Lecture notes:
Introduction to Generalized Descriptive Set Theory

Miguel Moreno

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1 Descriptive Set Theory

This is an intensive course, long proofs are discussed during the lectures but not included in the lecture notes.

Day 1

Definition 1.1 (The Baire space $\mathbf{B}$)**.** *The Baire space is the set ω^ω endowed with the following topology. For every $\eta \in \omega^n$ for some n, define the following basic open set*

$$N_\eta = \{f \in \omega^\omega \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

This topology is metrizable, let $d(f,g) = \frac{1}{n+1}$ where n is the least natural number that satisfies $f(n) \neq g(n)$, in case it does not exist then $f = g$ and $d(f,g) = 0$.

Definition 1.2 (The Cantor space $\mathbf{C}$)**.** *The cantor space is the set 2^ω with the relative subspace topology.*

Definition 1.3 (Borel class)**.** *Let $S \in \{\mathbf{B}, \mathbf{C}\}$. The class Borel$(S)$ of all Borel sets in S is the least collection of subsets of S which contains all open sets and is closed under complements, countable unions and countable intersections.*

Definition 1.4 (Borel hierarchy)**.** *Let $S \in \{\mathbf{B}, \mathbf{C}\}$. Define the classes $\Sigma_\alpha(S)$ and $\Pi_\alpha(S)$, $\alpha < \omega_1$, as follows.*

1. *$\Sigma_1(S)$ is the class of open sets.*
2. *$\Pi_1(S)$ is the class of closed sets.*
3. *For all $\alpha > 1$, $\Sigma_\alpha(S)$ is the class of of all countable unions of sets from $\bigcup_{\beta<\alpha} \Pi_\beta(S)$.*
4. *For all $\alpha > 1$, $\Pi_\alpha(S)$ is the class of of all countable unions of sets from $\bigcup_{\beta<\alpha} \Sigma_\beta(S)$.*

Exercise 1.1.
1. *For all $n < \omega$ and all $\eta \in \omega^n$ the set N_η is closed.*
2. *For all $\beta < \alpha < \omega_1$, $\Sigma_\beta(\mathbf{B}) \subseteq \Sigma_\alpha \mathbf{B}$.*
3. *$Borel(\mathbf{B}) = \bigcup_{0<\alpha<\omega_1} \Sigma_\alpha(\mathbf{B})$.*
4. *$\mid Borel(\mathbf{B}) \mid = 2^\omega$.*
5. *There are subsets of $\mathbf{B}$ that are not Borel.*

Definition 1.5. *Let $S \in \{\mathbf{B}, \mathbf{C}\}$. We say that $A \subseteq S$ is co-meager, if it contains a countable intersection of open and dense subsetes of S. A subset of S is meager, if the cmplement of it is co-meager.*

Definition 1.6. *Let $S \in \{\mathbf{B}, \mathbf{C}\}$. We say that $X \subseteq S$ has the property of Baire (PB) if there is an open set $U \subseteq S$ such that $X \Delta U$ is meager.*

Lemma 1.7. *Every Borel subset of $\mathbf{B}$ has the property of Baire.*

Exercise 1.2. *Prove Lemma 1.7. (Hint: prove that X has the PB if and only if $\mathbf{B}\backslash X$ has the PB.)*

Definition 1.8 (*Borel**-code)**.** *Let X be a non-emprty set.*

1. *A subset $T \subset X^{<\omega}$ is a tree if for all $f \in T$ with $n = dom(f) > 0$ and for all $m < n$, $f \restriction m \in T$.*

2. *A non-empty tree $T \subset X^{<\omega}$ is called an ω-tree if the following holds:*
   (a) *If $f : n \to X$ is in T and $n > 0$, then for all $x \in X$, $f \restriction (n-1) \cup \{(n-1, x)\} \in T$.*
   (b) *There is no $f : \omega \to X$ such that for all $n < \omega$, $f \restriction n \in T$.*
3. *We order T by $\subseteq$. The maximal elements of T are called leaves and the set of leaves is denoted by $L(T)$. The least element of T is called root ($\emptyset$). For every $f \in T$ that is not the root, we denote by f^- the immediate predecessor of f in T. We call node every element that is not a leaf.*
4. *A Borel*-code is a pair (T, π), where $T \subseteq (\omega \times \omega)^{<\omega}$ is an ω-tree and π is a function from $L(T)$ to the basic open sets of $\mathbf{B}$.*
5. *Given a Borel*-code (T, π) and $\eta \in \mathbf{B}$, we define the game $GB^*(\eta, (T, \pi))$ as follows. The game $GB^*(\eta, (T, \pi))$ is played by two players, **I** and **II**. In each move $0 \leq n < \omega$ the function $f_n : n+1 \to (\omega \times \omega)$ from T is chosen as follows: Suppose $f_{n-1} \in T$ is chosen, in case $n = 0$, $f_{-1} = \emptyset$. If f_{n-1} is not a leaf, then **I** choose some $i < \omega$ and then **II** choose some $j < \omega$. This determines $f_n = f_{n-1} \cup \{(n, (i, j))\}$. If f_{n-1} is a leaf, then the game ends and **II** wins if $\eta \in \pi(f_{n-1})$.*
6. *A function $W : \omega^{<\omega} \to \omega$ is a winning strategy of **II** in $GB^*(\eta, (T, \pi))$, if **II** wins by choosing $W(i_0, \ldots, i_n)$ on the move n, where $i_0, \ldots, i_n$ are the moves that **I** made on the moves $0, \ldots, n$.*
7. *A Borel*-code (T, π) is a Borel*-code for $X \subseteq \mathbf{B}$ if for all $\eta \in \mathbf{B}$, $\eta \in X$ if and only if **II** has a winning strategy in $GB^*(\eta, (T, \pi))$. We say that $X \subseteq \mathbf{B}$ is a Borel* set if it has a Borel*-code. We denote by Borel$^*(\mathbf{B})$ the class of Borel* sets.*

Theorem 1.9. *Borel$(\mathbf{B})$ = Borel$^*(\mathbf{B})$.*

Proof. Let us start by showing that *Borel*$(\mathbf{B}) \subseteq$ *Borel*$^*(\mathbf{B})$. We will prove this by showing that every open set is a *Borel** set and if $\{X_i\}_{i<\omega}$ is a countable collection of *Borel** sets, then $\bigcup_{i<\omega} X_i$ and $\bigcap_{i<\omega} X_i$ are *Borel** sets.

Suppose that X is an open set. Let $\{\xi_i\}_{i<\omega}$ be a collection of elements of $\omega^{<\omega}$ such that $X = \bigcup_{i<\omega} N_{\xi_i}$. Let $T = (\omega \times \omega)^{\leq 1}$ and π the fuction given by $\pi((0, (i, j))) = N_{\xi_j}$. It is clear that for every $\eta \in X$, **II** has a winning strategy in $GB^*(\eta, (T, \pi))$. Therefore (T, π) is a *Borel**-code for X.

