cdot \frac{[j_{i+1}]_q! [d-1-j_{i+1}+k]_q!}{[j_{i+1}-k]_q! [m]_q! [k-m]_q! [d-1-j_{i+1}]_q!} (w_{j_i+k} \otimes w_{j_{i+1}-k}),
\end{aligned}$$

where we denoted $k = m + n$. Observe now that

$$\begin{aligned}
&\frac{[j_{i+1}]_q![d-1-j_{i+1}+k]_q!}{[j_{i+1}-k]_q![m]_q![k-m]_q![d-1-j_{i+1}]_q!} \\
&= \frac{[j_{i+1}]_q!}{[j_{i+1}-k]_q![k]_q!}\frac{[k]_q!}{[m]_q![k-m]_q!}\frac{[d-1-j_{i+1}+k]_q!}{[d-1-j_{i+1}]_q!} \\
&= \begin{bmatrix} j_{i+1} \\ k \end{bmatrix}_q \begin{bmatrix} k \\ m \end{bmatrix}_q \prod_{u=1}^{k} (q^{d-1-j_{i+1}+u} - q^{-d+1+j_{i+1}-u})(q-q^{-1})^{-k} \\
&= \begin{bmatrix} j_{i+1} \\ k \end{bmatrix}_q \begin{bmatrix} k \\ m \end{bmatrix}_q \prod_{u=0}^{k-1} (q^{d-j_{i+1}+u} - q^{-d+j_{i+1}-u})(q-q^{-1})^{-k}
\end{aligned}$$

and that by lemma 5.19 with $\beta = 1$ we can write

$$\sum_{m=0}^{k} (-1)^m \begin{bmatrix} k \\ m \end{bmatrix}_q q^{m(1-k)} = q^{\frac{1}{2}k(1-k)} \prod_{s=0}^{k-1} (q^s - q^{-s}) = \delta_{k,0}.$$

It follows that

$$\begin{aligned}
&(M_{\gamma_{z_i,z_{i+1}}})_{i,i+1}(\check{R}(w_{j_i} \otimes w_{j_{i+1}})) \\
=&\sum_{k=0}^{j_{i+1}} \sum_{m=0}^{k} (-1)^m (q-q^{-1})^k q^{m(1-k)+2k(j_{i+1}-j_i-k)+\frac{1}{2}k(k-1)} \\
&\cdot \frac{[j_{i+1}]_q![d-1-j_{i+1}+k]_q!}{[j_{i+1}-k]_q![m]_q![k-m]_q![d-1-j_{i+1}]_q!}(w_{j_i+k} \otimes w_{j_{i+1}-k}) \\
=&\sum_{k=0}^{j_{i+1}} \begin{bmatrix} j_{i+1} \\ k \end{bmatrix}_q q^{2k(j_{i+1}-j_i-k)} \prod_{s=0}^{k-1} (q^s - q^{-s}) \\
&\cdot \prod_{u=0}^{k-1} (q^{d-j_{i+1}+u} - q^{-d+j_{i+1}-u})(w_{j_i+k} \otimes w_{j_{i+1}-k}) = w_{j_i} \otimes w_{j_{i+1}}.
\end{aligned}$$

Since by theorem 6.25 the operator $\check{R}$ is an R-matrix, it is invertible. By finding the left inverse of $\check{R}$ we have shown the following.

Proposition 6.26. *The operator* $(M_{\gamma_{z_i,z_{i+1}}})_{i,i+1} : W_d^1 \otimes W_d^1 \to W_d^1 \otimes W_d^1$ *defined by*

$$\begin{aligned}
(M_{\gamma_{z_i,z_{i+1}}})_{i,i+1}(w_{j_i} \otimes w_{j_{i+1}}) &= \sum_{m=0}^{j_i} \widetilde{r}^m_{j_{i+1},j_i} (w_{j_{i+1}+m} \otimes w_{j_i-m}) \quad \textit{with} \\
\widetilde{r}^m_{j_{i+1},j_i} &= \frac{[j_i]_q![d-1-j_i+m]_q!}{[m]_q![j_i-m]_q![d-1-j_i]_q!}(q-q^{-1})^m \\
&\quad \cdot q^{2(\frac{d-1}{2}-(j_{i+1}+m))(\frac{d-1}{2}-(j_i-m))+\frac{1}{2}m(m-1)},
\end{aligned}$$

