Lecture notes: Combinatorics

Miguel Moreno

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1 Counting

We will denote by $\mid A \mid$ the number of elements of the set A. There are four basic principles to count the elements of a set A without making a list of the elements. First we will take a look to the ideas behind these principles and then we state them properly.

- *Addition principle: If there are n ways to do a task $\mathcal{A}$ and m ways to do a task $\mathcal{B}$, then there are $n+m$ ways to do $\mathcal{A}$ or $\mathcal{B}$.*
- *Subtraction principle: If there are n ways to do a task $\mathcal{A}$ and m ways to do the tasks $\mathcal{A}$ and $\mathcal{B}$, then there are $n-m$ ways to do $\mathcal{A}$ without doing $\mathcal{B}$.*
- *Product principle: If there are n ways to do a task $\mathcal{A}$ and m ways to do a task $\mathcal{B}$, then there are $n \cdot m$ ways to do $\mathcal{A}$ and then $\mathcal{B}$.*
- *Division principle: If there are n ways to do a task $\mathcal{A}$, m ways to do a task $\mathcal{B}$, and every way to do a task $\mathcal{A}$ is first doing the task $\mathcal{C}$ and then the task $\mathcal{B}$, then there are $\frac{n}{m}$ to do the task $\mathcal{C}$.*

Theorem 1.1. *Addition principle: Let m be a non negative integer. If $\{A_i\}_{i\leq m}$ are pairwise disjoint finite sets, then $\mid \bigcup_{i\leq m} A_i \mid = \sum_{i=0}^{m} \mid A_i \mid$.*

Proof. Notice that in each side of the equation, each element is counted once because the sets are pairwise disjoint. □

Theorem 1.2. *Subtraction principle: If B is a finite set and $A \subseteq B$, then $\mid B - A \mid = \mid B \mid - \mid A \mid$.*

Proof. From the addition principle it follows that $\mid B - A \mid + \mid A \mid = \mid B \mid$. □

Theorem 1.3. *Product principle: Let A and B be two non-empty finite sets. Then the number of pairs (x, y) satisfying $x \in A$ and $y \in B$ is $\mid A \mid \times \mid B \mid$.*

Proof. Clearly $A \times B = \bigcup_{x\in B}(A \times \{x\})$. Since B is finite, by the addition principle $\mid A \times B \mid = \sum_{x\in B} \mid A \times \{x\} \mid$. On the other hand $\mid A \times \{x\} \mid = \mid A \mid$. Therefore $\mid A \times B \mid = \sum_{x\in B} \mid A \mid = \mid A \mid \times \mid B \mid$ □

Theorem 1.4. *Division principle: Let S and T be finite sets. If there is a surjective function $f : T \to S$ such that for all $x \in S$ the set $\{y \in T \mid f(y) = x\}$ has k elements, then $\mid S \mid = \frac{|T|}{k}$.*

Proof. For every $x \in S$ define the sets $R_x = \{y \in T \mid f(y) = x\}$. Since f is a function, we know that $T = \bigcup_{x\in S} R_x$. By the addition principle, $\mid T \mid = \sum_{x\in S} \mid R_x \mid$. Since $\mid R_x \mid = k$ for all $x \in S$, $\mid T \mid = \sum_{x\in S} k = \mid S \mid \cdot k$. □

Exercise 1.1. *There are n people in a conference, if everyone shake hands once with everyone at the beginning of the conference, how many hand shakes were at the beginning of the conference?*

Exercise 1.2. *How many numbers with 4 digits exist with at least one of the digits equal to 5?*

Exercise 1.3. *How many numbers with 4 digits exist with no two consecutive digits equal?*

Exercise 1.4. *In how many different ways can 10 people sit in a round table?*

Proposition 1.5. *Let A be a finite set. There are $2^{|A|}$ subsets of A.*

Proof. Let $\{x_i\}_{i\leq|A|}$ be an enumeration of the elements of A and $\mathcal{P}(A)$ the set of subsets of A. Define the function $f : \mathcal{P}(A) \to \Pi_{i\leq|A|}\{\emptyset, x_i\}$ by $f(S) = (y_1, y_2, \ldots, y_{|A|})$, where $y_i = x_i$ if $x_i \in S$, and $y_i = \emptyset$ otherwise. Clearly, f is one-to-one, by the division principle $\mid \mathcal{P}(A) \mid=\mid \Pi_{i\leq|A|}\{\emptyset, x_i\} \mid$. By the multiplication principle, $\mid \Pi_{i\leq|A|}\{\emptyset, x_i\} \mid= 2^{|A|}$, we conclude $\mid \mathcal{P}(A) \mid= 2^{|A|}$. □

Exercise 1.5. *How many numbers of ten digits that do not start zero have at least two digits equal?*

Exercise 1.6. *How many numbers of ten digits that do not start zero have exactly two digits equal?*