Suppose that $\{X_i\}_{i<\omega}$ is a countable collection of *Borel** sets. Let (T_i, π_i) be a *Borel**-code of X_i. Let T be the set of all functions $f : n \to (\omega \times \omega)$, for some $n < \omega$, such that if $f(0) = (i, j)$, then there is $g \in T_i$, $g : n-1 \to (\omega \times \omega)$ with $dom(f) = dom(g) + 1$, and $f(m) = g(m-1)$, for all $0 < m < dom(f)$. For every leaf f of T if $f(0) = (i, j)$, then there is $g \in L(T_i)$ such that $f(m) = g(m-1)$, for all $0 < m < dom(f)$; define $\pi(f) = \pi_i(g)$.

Claim 1.10. *(T, π) is a Borel*-code of $\bigcap_{i<\omega} X_i$, and $\bigcap_{i<\omega} X_i$ is a Borel* set.*

Proof. Let $\eta \in \bigcap_{i<\omega} X_i$. Then for all $i < \omega$, there is a winning strategy W_i of **II** in $GB^*(\eta, (T_i, \pi_i))$. Define $W : \omega^{<\omega} \to \omega$ by $W(i_0) = 0$ and $W(i_0, \ldots, i_n) = W_{i_0}(i_1, \ldots, i_n)$ for all $0 < n < \omega$. It is easy to see that W is a winning strategy of **II** in $GB^*(\eta, (T, \pi))$.

Let $\eta \in \mathbf{B}$ be such that **II** has a winning strategy, W, in $GB^*(\eta, (T, \pi))$. Define $W_i : \omega^{<\omega} \to \omega$ by $W_i(i_0, \ldots, i_n) = W(i, i_0, \ldots, i_n)$. It is easy to see that W_i is a winning strategy of **II** in $GB^*(\eta, (T_i, \pi_i))$. Since this holds for all $i < \omega$, we conclude that $\eta \in X_i$, for all $i < \omega$. □

Let (T_i, π_i) be a *Borel**-code of X_i. Let T be the set of all functions $f : n \to (\omega \times \omega)$, for some $n < \omega$, such that if $f(0) = (i, j)$, then there is $g \in T_j$, $g : n-1 \to (\omega \times \omega)$ with $dom(f) = dom(g) + 1$ and $f(m) = g(m-1)$, for all $0 < m < dom(f)$. For every leaf f of T if $f(0) = (i, j)$, then there is $g \in L(T_j)$ such that $f(m) = g(m-1)$, for all $0 < m < dom(f)$; define $\pi(f) = \pi_j(g)$.

Claim 1.11. *(T, π) is a Borel*-code of $\bigcup_{i<\omega} X_i$, and $\bigcup_{i<\omega} X_i$ is a Borel* set.*

Proof. Let $\eta \in \bigcup_{i<\omega} X_i$. Then there is $j < \omega$, such that there is a winning strategy W_j of **II** in $GB^*(\eta, (T_j, \pi_j))$. Define $W : \omega^{<\omega} \to \omega$ by $W(i_0) = j$ and $W(i_0, \ldots, i_n) = W_j(i_1, \ldots, i_n)$ for all $0 < n < \omega$. It is easy to see that W is a winning strategy of **II** in $GB^*(\eta, (T, \pi))$.

Let $\eta \in \mathbf{B}$ be such that **II** has a winning strategy, W, in $GB^*(\eta, (T, \pi))$. Define $W' : \omega^{<\omega} \to \omega$ by $W'(i_1, \ldots, i_n) = W(0, \ldots, i_n)$. It is easy to see that W' is a winning strategy of **II** in $GB^*(\eta, (T_{W(0)}, \pi_{W(0)}))$. Therefore $\eta \in X_{W(0)}$. □

To show that *Borel*$^*(\mathbf{B}) \subseteq$ *Borel*$(\mathbf{B})$ we will define the rank of an ω-tree and the rank of the elements of an ω-tree.

Given an ω-tree T, we define the rank function, rk, as follows:

- If $\eta \in L(T)$, then $rk(\eta) = 0$.
- If $\eta \notin L(T)$, then $rk(\eta) = \bigcup\{rk(f) + 1 \mid f^- = \eta\}$.

The rank of a tree T is defined by $rk(T) = rk(\emptyset)$.

Exercise 1.3. *1. Show that the rank of an ω-tree is smaller than ω_1.*

2. Find an ω-tree with infinite rank.

Let X be a *Borel** set, and (T, π) a *Borel**-code of X. We will prove by induction on $rk(T)$ that X is a Borel set.

Case $rk(T) = 0$. It is clear that $T = \{\emptyset\}$ and $X = \pi(\emptyset)$, therefore X is a Borel set.

Suppose $rk(T) = \alpha$ and if Y is *Borel** set with *Borel**-code (T', π') with $rk(T) < \alpha$, then Y is a Borel set.

Let T_{ij} be the set of all functions $f : n \to \omega$ such that there is a function $g \in T$ with $g(0) = (i, j)$, $dom(g) = dom(f) + 1$ and $f(m) = g(m + 1)$ for all $m \in dom(f)$. Define π_{ij} by $\pi_{ij}(f) = \pi(g)$, where $g \in T$ is such that $g(0) = (i, j)$, $dom(g) = dom(f) + 1$ and $f(m) = g(m + 1)$ for all $m \in dom(f)$. Notice that for all $i, j < \omega$, $rk(T_{ij}) < \alpha$. By the induction hypothesis, for all $i, j < \omega$, (T_{ij}, π_{ij}) is a *Borel**-code of a Borel set. Denote by B_{ij} the Borel set with *Borel**-code (T_{ij}, π_{ij}).

Claim 1.12. $X = \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$

Proof. Let $\eta \in X$, then **II** has a winning strategy, W, in $GB^*(\eta, (T, \pi))$. Define $W_{iW(i)} : \omega^{<\omega} \to \omega$ by $W_{iW(i)}(i_0, \dots, i_n) = W(i, i_0, \dots, i_n)$, it is clear that $W{-}iW(i)$ is a winning strategy of **II** in $GB^*(\eta, (T_{iW(i)}, \pi_{iW(i)}))$, so $\eta \in B_{iW(i)}$. Therefore, for all $i < \omega$ there is $j < \omega$ such that $\eta \in B_{ij}$, we conclude that $\eta \in \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$.

Let $\eta \in \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$. Then for all $i < \omega$ there is $j < \omega$ such that $\eta \in B_{ij}$, denote by $h(i)$ this j. So there is $W_{ih(i)}$ a winning strategy of **II** in $GB^*(\eta, (T_{ih(i)}, \pi_{ih(i)}))$. Define $W : \omega^{<\omega} \to \omega$ by $W(i_0) = h(i_0)$ and $W(i_0, \dots, i_n) = W_{h(i_0)}(i_1, \dots, i_n)$. It is clear that W is a winning strategy of **II** in $GB^*(\eta, (T_{iW(i)}, \pi_{iW(i)}))$ and $\eta \in X$. □

□

At the beginning the *Borel**-codes look very artificial and complicated, but this codes will be very helpful in the future. In order to give a better understanding of the motivation behind the *Borel**-codes we will define the *Borel***-codes. This codes use intersections and unions as part of the coding of sets, this gives a better understanding on what is going on in the coding.