is the inverse of the operator $\check{R} : W_d^1 \otimes W_d^1 \to W_d^1 \otimes W_d^1$ *defined by*

$$\begin{aligned} \check{R}(w_{j_i} \otimes w_{j_{i+1}}) &= \sum_{m=0}^{j_{i+1}} \check{r}^m_{j_i,j_{i+1}} (w_{j_{i+1}-m} \otimes w_{j_i+m}) \quad \textit{with} \\ \check{r}^m_{j_i,j_{i+1}} &= \frac{[j_{i+1}]_q![d+m-j_{i+1}-1]_q!}{[j_{i+1}-m]_q![d-j_{i+1}-1]_q![m]_q!}(q^{-1}-q)^m \\ &\quad \cdot q^{2(j_i-\frac{d-1}{2})(\frac{d-1}{2}-j_{i+1})-\frac{1}{2}m(m-1)}. \end{aligned}$$

In particular, the representation

$$\rho_N^{KZ} : B_N \to Aut(V_d^{\otimes N}), \quad \sigma_i \mapsto M^{-1}_{\gamma_{z_i,z_{i+1}}}$$

of the braid group B_N *defined by the monodromy of KZ(*$\mathfrak{sl}_2$*) under the identification*

$$\Psi_l^{(\mathcal{P}_{j_1,\ldots,j_N})} \leftrightarrow w_{j_N} \otimes \cdots \otimes w_{j_1}$$

coincides with the representation

$$\rho_N^{\check{R}} : B_N \to Aut((W_d^1)^{\otimes N}), \quad \sigma_i \mapsto \check{R}_{i,i+1},$$

arising from the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$*.*

Next we compute the action of $\check{R}$ on some tensor products of irreducible representations of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ and compare it with the monodromy representation of the braid group associated to the corresponding KZ-equation. We will see concretely that these two representations are equivalent.

6.9 The braid group representation on $W_2^1 \otimes W_2^1$

By example 4.18

$$W_2^1 \otimes W_2^1 \cong W_1^1 \oplus W_3^1$$

with the highest weight vectors

$$\begin{aligned} w^{(1)} &= w_0 \otimes w_1 - q^{-1} w_1 \otimes w_0 \in W_1^1, \quad \text{and} \\ w^{(0)} &= w_0 \otimes w_0 \in W_3^1. \end{aligned}$$

By theorem 6.23 the action of $\check{R}$ on $W_2^1 \otimes W_2^1$ commutes with the action of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, and Schur's lemma 4.10 implies that $\check{R}$ acts as a scalar on every component of the direct sum decomposition of $W_2^1 \otimes W_2^1$ separately. Indeed, using the formula

$$\begin{aligned} \check{R}(w_i \otimes w_j) &= \sum_{m=0}^{j} \check{r}^m_{i,j} (w_{j-m} \otimes w_{i+m}) \quad \text{with} \\ \check{r}^m_{i,j} &= \frac{[j]_q![d+m-j-1]_q!}{[j-m]_q![d-j-1]_q![m]_q!}(q^{-1}-q)^m q^{2(i-\frac{d-1}{2})(\frac{d-1}{2}-j)-\frac{1}{2}m(m-1)} \end{aligned}$$

of theorem 6.25 in the dimension $d = 2$, we compute

$$\check{R}(w^{(0)}) = q^{2(-\frac{d-1}{2})(\frac{d-1}{2})}(w_0 \otimes w_0) = q^{-\frac{1}{2}} w^{(0)} \text{ and}$$

$$\begin{aligned}\check{R}(w^{(1)}) &= \sum_{m=0}^{1} \frac{[d+m-2]_q!}{[d-2]_q!}(q^{-1}-q)^m q^{2(-\frac{d-1}{2})(\frac{d-1}{2}-1)-\frac{1}{2}m(m-1)}(w_{1-m} \otimes w_m) \\ &\quad - q^{-1} q^{2(1-\frac{d-1}{2})(\frac{d-1}{2})}(w_0 \otimes w_1) \\ &= q^{\frac{1}{2}}(w_1 \otimes w_0) + (q^{-1}-q)q^{\frac{1}{2}}(w_0 \otimes w_1) - q^{-\frac{1}{2}}(w_0 \otimes w_1) \\ &= q^{\frac{1}{2}}(w_1 \otimes w_0) - q^{\frac{3}{2}}(w_0 \otimes w_1) = -q^{\frac{3}{2}} w^{(1)},\end{aligned}$$