Definition 1.13. *1. A pair (T, π) is a* Borel**-code *if $T \subseteq \omega^{<\omega}$ is an ω-tree and π is a function with domain T such that if $f \in T$ is a leaf, then $\pi(f)$ is an open set, and in case f is a node, $\pi(f) = \cap$ if $\mid dom(f) \mid$ is an even number and $\pi(f) = \cup$ if $\mid dom(f) \mid$ is an odd number.*

2. For an element $\eta \in \mathbf{B}$ and a Borel**-code (T, π), the game $B^*(\eta, (T, \pi))$ is played as follows. There are two players, **I** *and* **II**. *The game starts from the root of T. At each move, if the game is at node $f \in T$ and $\pi(f) = \cap$, then* **I** *chooses an immediate successor g of f and the game continues from this g. If $\pi(f) = \cup$, then* **II** *makes the choice. Finally, if $\pi(f)$ is an open set, then the game ends, and* **II** *wins if and only if $\eta \in \pi(x)$.*

3. A set $X \subseteq \omega^\omega$ is a Borel**-set *if there is a Borel**-code (T, π) such that for all $\eta \in \omega^\omega$, $\eta \in X$ if and only if* **II** *has a winning strategy in the game $B^*(\eta, (T, \pi))$. We denote by Borel$^{**}(\mathbf{B})$ the set of Borel** sets.*

Exercise 1.4. *Borel*$^*(\mathbf{B}) =$ *Borel*$^{**}(\mathbf{B})$.

Notice that the rank was defined for ω-trees in general. For every *Borel*** set, X, as the least ordinal α such that there is a *Borel***-code of X.

Exercise 1.5. *What is the relation between the rank of a Borel** set and the Borel hierarchy?*

Day 2

Definition 1.14.
- *$X \subseteq \mathbf{B}$ is $\Sigma^1_1(\mathbf{B})$ if there is $Y \subseteq \mathbf{B} \times \mathbf{B}$ a Borel set such that $pr(Y) = X$.*
- *$X \subseteq \mathbf{B}$ is $\Pi^1_1(\mathbf{B})$ if $\mathbf{B} \backslash X$ is $\Sigma^1_1(\mathbf{B})$.*
- *$X \subseteq \mathbf{B}$ is $\Delta^1_1(\mathbf{B})$ if it is $\Sigma^1_1(\mathbf{B})$ and $\Pi^1_1(\mathbf{B})$.*

Lemma 1.15. *The following are equivalent:*

- *X is $\Sigma^1_1(\mathbf{B})$.*
- *$X = pr(Y)$ for some closed $y \subseteq \mathbf{B} \times \mathbf{B}$.*

Lemma 1.16. *If $X \subseteq \mathbf{B}$ is Borel, then X is $\Delta^1_1(\mathbf{B})$.*

Proof. Let $X \subseteq \mathbf{B}$ be a Borel set and (T, π) a $Borel^*$-code for X. Let $h : \omega^{<\omega} \to \omega$ be one-to-on and onto. For all $f \in \omega^\omega$ define $W_f : \omega^{<\omega} \to \omega$ by $W_f(i_0, \dots, i_n) = f(h(i_0, \dots, i_n))$. Let P be the set of all the tuples $(\eta, f) \in \omega^\omega \times \omega^\omega$ such that W_f is a winning strategy for **II** in the game $GB^*(\eta, (T, \pi))$. It is clear that $pr(P) = X$.

Claim 1.17. *P is closed*

Proof. Let $(\eta, f) \notin P$ then there are $n < \omega$ and $\{j_0, \dots, j_n\}$ such that if **I** choose j_m in the m-move and **II** choose $W_f(j_0 \dots, j_m)$ in the m-move, then after n moves the game stops in a leaf g and $\eta \notin \pi(g)$. Therefore, there is $r < \omega$, such that $N_{\eta\restriction r} \cap \pi(g) = \emptyset$, so $(N_{\eta\restriction r} \times N_{f\restriction m}) \cap P = \emptyset$. □

We conclude that X is $\Sigma^1_1(\mathbf{B})$ and since $Borel(\mathbf{B})$ is closed under complements, we conclude that $\mathbf{B}\backslash X$ is Borel, therefore it is $\Sigma^1_1(\mathbf{B})$. We conclude that X is $\Delta^1_1(\mathbf{B})$. □

Exercise 1.6. *Prove the claims of the following proof.*

Theorem 1.18 (Separation)**.** *If $X, Y \subseteq \mathbf{B}$ are $\Sigma^1_1(\mathbf{B})$ disjoint sets, then there is a Borel set $Z \subseteq \mathbf{B}$ that satisfies $X \subseteq Z \subseteq \mathbf{B}\backslash Y$.*

Proof. Choose $X^*, Y^* \subseteq \mathbf{B} \times \mathbf{B}$ such that $pr(X^*) = X$ and $pr(Y^*) = Y$. For all $\eta \in \mathbf{B}$, let X_η be the set of all $\xi \in \omega^\omega$ that satisfy the following: If $dom(\xi) = n$, then there are $\eta'\xi' \in \mathbf{B}$, $(\eta', \xi') \in X^*$, and $\eta' \restriction n = \eta \restriction n$ and $\xi \subseteq \xi'$. Define Y_η in the same way. We denote by $X_{\eta\restriction n}$ the set of functions $\xi \in \omega^n$ such that there is $\eta' \in \mathbf{B}$, and $\xi \in X_{\xi'}$ and $\eta \restriction n \subseteq \eta'$. It is clear that $X_\eta = \bigcup_{n<\omega} X_{\eta\restriction n}$.

Given two trees $T, T' \subseteq \omega^{<\omega}$, we say that $T \leq T'$ if there is a function $f : T \to T'$ that satisfies the following: for all $\eta, \xi \in T$, if $\eta \subsetneq \xi$, then $f(\eta) \subsetneq f(\xi)$. Let Z be the set of $\eta \in \mathbf{B}$ that satisfy $Y_\eta \leq X_\eta$.

Claim 1.19.
- *If $\eta \in X$, then $Y_\eta \leq X_\eta$.*
- *If $Y_\eta \leq X_\eta$, then $\eta \notin Y$.*
- *$X \subseteq Z \subseteq \mathbf{B}\backslash Y$.*

for all $T, T' \subseteq \omega^{<\omega}$ we define the game $GC(T, T')$ as follows: in the n-th movement, **I** chooses $t_n \in T$ such that $t_m \subseteq t_n$ holds for all $m < n$, and **II** chooses $t'_n \in T'$ such that $t'_m \subseteq t'_n$ holds for all $m < n$. The game ends when a player cannot make a choice, the player that cannot make a choice looses.

Claim 1.20. *$T \leq T'$ si y solo si **II** has a winning strategy for the game $GC(T, T')$.*

Let T be the set of all functions with finite domain, $f : n \to \bigcup_{m<\omega} (\omega^m)^3$ such that for all $i < n$ the following holds:

- $f(i) \in (\omega^i)^3$.
- If $j + 1 < n$ and $f(j) = (\xi_k)_{k<3}$, then $\xi_1 \in X_{\xi_0}$ and $\xi_2 \in X_{\xi_0}$.
- If $j < l < n$, $f(j) = (\xi_k)_{k<3}$, and $f(l) = (\xi'_k)_{k<3}$, then for all $k < 3$, $\xi_k \subseteq \xi'_k$.

Define π with domain $L(T)$ as $\pi(f) = N_{\xi_0}$ if $dom(f) = n + 1$, $f(n) = (\xi_k)_{k<3}$, and $\xi_2 \notin Y_{\xi_0}$. And $\pi(f) = \emptyset$ in other case.

Claim 1.21. *There is a $Borel^*$-code (T', π') such that there is a tree isomorphism $h : T' \to T$ that satisfies $\pi'(f) = \pi(h(f))$.*

Claim 1.22. ***II** has a winning strategy in $GB^*(\eta, (T', \pi'))$ if and only if $GC(Y_\eta, X_\eta)$.*

□

The following is a standard way to code structures with domain ω with elements of 2^ω. Fix a countable relational vocabulary $\mathcal{L} = \{P_n \mid n < \omega\}$.

Definition 1.23. *Fix a bijection $\pi \colon \omega^{<\omega} \to \omega$. For every $\eta \in 2^\omega$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain ω as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \dots, a_n)$ in ω^n satisfies*

$$(a_1, a_2, \dots, a_n) \in P_m^{\mathcal{A}_\eta} \iff \eta(\pi(m, a_1, a_2, \dots, a_n)) = 1.$$

Definition 1.24 (The isomorphism relation)**.** *Assume T is a complete first order theory in a countable vocabulary. We define $\cong^\omega_T$ as the relation*

$$\{(\eta,\xi)\in 2^\omega\times 2^\omega \mid (\mathcal{A}_\eta\models T,\mathcal{A}_\xi\models T,\mathcal{A}_\eta\cong\mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta\not\models T,\mathcal{A}_\xi\not\models T)\}.$$

A function $f\colon 2^\omega\to 2^\omega$ is Borel, if for every open set $A\subseteq 2^\omega$ the inverse image $f^{-1}[A]$ is a Borel subset of 2^ω. Let E_1 and E_2 be equivalence relations on 2^ω. We say that E_1 is Borel reducible to E_2, if there is a Borel function $f\colon 2^\omega\to 2^\omega$ that satisfies $(x,y)\in E_1\Leftrightarrow (f(x),f(y))\in E_2$, we denote it by $E_1\leq_B E_2$.

Exercise 1.7. *A function f is Borel if and only if for all Borel set X, $f^{-1}[X]$ is Borel.*

Example 1.1. *Let T_1 be the theory of the order of the rational numbers, $\cong^\omega_{T_1}$ has only two equivalent classes. Let T_2 be the theory of a vector space over the field of rational numbers. $\cong^\omega_{T_1}\leq_B\cong^\omega_{T_2}$.*

This can be use to compare the complexity of two theories, from Example 1.1 we conclude that T_1 is less complex than T_2, in the Borel reducibility sense.

Question 1.25. *Is there an equivalence relation E on 2^ω such that for every complete first order theory in a countable vocabulary T, either $E\not\leq_B\cong^\omega_{T_1}$ or $\cong^\omega_{T_1}\not\leq_B E$.*

Let T be a complete countable theory, we will denote by $I(\lambda,T)$ the amount of non-isomorphic models of T of size λ. The following is the main theorem of [12].

Theorem 1.26 (The Main Gap Theorem, [12])**.** *Let T be a complete countable theory.*

- *If T is not superstable, or deep, or with DOP or OTOP then for every uncountable cardinal λ, $I(\lambda,T)=2^\lambda$.*
- *If T is shallow superstable without DOP and without OTOP, then for every $\alpha>0$, $I(\aleph_\alpha,T)\leq\beth_{\omega_1}(|\alpha|)$.*

Let T be a complete countable theory, we say that T is a classifiable theory if T is superstable without DOP and without OTOP. T_1 in Example 1.1 is not classifiable and T_2 is classifiable. The Main Gap Theorem tells us that classifiable theories are less complex than non-classifiable ones, in the stability sense.

2 Generalized Descriptive Set Theory

Day 3

Definition 2.1 (The Generalized Baire space $\mathbf{B}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Baire space is the set κ^κ endowed with the following topology. For every $\eta\in\kappa^{<\kappa}$, define the following basic open set*

$$N_\eta=\{f\in\kappa^\kappa\mid\eta\subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 2.2 (The Generalized Cantor space $\mathbf{C}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Cantor space is the set 2^κ with the relative subspace topology.*

From now on κ is an uncountable cardinal that satisfies κ^κ.

Definition 2.3 (κ-Borel class)**.** *Let $S\in\{\mathbf{B}(\kappa),\mathbf{C}(\kappa)\}$. The class κ-Borel(S) of all κ-Borel sets in S is the least collection of subsets of S which contains all open sets and is closed under complements, unions and intersections both of length at most κ.*

Definition 2.4 (κ-Borel*-set in $\mathbf{C}(\kappa)$)**.**

1. *A tree T is a κ^+,κ-tree if does not contain chains of length κ and its cardinality is less than κ^+. It is closed if every chain has a unique supremum.*
2. *A pair (T,h) is a κ-Borel*-code if T is a closed κ^+,κ-tree and h is a function with domain T such that if $x\in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x)\in\{\cup,\cap\}$.*
3. *For an element $\eta\in 2^\kappa$ and a κ-Borel*-code (T,h), the κ-Borel*-game $B^*(T,h,\eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T. At each move, if the game is at node $x\in T$ and $h(x)=\cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x)=\cup$, then **II** makes the choice. At limits the game continues from the (unique) supremum of the previous moves by Player **I**. Finally, if $h(x)$ is a basic open set, then the game ends, and **II** wins if and only if $\eta\in h(x)$.*
4. *A set $X\subseteq 2^\kappa$ is a κ-Borel*-set if there is a κ-Borel*-code (T,h) such that for all $\eta\in 2^\kappa$, $\eta\in X$ if and only if **II** has a winning strategy in the game $B^*(T,h,\eta)$.*

We can define the κ-Borel*-set in the generalized Baire space too, by using the same coding but with basic open sets of the generalized Baire space. Given two sets $X, Y \subset \kappa^\kappa$ we say that X and Y are duals if there is a κ-Borel*-code (T,h) such that for all $\eta \in \kappa^\kappa$, $\eta \in X$ if and only if **II** has a winning strategy in the game $B^*(T,h,\eta)$, and $\eta \in Y$ if and only if **I** has a winning strategy in the game $B^*(T,h,\eta)$. We will write $\mathbf{II} \uparrow B^*(T,h,\eta)$ when **II** has a winning strategy in the game $B^*(T,h,\eta)$, and $\mathbf{I} \uparrow B^*(T,h,\eta)$ when **I** has a winning strategy in the game $B^*(T,h,\eta)$.