that is

$$\check{R}|_{W_1^1} = -q^{\frac{3}{2}} id_{W_1^1}, \quad \check{R}|_{W_3^1} = q^{-\frac{1}{2}} id_{W_3^1}.$$

Recall from section 5.3.2 the braid group representation associated to the solutions of KZ($\mathfrak{sl}_2$) taking values in the $\mathfrak{sl}_2$-module

$$V_2 \otimes V_2 \cong V_1 \oplus V_3,$$

namely $\rho_2^{KZ} : \sigma_1 \mapsto M_{\gamma_1}^{-1}$, where

$$M_{\gamma_1}^{-1}|_{V_1} = -q^{\frac{3}{2}} id_{V_1}, \quad M_{\gamma_1}^{-1}|_{V_3} = q^{-\frac{1}{2}} id_{V_3}.$$

Notice that there is indeed a one-to-one correspondence between the actions of $\check{R}$ and $M_{\gamma_1}^{-1}$ on the direct sum components W_j^1 and V_j, $j \in \{1, 3\}$, of the corresponding $U_q(\mathfrak{sl}_2)[\sqrt{K}]$- and $\mathfrak{sl}_2$-modules, respectively. Notice also that the value of the parameter $\kappa \in \mathbb{C} \setminus \mathbb{Q}$ determines the value of the deformation parameter $q = e^{\frac{\pi i}{\kappa}} \in \mathbb{C} \setminus \{0\}$, and from the fact that $\kappa \notin \mathbb{Q}$ it follows that q is not a root on unity.

6.10 The braid group representation on $W_2^1 \otimes W_2^1 \otimes W_2^1$

Studying the braid group representations of braids with more than two strands induced by $\check{R}$, the action of $\check{R}$ on the tensor products of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-modules has to be considered via the operators $\check{R}_{ij}$ acting on the i:th and j:th tensor component. In such tensor products the operators $\check{R}_{ij}$ are not necessarily diagonal in the natural basis of the direct sum decomposition, generated by the highest weight vectors. They are, however, diagonalizable, but not simultaneously since they do not commute.

Using proposition 4.17 (quantum Clebsch-Gordan) repeatedly we obtain a direct sum decomposition of irreducible $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-modules for the representation $W_d^1 \otimes W_d^1 \otimes W_d^1$. However, it may contain more than one irreducibles with the same dimension. By Schur's lemma 4.10 the operator $\check{R}_{ij}$ shuffles the components of the same dimension of the direct sum. When $N = 3$, we have to consider the two operators $\check{R}_{12}$ and $\check{R}_{23}$ corresponding to the two braid group generators $\sigma_1, \sigma_2 \in B_3$.

For simplicity, we consider only the case $d = 2$, from which the structure of the braid group representations

$$\rho_3^{\check{R}} : B_3 \to Aut((W_d^1)^{\otimes 3}), \quad \rho_n^{\check{R}}(\sigma_i) = \check{R}_{i,i+1},$$

$i = 1, 2$, $d > 1$, can be seen. Recall that the $\mathfrak{sl}_2$-modules

$$V_2 \otimes V_2 \otimes V_2 \cong V_4 \oplus V_2 \oplus V_2,$$

where the highest weight vectors are

$$u_0 = v_0 \otimes v_0 \otimes v_0$$

of weight 3, and two linearly independent highest weight vectors of weight 1, which by lemma 5.11 are of the form

$$u_1 = a_1 v_1 \otimes v_0 \otimes v_0 + a_2 v_0 \otimes v_1 \otimes v_0 + a_3 v_0 \otimes v_0 \otimes v_1 = \sum_{k=1}^{3} f_k a_k u_0,$$

where the coefficients satisfy $\sum_{k=1}^{3} a_k = 0$. We want first to find the corresponding highest weight vectors in the $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-module