Exercise 2.1. *X is a κ-Borel set if and only if there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree.*

Definition 2.5.
- *$X \subseteq \mathbf{B}(\kappa)$ is $\Sigma_1^1(\kappa)$ if there is $Y \subseteq \mathbf{B}(\kappa) \times \mathbf{B}(\kappa)$ a closed set such that $pr(Y) = X$.*
- *$X \subseteq \mathbf{B}(\kappa)$ is $\Pi_1^1(\kappa)$ if $\mathbf{B}(\kappa)\backslash X$ is $\Sigma_1^1(\kappa)$.*
- *$X \subseteq \mathbf{B}(\kappa)$ is $\Delta_1^1(\kappa)$ if it is $\Sigma_1^1(\kappa)$ and $\Pi_1^1(\kappa)$.*

Theorem 2.6 ([2], Theorem 17)**.**
1. *κ-Borel$\subseteq$ κ-Borel*.*
2. *κ-Borel$\subseteq \Delta_1^1(\kappa)$.*
3. *κ-Borel$\subseteq \Sigma_1^1(\kappa)$.*
4. *κ-Borel* $\subseteq \Sigma_1^1(\kappa)$.*

Proof. (Sketch). From Exercise 2.1 we conclude that (1) holds. (2) follows from (3) and tha fact that κ-Borel is closed under complement. (3) follows from (1) and (4). To prove (4), code the winning strategies $\sigma : T \to T$ by elements of κ^κ, notice that the assumption $\kappa^{<\kappa}$ is needed. Then, if X is κ-*Borel**, then there is a κ-Borel*-code (T,h) that codes X. The set $Y = \{(\eta,\xi) \mid \xi$ is a code of a winning strategy for **II** in $B^*(T,h,\eta)\}$ is closed and $pr(Y) = X$. □

Exercise 2.2. *Complete the details in the proof of Theorem 2.6.*

The following theorem is the separation theorem and the proof can be found in [10].

Theorem 2.7 ([10], Corollary 34)**.** *Suppose A and B are disjoint $\Sigma_1^1(\kappa)$ sets. There are κ-Borel* sets C_0 and C_1 such that $A \subseteq C_0$, $B \subseteq C_1$, and C_0 and C_1 are duals.*

Theorem 2.8 ([2], Theorem 17)**.** *$\Delta_1^1(\kappa) \subseteq \kappa$-Borel**

Proof. Let A be a $\Delta_1^1(\kappa)$ set. Let $B = \mathbf{B}(\kappa)\backslash A$, by 2.7, there are κ-Borel* sets C_0 and C_1 such that $A \subseteq C_0$, $B \subseteq C_1$, and C_0 and C_1 are duals. Since C_0 and C_1 are duals, C_0 and C_1 are disjoint. So $A = C_0$, $B = C_1$. □

Corollary 2.9 ([10], Corollary 35)**.** *X is $\Delta_1^1(\kappa)$ if there is a κ-Borel*-code (T,h) that codes X and*

$$\mathbf{II} \uparrow B^*(T,h,\eta) \Leftrightarrow \mathbf{I} \not\uparrow B^*(T,h,\eta)$$

for all $\eta \in \kappa^\kappa$ the game is determined.

Exercise 2.3. *Prove the claims of the following proof.*

Theorem 2.10 ([2], Theorem 18)**.**
1. *κ-Borel$\subsetneq \Delta_1^1(\kappa)$*
2. *$\Delta_1^1(\kappa) \subsetneq \Sigma_1^1(\kappa)$*

Proof.
1. Let $\xi \mapsto (T_\xi, h_\xi)$ be a continuous coding of the κ-Borel*-codes with T a $\kappa^+\omega$-tree, such that for all $\kappa^+\omega$-tree, T, and h, there is ξ such that $T_\xi, h_\xi = (T,h)$.

   Claim 2.11. *The set $B = \{(\eta,\xi) \mid \eta$ is in the set coded by $(T_\xi, h_\xi)\}$ is $\Sigma_1^1(\kappa)$ and is not κ-Borel, otherwise $D = \{\eta \mid (\eta,\eta) \notin B\}$ would be Borel (Hint: use the set $C = \{(\eta,\xi,\sigma) \mid \sigma$ is a winning strategy for **II** in $B^*(T_\xi, h_\xi, \eta)\}$).*

2.

   Claim 2.12. *There is $A \subseteq 2^\kappa \times 2^\kappa$ such that if $B \subseteq 2^\kappa$ is a $\Sigma_1^1(\kappa)$ set, then there is $\eta \in 2^\kappa$ such that $B = \{\xi \mid (\xi,\eta) \in A\}$ (Hint: the construction used in the classical case works too).*

   The set $D = \{\eta \mid (\eta,\eta) \in A\}$ is $\Sigma_1^1(\kappa)$ but not $\Pi_1^1(\kappa)$.

□

Exercise 2.4. *Prove the claims of the following proof.*

Lemma 2.13 ([5], Lemma 5)**.** *Assume $V = L$. Suppose $\psi(x,\xi)$ is a Σ_1-formula in set theory with parameter $\xi \in 2^\kappa$ and that $r(\alpha)$ is a formula of set theory that says that "α is a regular cardinal". Then for $x \in 2^\kappa$ we have $\psi(x,\xi)$ if and only if the set*

$$A = \{\alpha < \kappa \mid \exists\beta > \alpha(L_\beta \models ZF^- \wedge \psi(x \restriction \alpha, \xi \restriction \alpha) \wedge r(\alpha))\}$$

contains a club.

Proof. Suppose that $x \in 2^\kappa$ is such that $\psi(x,\xi)$ holds. Let θ be a large enough cardinal such that

$$L_\theta \models ZF^- \wedge \psi(x,\xi) \wedge r(\alpha).$$

For each $\alpha < \kappa$, let

$$H(\alpha) = Sk(\alpha \cup \{\kappa, \xi, x\})^{L_\theta}$$

and $\bar{H}(\alpha)$ the Mostowski collapse of $H(\alpha)$. Let

$$D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha\}.$$

Claim 2.14. *D is a club set and $D \subseteq A$.*

Suppose $x \in 2^\kappa$ is such that $\psi(x,\xi)$ does not hold. Let $\mu < \kappa$ be a regular cardinal. Take θ as above and let C be an unbounded set, closed under μ-limits (i.e. if $(\gamma_i)_i < \mu$ is an increasing succession of elements of C, then $\bigcup\{\gamma_i \mid i < \mu\} \in C$). Let

$$K(\alpha) = Sk(\alpha \cup \{\kappa, C, \xi, x\})^{L_\theta}$$

and

$$D = \{\alpha \in S^\kappa_\mu \mid K(\alpha) \cap \kappa = \alpha\}.$$

Claim 2.15. *D is an unbounded set, closed under μ-limits.*

Let $\alpha_0 \in D$ be the least ordinal that is a μ-cofinal limit of elements of D.

Claim 2.16. *$\alpha_0 \in C$ and $\alpha_0 > \mu$ (Hint: Use the elementarity of $K(\alpha)$ and the fact that $D \subseteq S^\kappa_\mu$).*

Let $\bar{\beta}$ be such that $L_{\bar{\beta}}$ is equal to the Mostowski collapse of $K(\alpha_0)$. We will show that $\alpha_0 \notin A$. Suppose, towards a contradiction, that $\alpha_0 \in A$. There exists $\beta > \alpha$ such that

$$L_\beta \models ZF^- \wedge \psi(x \restriction \alpha, \xi \restriction \alpha) \wedge r(\alpha).$$

Claim 2.17. *β is a limit ordinal greater than $\bar{\beta}$ and L_β satisfies "there exists a $\gamma \leq \alpha_0$ and an order-preserving bijection from γ to $D \cap \alpha_0$" (Hint: Show that $K(\alpha_0)$ is a definable subset of L_θ and $D \cap \alpha_0$ is a definable subset of $K(\alpha_0)$, to conclude that $D \cap \alpha_0$ is a definable subset of $L_{\bar{\beta}}$ and $D \cap \alpha_0 \in L_\beta$).*

By the way α_0 was chosen, $D \cap \alpha_0$ has order type μ. Hence, by Claim 2.16 α_0 is singular in L_β but this contradicts that $L_\beta \models r(\alpha)$. ☐

Day 4

Let μ be a regular cardinal, we say that $X \subseteq \kappa$ is a μ-club if X is unbounded set and closed under μ-limits.

Definition 2.18 ($E^\kappa_{\mu\text{-club}}$)**.** *Let $\mu < \kappa$ be a regular cardinal. For all $\eta, \xi \in \kappa^\kappa$ we say that η and ξ are $E^\kappa_{\mu\text{-club}}$ equivalent ($\eta\ E^\kappa_{\mu\text{-club}}\ \xi$) if the set $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$ contains a μ-club.*

Definition 2.19 ($E^2_{\mu\text{-club}}$)**.** *Let $\mu < \kappa$ be a regular cardinal. For all $\eta, \xi \in 2^\kappa$ we say that η and ξ are $E^2_{\mu\text{-club}}$ equivalent ($\eta\ E^2_{\mu\text{-club}}\ \xi$) if the set $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$ contains a μ-club.*

An equivalence relation E on $X \in \{\kappa^\kappa, 2^\kappa\}$ is *$\Sigma^1_1(\kappa)$-complete* if every $\Sigma^1_1(\kappa)$ equivalence relation is κ-Borel reducible to it.

Exercise 2.5. *Prove the claims of the following proof.*

Theorem 2.20 ([5], Theorem 7)**.** *Suppose that $V = L$. Then $E^\kappa_{\mu\text{-club}}$ is $\Sigma^1_1(\kappa)$-complete, for every regular μ.*

Proof. Suppose E is a $\Sigma^1_1(\kappa)$ equivalence relation on κ^κ. Let $a : \kappa^\kappa \to 2^{\kappa\times\kappa}$ the map defined by

$$a(\eta)(\alpha,\beta) = 1 \Leftrightarrow \eta(\alpha) = \beta.$$

Let b be a continuous bijection from $2^{\kappa\times\kappa}$ to 2^κ, and $c = b \circ a$. Define E' by

$$(\eta,\xi) \in E' \Leftrightarrow (\eta = \xi) \vee (\eta, \xi \in ran(c) \wedge (c^{-1}(\eta), c^{-1}(\xi)) \in E)$$

Claim 2.21. *c is a continuous reduction of E to E' and E' is a $\Sigma_1^1(\kappa)$ equivalence relation.*

We can assume without loss of generality, that E is an equivalence relation on 2^κ. It is enough to define $f : 2^\kappa \to (2^{<\kappa})^\kappa$ such that for all $\eta, \xi \in 2^\kappa$, $(\eta, \xi) \in E$ if and only if the set $\{\alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha)\}$ contains a μ-club and f is continuous in the topology generated by the sets

$$\{\eta \mid \eta \restriction \alpha = p\}, p \in (2^{<\kappa})^\alpha, \alpha < \kappa.$$

Claim 2.22. *f can be coded by a κ-Borel function $\mathcal{F} : 2^\kappa \to \kappa^\kappa$.*

Claim 2.23. *There is a Σ_1-formula of set theory $\psi(\eta, \xi) = \psi(\eta, \xi, x) = \exists k\varphi(k, \eta, \xi, x)$ with $x \in 2^\kappa$, such that for all $\eta, \xi \in 2^\kappa$,*

$$(\eta, \xi) \in E \Leftrightarrow \psi(\eta, \xi).$$

Let $r(\alpha)$ be the formula "α is a regular cardinal" and $\psi^E = \psi^E(\kappa)$ be the sentence with parameter κ that asserts that $\psi(\eta, \xi)$ defines an equivalence relation on 2^κ. For all $\eta \in 2^\kappa$ and $\alpha < \kappa$, let

$$T_{\eta,\alpha} = \{p \in 2^\alpha \mid \exists\beta > \alpha(L_\beta \models ZF^- \wedge \psi(p, \eta \restriction \alpha, x) \wedge r(\alpha) \wedge \psi^E)\}$$

and let

$$f(\eta)(\alpha) = \begin{cases} min_L\ T_{\eta,\alpha} & \text{if } T_{\eta,\xi} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

We will show that $(\eta, \xi) \in E$ if and only if the set $\{\alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha)\}$ contains a μ-club.

Suppose $\psi(\eta, \xi, x) = \exists k\varphi(k, \eta, \xi, x)$ holds and let k witnesses that. Let θ be a cardinal large enough such that $L_\theta \models ZF^- \wedge \varphi(k, \eta, \xi, x) \wedge r(\alpha)$. For all $\alpha < \kappa$ let $H(\alpha) = Sk(\alpha \cup \{\kappa, k, \eta, \xi, x\})^{L_\theta}$. The set $D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha \wedge H(\alpha) \models \psi^E\}$ is a club. Using the Mostowski collapse we have that

$$D' = \{\alpha < \kappa \mid \exists\beta > \alpha(L_\beta \models ZF^- \wedge \varphi(k \restriction \alpha, \eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^E)\}$$

contains a club. For all $\alpha \in D'$ and $p \in T_{\eta,\alpha}$ we have that

$$\exists\beta_1 > \alpha(L_{\beta_1} \models ZF^- \wedge \psi(p, \eta \restriction \alpha) \wedge r(\alpha) \wedge \psi^E)$$

and

$$\exists\beta_2 > \alpha(L_{\beta_2} \models ZF^- \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha) \wedge r(\alpha) \wedge \psi^E).$$

Therefore, for $\beta = max\{\beta_1, \beta_2\}$ we have that

$$L_\beta \models ZF^- \wedge \psi(p, \eta \restriction \alpha) \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha) \wedge r(\alpha) \wedge \psi^E.$$

Since ψ^E holds and so transitivity holds for $\psi(\eta, \xi)$, we conclude that

$$L_\beta \models ZF^- \wedge \psi(p, \xi \restriction \alpha) \wedge r(\alpha) \wedge \psi^E$$

so $p \in T_{\xi,\alpha}$ and $T_{\eta,\alpha} \subseteq T_{\xi,\alpha}$. Using the same argument we can show that $T_{\xi,\alpha} \subseteq T_{\eta,\alpha}$ holds for all $\alpha \in D'$. We conclude that for all $\alpha \in D'$ it holds that $T_{\xi,\alpha} = T_{\eta,\alpha}$, and the set $\{\alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha)\}$ contains a μ-club.