$$W_2^1 \otimes W_2^1 \otimes W_2^1 \cong W_4^1 \oplus W_2^1 \oplus W_2^1.$$

Clearly the vector of weight q^3 is $u_0' := w_0 \otimes w_0 \otimes w_0$, and similarly as in the proof of lemma 5.11 there are two linearly independent vectors of weight q, which are of the form

$$u_1' := q^{-2} a_1 w_1 \otimes w_0 \otimes w_0 + q^{-1} a_2 w_0 \otimes w_1 \otimes w_0 + a_3 w_0 \otimes w_0 \otimes w_1,$$

where $\sum_{k=1}^{3} a_k = 0$. This can be checked by direct computation from the formulas that highest weight vectors must satisfy, namely

$$K.u_1' = q u_1' \text{ and } E.u_1' = 0.$$

We compute the action of the R-matrix $\check{R}$ on the highest weight vectors using the formula

$$\check{R}(w_i \otimes w_j) = \sum_{m=0}^{j} \check{r}_{i,j}^m (w_{j-m} \otimes w_{i+m}) \text{ with}$$
$$\check{r}_{i,j}^m = \frac{[j]_q! [d+m-j-1]_q!}{[j-m]_q! [d-j-1]_q! [m]_q!} (q^{-1} - q)^m q^{2(i - \frac{d-1}{2})(\frac{d-1}{2} - j) - \frac{1}{2} m(m-1)}$$

of theorem 6.25 in the dimension $d = 2$. First,

$$\begin{aligned}
\check{R}(w_0 \otimes w_0) &= q^{2(-\frac{d-1}{2})(\frac{d-1}{2})}(w_0 \otimes w_0) = q^{-\frac{1}{2}}(w_0 \otimes w_0),\\
\check{R}(w_0 \otimes w_1) &= \sum_{m=0}^{1} \frac{[d+m-2]_q!}{[d-2]_q!}(q^{-1}-q)^m \\
&\quad \cdot q^{2(-\frac{d-1}{2})(\frac{d-1}{2}-1)-\frac{1}{2}m(m-1)}(w_{1-m} \otimes w_m)\\
&= q^{\frac{1}{2}}(w_1 \otimes w_0) + (q^{-1}-q)q^{\frac{1}{2}}(w_0 \otimes w_1)\\
&= q^{\frac{1}{2}}((q^{-1}-q)w_0 \otimes w_1 + w_1 \otimes w_0),\\
\check{R}(w_1 \otimes w_0) &= q^{2(1-\frac{d-1}{2})(\frac{d-1}{2})}(w_0 \otimes w_1) = q^{\frac{1}{2}}(w_0 \otimes w_1).
\end{aligned}$$

Using these, we obtain for $i = 1, 2$

$$\begin{aligned}
&\check{R}_{i,i+1}u'_0 = \ q^{-\frac{1}{2}}u'_0,\\
&\check{R}_{12}u'_1 = \ q^{-\frac{1}{2}}u'_1 + q^{\frac{1}{2}}(\tau_{W_2^1,W_2^1})_{12}(u'_1)\\
&\quad -q^{\frac{1}{2}}(q^{-3}a_1 w_1 \otimes w_0 \otimes w_0 + a_2 w_0 \otimes w_1 \otimes w_0 + a_3 w_0 \otimes w_0 \otimes w_1),\\
&\check{R}_{23}u'_1 = \ q^{-\frac{1}{2}}u'_1 + q^{\frac{1}{2}}(\tau_{W_2^1,W_2^1})_{23}(u'_1)\\
&\quad -q^{\frac{1}{2}}(q^{-2}a_1 w_1 \otimes w_0 \otimes w_0 + q^{-2}a_2 w_0 \otimes w_1 \otimes w_0 + q a_3 w_0 \otimes w_0 \otimes w_1).
\end{aligned}$$

We will consider the operator $\check{R}_{12}$, which represents the braid group generator $\sigma_1 \in B_3$; the operator $\check{R}_{23}$ is similar.