Suppose that $\neg\psi(\eta, \xi, x)$ holds. Then by Lemma 2.13 there is no μ-club inside

$$\{\alpha < \kappa \mid \exists\beta > \alpha(L_\beta \models ZF^- \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha) \wedge r(\alpha))\}.$$

Notice that $\{\alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha)\} = \{\alpha \mid min_L T_{\eta,\alpha} = min_L T_{\xi,\alpha}\}$, so $\{\alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha)\} \subseteq \{\alpha \mid T_{\eta,\alpha} \cap T_{\xi,\alpha} \neq \emptyset\}$, therefore

$$\{\alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha)\} \subseteq \{\alpha \mid \exists p\exists\beta > \alpha(L_\beta \models ZF^- \wedge \psi(p, \xi \restriction \alpha) \wedge \psi(p, \eta \restriction \alpha) \wedge r(\alpha) \wedge \psi^E)\}.$$

We conclude that $\{\alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha)\} \subseteq \{\alpha < \kappa \mid \exists\beta > \alpha(L_\beta \models ZF^- \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha) \wedge r(\alpha))\}$, so $\{\alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha)\}$ does not contain a μ-club. □

Exercise 2.6. *$E^\kappa_{\omega\text{-}club}$ is a κ-Borel* set.*

A function $f\colon 2^\kappa \to 2^\kappa$ is κ-Borel, if for every open set $A \subseteq 2^\kappa$ the inverse image $f^{-1}[A]$ is a κ-Borel subset of 2^κ. Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is κ-Borel reducible to E_2, if there is a κ-Borel function $f\colon 2^\kappa \to 2^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$, we denote it by $E_1 \leq_B E_2$. In the same way it can be define κ-Borel function and κ-Borel reducibility in $\mathbf{B}(\kappa)$.

Exercise 2.7. *Assume $f\colon 2^\kappa \to 2^\kappa$ is κ-Borel function and B is a κ-Borel* set. Prove that $f^{-1}[B]$ is a κ-Borel* set.*

Corollary 2.24 ([2], Theorem 18)**.** *Suppose that $V = L$. Then κ-Borel* $= \Sigma^1_1(\kappa)$.*

Proof. It follows from Exercise 2.7, Exercise 2.6, and Theorem 2.20. □

Corollary 2.25 ([2], Theorem 18)**.** *Suppose that $V = L$. Then $\Delta^1_1(\kappa) \neq \kappa$-Borel*.*

Proof. It follows from Theorem 2.10 and Corollary 2.24. □

Question 2.26. *Is it consistent that $\Delta^1_1(\kappa) = \kappa$-Borel*?*

Question 2.27. *An equivalence relation E on $X \in \{\kappa^\kappa, 2^\kappa\}$ is* κ-Borel*-complete *if every κ-Borel* equivalence relation is κ-Borel reducible to it. Does there exists a κ-Borel*-complete relation that is not a Σ^1_1–complete relation?*

The following lemma shows that there is a model of set theory in which $\Delta^1_1(\kappa)$, κ-Borel*, and $\Sigma^1_1(\kappa)$ are different. The proof can be found in [4].

Lemma 2.28 ([4], Corollary 3.2)**.** *It is consistently that $\Delta^1_1(\kappa) \subsetneq \kappa$-Borel* $\subsetneq \Sigma^1_1(\kappa)$.*

3 The Main Gap in $\mathrm{B}(\kappa)$

Session in the logic seminar

Definition 3.1. *For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ as follows. For every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n*

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) > 0.$$

Definition 3.2. *For every $\eta \in 2^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ as follows. For every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n*

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$

Notice that the structure $\mathcal{A}_\eta \restriction \alpha$ is not necessary coded by the function $\eta \restriction \alpha$.

Exercise 3.1. *There is a club C_π such that for all $\alpha \in C_\pi$, $\mathcal{A}_\eta \restriction \alpha = \mathcal{A}_{\eta\restriction\alpha}$*

With the structures coded by the elements of 2^κ and κ^κ, it is easy to define the isomorphism relation of structures of size κ in both spaces.

Definition 3.3 (The isomorphism relation)**.** *Assume T is a complete first order theory in a countable vocabulary. We define $\cong^\kappa_T$ as the relation*

$$\{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

Definition 3.4. *Assume T is a complete first order theory in a countable vocabulary. We define $\cong^2_T$ as the relation*

$$\{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

Notice that $\cong^\kappa_T \leq_c \cong^2_T$ holds for every theory T.

Definition 3.5. *(Ehrenfeucht-Fraïssé game) Fix $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration of all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. For every $\alpha < \kappa$ we define the game $EF^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ for structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ, as follows. The game is played by two players,* **I** *and* **II***. In the n-th move,* **I** *choose an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$, $X_{\beta_{n-1}} \subseteq X_{\beta_n}$, and then* **II** *chooses an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), rang(f_{\theta_n}) \subset \alpha$, $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$ (if $n = 0$ then $X_{\beta_{n-1}} = \emptyset$ and $f_{\theta_{n-1}} = \emptyset$). The game finishes after ω moves. The player* **II** *wins if $\cup_{i<\omega} f_{\theta_i} : A\restriction_\alpha \to B\restriction_\alpha$ is a partial isomorphism, otherwise the player* **I** *wins.*

We will write $\mathbf{I} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ when **I** has a winning strategy in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$, similarly we write $\mathbf{II} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ when **II** has a winning strategy.

Theorem 3.6. *[12] If T is a classifiable theory, then for every two models of T with domain κ, $\mathcal{A}, \mathcal{B}$, it holds that $\mathbf{II} \uparrow EF^\kappa_\omega(\mathcal{A}, \mathcal{B}) \Longleftrightarrow \mathcal{A} \cong \mathcal{B}$.*

Corollary 3.7 ([2], Theorem 70)**.** *If T is a classifiable theory, then $\cong^\kappa_T$ is Δ^1_1.*

Lemma 3.8 ([7], Lemma 2.4)**.** *If $\mathcal{A}$ and $\mathcal{B}$ are structures with domain κ, then the following hold:*

- $\mathbf{II}\uparrow EF^\kappa_\omega(\mathcal{A},\mathcal{B}) \iff \mathbf{II}\uparrow EF^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ *for club-many α.*
- $\mathbf{I}\uparrow EF^\kappa_\omega(\mathcal{A},\mathcal{B}) \iff \mathbf{I}\uparrow EF^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ *for club-many α.*

Exercise 3.2. *Prove Lemma 3.8 (Hint: look at the closed points of a winning strategy).*

Definition 3.9. *Assume T is a complete first order theory in a countable vocabulary. For every $\alpha<\kappa$ and $\eta,\xi\in\kappa^\kappa$, we write $\eta\ R^\alpha_{EF}\ \xi$ if one of the following holds, $\mathcal{A}_\eta\restriction_\alpha\not\models T$ and $\mathcal{A}_\xi\restriction_\alpha\not\models T$, or $\mathcal{A}_\eta\restriction_\alpha\models T$, $\mathcal{A}_\xi\restriction_\alpha\models T$ and $\mathbf{II}\uparrow EF^\kappa_\omega(\mathcal{A}_\eta\restriction_\alpha,\mathcal{A}_\xi\restriction_\alpha)$.*

Exercise 3.3. *Let T be a complete first order theory in a countable vocabulary. There are club many α such that R^α_{EF} is an equivalence relation.*

Theorem 3.10 ([7], Theorem 2.8)**.** *If T is a classifiable theory and $\mu<\kappa$ a regular cardinal, then $\cong_T$ is continuously reducible to $E^\kappa_{\mu\text{-club}}$ ($\cong^\kappa_T\ \leq_c\ E^\kappa_{\mu\text{-club}}$).*

Proof. Define the reduction $\mathcal{F}:\kappa^\kappa\to\kappa^\kappa$ by,

$$\mathcal{F}(\eta)(\alpha)=\begin{cases} f_\eta(\alpha) & \text{if } cf(\alpha)=\mu, \mathcal{A}_\eta\restriction_\alpha\models T \text{ and } R^\alpha_{EF} \text{ is an equivalence relation}\\ 0 & \text{in other case}\end{cases}$$

where $f_\eta(\alpha)$ is a code in $\kappa\backslash\{0\}$ for the R^α_{EF} equivalence class of $\mathcal{A}_\eta\restriction_\alpha$. The proof follows from Lemma 3.8 and Exercise 3.3. □

Question 3.11. *Is it provable in ZFC that $E^\kappa_{\mu\text{-club}}\ \leq_B\ \cong^\kappa_T$ holds for every non-classifiable theory T and regular cardinal μ?*

Model theory session

Exercise 3.4. *Prove the claim below (Hint: Use the proof of Theorem 3.10).*

Lemma 3.12 ([6], Lemma 2)**.** *Assume T is a classifiable theory and $\mu<\kappa$ is a regular cardinal. If $\diamondsuit_\kappa(S^\kappa_\mu)$ holds then $\cong^\kappa_T$ is continuously reducible to $E^2_{\mu\text{-club}}$.*

Proof. Let $\{S_\alpha\mid\alpha\in X\}$ be a sequence testifying $\diamondsuit_\kappa(S^\kappa_\mu)$ and define the function $\mathcal{F}\colon 2^\kappa\to 2^\kappa$ by

$$\mathcal{F}(\eta)(\alpha)=\begin{cases} 1 & \text{if } \alpha\in S^\kappa_\mu\cap C_\pi\cap C_{EF},\ \mathbf{II}\uparrow EF^\kappa_\omega(\mathcal{A}_\eta\restriction_\alpha,\mathcal{A}_{S_\alpha}) \text{ and } \mathcal{A}_\eta\restriction_\alpha\models T\\ 0 & \text{otherwise.}\end{cases}$$

Claim 3.13. *$\eta\ \ \xi$ if and only $\mathcal{F}(\eta)\ E^2_{\mu\text{-club}}\ \mathcal{F}(\xi)$.*

□

The proof of the following theorems can be found in [2].

Theorem 3.14 ([2], Theorem 79)**.** *Suppose that $\kappa=\lambda^+=2^\lambda$ and $\lambda^{<\lambda}=\lambda$.*

1. *If T is unstable or superstable with OTOP, then $E^2_{\lambda\text{-club}}\leq_c\ \cong^\kappa_T$.*
2. *If $\lambda\geq 2^\omega$ and T is superstable with DOP, then $E^2_{\lambda\text{-club}}\leq_c\ \cong^\kappa_T$.* □

Theorem 3.15 ([2], Theorem 86)**.** *Suppose that for all $\gamma<\kappa$, $\gamma^\omega<\kappa$ and T is a stable unsuperstable theory. Then $E^2_{\omega\text{-club}}\leq_c\ \cong^\kappa_T$.* □

Theorem 3.16 ([6], Theorem 4)**.** *Suppose that $\kappa=\lambda^+=2^\lambda$, $\lambda^{<\lambda}=\lambda$ and $\diamondsuit_\kappa(S^\kappa_\lambda)$ holds.*

1. *If T_1 is classifiable and T_2 is unstable or superstable with OTOP, then $\cong^\kappa_{T_1}\ \leq_c\ \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2}\ \not\leq_B\ \cong^\kappa_{T_1}$.*
2. *If $\lambda\geq 2^\omega$, T_1 is classifiable and T_2 is superstable with DOP, then $\cong^\kappa_{T_1}\ \leq_c\ \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2}\ \not\leq_B\ \cong^\kappa_{T_1}$.*

Notice that if $V=L$, then $\diamondsuit_\kappa(S^\kappa_\lambda)$ holds for all $\lambda<\kappa$. Therefore in L it holds that If T is classifiable and T' not, then $\cong^\kappa_T\ \leq_c\ \cong^\kappa_{T'}$.

The last session was used to study Question 3.11. The following results answer Question 3.11 for two kind of non-classifiable theories, the proofs are omitted in this notes, due to the length of them. The proofs can be found in [7] and [11]. The main ideas of these proofs is the use of coloured trees, as it was discussed during the lecture. Coloured trees has been used to obtain Borel-reducibility results of isomorphism relations (see [2], [5], [7], and [11]).

Definition 3.17. *Let T be a stable theory. T has the orthogonal chain property (OCP), if there exist $\lambda_r(T)$-saturated models of T of power $\lambda_r(T)$, $\{\mathcal{A}_i\}_{i<\omega}$, $a \notin \cup_{i<\omega}\mathcal{A}_i$, such that $t(a, \cup_{i<\omega}\mathcal{A}_i)$ is not algebraic for every $j < \omega$, $t(a, \cup_{i<\omega}\mathcal{A}_i) \perp \mathcal{A}_j$, and for every $i \leq j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$.*

Exercise 3.5. *If T has the OCP, then T is unsuperstable.*

Lemma 3.18 ([7], Corollary 5.10)**.** *Assume T is stable and has the OCP, then $E^{\kappa}_{\omega\text{-}club} \leq_c \cong_T$.*

Corollary 3.19 ([7], Corollary 5.11)**.** *Assume T_1 is a classifiable theory and T_2 is a stable theory with the OCP, then $\cong_{T_1} \leq_c \cong_{T_2}$.*

Question 3.20. *Does there exists a stable unsuperstable theory that doesn't have OCP?*

Definition 3.21. *We say that a superstable theory T has the strong dimensional order property (S-DOP) if the following holds:*
There are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 F^a_ω-prime model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, such that it does not fork over $M_1 \cup M_2$.

Lemma 3.22 ([11], Corollary 4.15)**.** *Assume T is a theory with S-DOP and let λ be $(2^\omega)^+$, then $E^{\kappa}_{\lambda\text{-}club} \leq_c \cong_T$.*

Corollary 3.23 ([11], Corollary 4.16)**.** *Assume T_1 is a classifiable theory and T_2 is a superstable theory with S-DOP, then $\cong_{T_1} \leq_c \cong_{T_2}$.*

Question 3.24. *Does there exists a superstable theory with DOP that doesn't have S-DOP?*

Remark 3.25. *By Theorem 2.20 we conclude from Lemma 3.18 and Lemma 3.22 that, if $V = L$, then $\cong_T$ is Σ^1_1-complete for every T stable with the OCP or superstable theory with S-DOP.*

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