Choose the coefficients $a_1, a_2, a_3 \in \mathbb{C}$ so that the $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-module $W_2^1 \otimes W_2^1 \otimes W_2^1 \cong W_4^1 \oplus W_2^1 \oplus W_2^1$ has the basis

$$\{u'_0,\ F.u'_0,\ F^2.u'_0,\ F^3.u'_0;\ u'_1,\ F.u'_1;\ \widehat{u}'_1,\ F.\widehat{u}'_1\},$$

that is a natural basis of the direct sum decomposition, where

$$\begin{aligned}
u'_1 &:= w_1 \otimes w_0 \otimes w_0 - q w_0 \otimes w_1 \otimes w_0, \ \text{ i.e. } \ a_1 = q^2,\ a_2 = -q^2,\ a_3 = 0,\\
\widehat{u}'_1 &:= w_0 \otimes w_1 \otimes w_0 - q w_0 \otimes w_0 \otimes w_1, \ \text{ i.e. } \ a_1 = 0\ , a_2 = q,\ a_3 = -1.
\end{aligned}$$

The action of the tensor flip in the first two tensor components is

$$\begin{aligned}
(\tau_{W_2^1,W_2^1})_{12}(u'_1) &= w_0 \otimes w_1 \otimes w_0 - q w_1 \otimes w_0 \otimes w_0,\\
(\tau_{W_2^1,W_2^1})_{12}(\widehat{u}'_1) &= w_1 \otimes w_0 \otimes w_0 - q w_0 \otimes w_0 \otimes w_1,
\end{aligned}$$

and we obtain

$$\begin{aligned}
\check{R}_{12}(u_1') &= q^{-\frac{1}{2}}u_1' + q^{\frac{1}{2}}(w_0\otimes w_1\otimes w_0 - qw_1\otimes w_0\otimes w_0)\\
&\quad - q^{\frac{1}{2}}(q^{-1}w_1\otimes w_0\otimes w_0 - q^2 w_0\otimes w_1\otimes w_0)\\
&= (q^{-\frac{1}{2}} - q^{\frac{3}{2}} - q^{-\frac{1}{2}})w_1\otimes w_0\otimes w_0 + (-q^{\frac{1}{2}} + q^{\frac{1}{2}} + q^{\frac{5}{2}})w_0\otimes w_1\otimes w_0\\
&= \, - q^{\frac{3}{2}}(w_1\otimes w_0\otimes w_0 - qw_0\otimes w_1\otimes w_0) = -q^{\frac{3}{2}}u_1',\\
\check{R}_{12}(\widehat{u}_1') &= q^{-\frac{1}{2}}\widehat{u}_1' + q^{\frac{1}{2}}(w_1\otimes w_0\otimes w_0 - qw_0\otimes w_0\otimes w_1)\\
&\quad - q^{\frac{1}{2}}(qw_0\otimes w_1\otimes w_0 - w_0\otimes w_0\otimes w_1) = q^{\frac{1}{2}}w_1\otimes w_0\otimes w_0\\
&\quad + (q^{-\frac{1}{2}} - q^{\frac{3}{2}})w_0\otimes w_1\otimes w_0 + (-q^{-\frac{1}{2}} - q^{\frac{3}{2}} + q^{\frac{3}{2}})w_0\otimes w_0\otimes w_1\\
&= q^{\frac{1}{2}}u_1' + q^{-\frac{1}{2}}\widehat{u}_1'
\end{aligned}$$

The matrix of $\check{R}_{12}$ in the above basis of the direct sum decomposition of the $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-module $W_4^1\oplus W_2^1\oplus W_2^1$ is

$$\check{R}_{12} = \begin{pmatrix}
q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -q^{\frac{3}{2}} & 0 & q^{\frac{1}{2}} & 0\\
0 & 0 & 0 & 0 & 0 & -q^{-\frac{3}{2}} & 0 & q^{\frac{1}{2}}\\
0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}}
\end{pmatrix}$$

Recall the braid group representation associated to the solutions of KZ($\mathfrak{sl}_2$) taking values in the $\mathfrak{sl}_2$-module $V_2\otimes V_2\otimes V_2$, computed in section 5.6.2. With a suitable choice of basis we obtained exactly the same matrix representation of the braid group generator $\sigma_1 \in B_3$. The matrix representation of the other generator σ_2 can be computed similarly. Actually, with this choice of basis also the matrix representation of σ_2 will be the same as in section 5.6.2.

7 Conclusion

In this thesis we constructed solutions $\Psi_l^{(\mathcal{P})} : Y_N \to V_{d_1} \otimes \cdots \otimes V_{d_N}$,

$$\Psi_l^{(\mathcal{P})}(z) = \psi_0(z) \int_{\mathcal{P}} \eta_{z,l} u_0, \quad \text{with}$$
$$\psi_0(z) = \prod_{1 \le i < j \le N} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}} \quad \text{and}$$
$$\eta_{z,l} = \prod_{1 \le i < j \le l} (w_i - w_j)^{\frac{2}{\kappa}} \prod_{j,k} (w_j - z_k)^{-\frac{\mu_k}{\kappa}} \prod_{i=1}^{l} \sum_{k=1}^{N} \frac{f_k}{w_i - z_k} dw,$$

of the KZ-equations in the case of the (semi)simple Lie algebra $\mathfrak{sl}_2$ in integral form (proposition 5.15). Moreover, we saw that for $d_1 = \cdots = d_N = d$ the monodromy of KZ($\mathfrak{sl}_2$) indeed defines a linear representation of the whole braid group B_N, although the fundamental group of the domain

$$Y_N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N\} \setminus \bigcup_{i<j} \{z_i = z_j\}$$

where the equations are analytically defined is only a subgroup of B_N. This was achieved by incorporating an action of the symmetric group S_N on the trivial vector bundle $Y_N \times V_d^{\otimes N}$ over the manifold Y_N.

We also showed that the monodromy action of KZ($\mathfrak{sl}_2$) on any $\mathfrak{sl}_2$-module $V^{\otimes N}$ commutes with the action of the universal enveloping algebra $U(\mathfrak{sl}_2)$. This enabled us by Schur's lemma 4.10 and semisimplicity of $\mathfrak{sl}_2$ to concentrate on the monodromy of the solutions of KZ($\mathfrak{sl}_2$) (pointwise) proportional to highest weight vectors in a tensor product $V_d^{\otimes N}$ of irreducible $\mathfrak{sl}_2$-modules.

We also found an expression of an R-matrix (theorem 6.25)

$$\check{R}(w_i \otimes w_j) = \sum_{m=0}^{j} \check{r}_{i,j}^m (w_{j-m} \otimes w_{i+m}) \quad \text{with}$$
$$\check{r}_{i,j}^m = \frac{[j]_q! [d+m-j-1]_q!}{[j-m]_q! [d-j-1]_q! [m]_q!} (q^{-1} - q)^m q^{2(i - \frac{d-1}{2})(\frac{d-1}{2} - j) - \frac{1}{2} m(m-1)}$$

for the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ and saw that this operator defines for any $n > 0$ a linear representation of the braid group B_n on the tensor product $(W_d^1)^{\otimes n}$ of irreducible $U_q(\mathfrak{sl}_2)[\sqrt{K}]$-modules, commuting with the action of the extended quantum group $U_q(\mathfrak{sl}_2)[\sqrt{K}]$. By Schur's lemma 4.10 and semisimplicity of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$ we were again able to concentrate on the action on tensor products of irreducibles.

Especially, we showed concretely that the monodromy of KZ($\mathfrak{sl}_2$) (theorem 5.21) indeed coincides with the braid group representation induced by the R-matrix of $U_q(\mathfrak{sl}_2)[\sqrt{K}]$, via a suitable correspondence of solutions of

KZ($\mathfrak{sl}_2$) and vectors in $(W_d^1)^{\otimes n}$. We left some details unproven, mostly because the proofs would require more theory of homology and cohomology groups and local systems that would have been possible to cover in this thesis. We also left out some long and technical proofs since they would not have been very illustrating or useful for our purposes. For more details on this subject we refer to [EFK98] and [FW90].

7.1 The Drinfeld-Kohno theorem

To compare our results with the Drinfeld-Kohno theorem (DK), we will briefly present the statement of the DK-theorem and discuss differences between our concrete method and the topological method of the proof of DK. The Drinfeld-Kohno theorem establishes the link between the two braid group representations arising on the one hand from "universal" R-matrices associated to the quantum enveloping algebras $U_h(\mathfrak{g})$ of semisimple Lie algebras, and on the other hand from solutions of the KZ-equations associated to the same semisimple Lie algebra $\mathfrak{g}$. It is important to notice that in the formulation of the DK-theorem the deformation parameter h of the quantum group has a different meaning than the parameter $q \in \mathbb{C} \setminus \{0, \pm 1\}$ we have used in this thesis. Namely, h is considered as a variable of formal power series, whereas the parameter q is just a complex number. The DK-theorem can be formulated as follows.

Let $\mathbb{C}[[h]]$ be the algebra of formal power series in an indeterminate h with coefficients in $\mathbb{C}$. Define the *topological module* $V[[h]]$ as the set of all power series

$$\sum_{n \geq 0} v_n h^n,$$

$v_n \in V$, with the obvious structure as a $\mathbb{C}[[h]]$-module. The vector space $V[[h]]$ is equipped with the *h-adic topology*, that is, the open neighbourhoods of zero have a basis $\{h^n V[[h]]\}_{n \geq 0}$, and translations are continuous. The reader can find more information about topological modules in [Kas95].

The quantum enveloping algebra $U_h(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ can be defined as a topological algebra over $\mathbb{C}[[h]]$, and it has a "universal" R-matrix R_h in the sense of *braided quasi-bialgebras* (see [Kas95]) which generalise the notion of braided bialgebras, but the coproduct is not always coassociative. The R-matrix R_h induces a $\mathbb{C}[[h]]$-linear representation

$$\rho_n^{R_h} : B_n \to Aut_{\mathbb{C}[[h]]}(V^{\otimes n}[[h]])$$

as before, satisfying $\rho_n^{R_h}(\sigma_i) = (R_h)_{i,i+1}$, $i = 1, \dots, n-1$. On the other hand, the monodromy of solutions of the KZ-equations taking values in a $\mathfrak{g}$-module $W = V^{\otimes n}$ yields a braid group representation

$$\rho_n^{KZ} : B_n \to Aut(V^{\otimes n}).$$

Writing $h := \frac{2\pi i}{\kappa}$, where κ is the parameter in the KZ-equations, we can state the Drinfeld-Kohno theorem.

Theorem 7.1. *(Drinfeld-Kohno) The braid group representations ρ_n^{KZ} and $\rho_n^{R_h}$ are equivalent for any $n > 1$ and any finite dimensional $\mathfrak{g}$-module V. That is, there exists a $\mathbb{C}[[h]]$-linear automorphism $\pi \in Aut_{\mathbb{C}[[h]]}(V^{\otimes n}[[h]])$ such that*

$$\rho_n^{KZ}(\sigma_i) = \pi \circ (R_h)_{i,i+1} \circ \pi^{-1}$$

for all $i = 1, \ldots, n-1$.

The proof of the Drinfeld-Kohno theorem is long and tedious, and we leave it to the reader. It is presented using topological modules and braided quasi-bialgebras in [Kas95] and [Mee05], for instance. The DK-theorem was originally proved in [Dri90], and the relation between the two braid group representations was first established by Toshitake Kohno in [Koh87].

Notice that considering topological modules requires that the deformation parameter $h = \frac{2\pi i}{\kappa}$ is near one in order the formal power series to converge. Hence the proof of the DK-theorem is valid only for such h. However, the topological method yields, at least in a topological sense, a "universal" R-matrix for the quantum enveloping algebra $U_h(\mathfrak{g})$. As we have seen in this thesis, an expression of an R-matrix inducing braid group representations can be obtained for all values of the deformation parameter $q = e^{\frac{\pi \mathrm{i}}{\kappa}}$ which are not roots of unity. Moreover, we found an *explicit relation* between the monodromy of KZ($\mathfrak{sl}_2$) and the R-matrix, based on the fact that solutions of KZ($\mathfrak{sl}_2$) can be written in integral form. In particular, using the quantum R-matrix we are able to compute also monodromy of other systems of linear partial differential equations whose solutions have similar integral expressions. Systems of PDE's of this kind arise for instance in the theory of Schramm-Loewner evolutions.

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