Exercise 1.7. *In how many ways can be distributed n gifts into k kids if not every kid needs to get a gift and kids can get multiple gifts?*

Exercise 1.8. *How many odd numbers have all the digits different?*

Exercise 1.9. *There are n teams in a football tournament. How many matches will have the tournament if every team has to play exactly once against any other team?*

Exercise 1.10. *How many pairs (X, Y) exists such that $X, Y \subseteq \{0, 1, \ldots, n\}$ and X and Y are disjoint?*

2 Combinations and Permutations

Definition 2.1. *$n!$ is n-factorial and it is defined as $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$. For $n = 0$ we define $0! = 1$.*

Proposition 2.2. *Let S be a finite set, $\mid S \mid= n$. The number of ways to arrange all elements of S is $n!$.*

Proof. Let $\mathcal{A}_i$ be the task of choosing the i-th element of the arrangement after choosing the first $i-1$ elements of the arrangement. For every $i \leq n$ there are $n-(i-1)$ ways of choosing the i-th element of the arrangement, $i-1$ elements have been chose and the chose of the i-th element is made on the remaining $n-(i-1)$ elements. By the product principle, there are $n!$ ways to do the task $\mathcal{A}_1$, then $\mathcal{A}_2$, then $\mathcal{A}_3$, and so on until the task $\mathcal{A}_n$. □

A permutation π of a set A is a bijective function from A to A, $\pi : A \to A$. We will denote by S_n the set of permutations of $\{1, 2, \ldots, n\}$. So $\mid S_n \mid= n!$ for all n.

Proposition 2.3. *Let S be a finite set, $\mid S \mid= n$, and k a positive integer such that $k \leq n$. The number of ways to make a list of k elements from the set S, without repetitions, is $\frac{n!}{(n-k)!}$.*

Proof. For all $i \leq k$, define the task $\mathcal{A}_i$ as Proposition 2.2. There are $n-(i+1)$ ways to do the task $\mathcal{A}_i$, therefore there are $n \cdot (n-1) \cdot \cdots \cdot (n-(k-2)) \cdot (n-(k-1))$ ways to do the task $\mathcal{A}_1$, then $\mathcal{A}_2$, then $\mathcal{A}_3$, and so on until the task $\mathcal{A}_k$. By multiplying by $\frac{(n-k)!}{(n-k)!}$ we obtain $\frac{n!}{(n-k)!}$. □

We will denote by $[n]_k$ the set of lists of k elements from the set $\{1, 2, \ldots, n\}$, and $[n] = \bigcup_{k=1}^{n}[n]_k$.

Proposition 2.4. *Let S be a finite set, $\mid S \mid= n$, and k a positive integer such that $k \leq n$. The number of subsets of S with k elements is $\frac{n!}{k!\cdot(n-k)!}$.*

Proof. By Proposition 2.3 there are $\frac{n!}{(n-k)!}$ lists of k elements from the set S, without repetitions. Denote by $\mathcal{P}^k(A)$ the set $\{x \subseteq A \mid\mid x \mid= k\}$. Define the function $f : [n]_k \to \mathcal{P}^k(S)$ by $f((y_1, y_2, \ldots, y_k)) = \{y_1, y_2, \ldots, y_k\}$. By 2.2 there are $k!$ ways to arrange k elements, so $k! =\mid f^{-1}[x] \mid$ for all $x \in \mathcal{P}^k(S)$. By the division principle, $\mid \mathcal{P}^k(A) \mid= \frac{|[n]_k|}{k!}$. By Proposition 2.3 we conclude that $\mid \mathcal{P}^k(A) \mid= \frac{n!}{k!\cdot(n-k)!}$. □

We define $\binom{n}{k}$ by $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$.

Fact 2.5. $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

Proof. Let A be a set with n elements. By Proposition 1.5 we know that $\mid \mathcal{P}(A) \mid= 2^n$. On the other side by $\mid \mathcal{P}(A) \mid= \sum_{k=0}^{n} \mid \mathcal{P}^k(A) \mid= \sum_{k=0}^{n} \binom{n}{k}$. □

Theorem 2.6 (Binomial Theorem)**.** $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof. We will prove the theorem by induction over n. The case $n = 0$, we know that $1 = (x+y)^0$. On the other hand we know that $\binom{0}{0} = \frac{0!}{0!} = 1$, so $\binom{0}{0}x^0y^0 = 1$. Let us suppose that n is such that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$.

Successor step. We know that $(x+y)^{n+1} = (x+y)(x+y)^n$, by the induction hypothesis, $(x+y)^{n+1} = (x+y)\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$. Therefore $(x+y)^{n+1} = \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^ky^{n+1-k}$. We conclude that $(x+y)^{n+1} = \binom{0}{0}y^{n+1} + \sum_{k=0}^{n}(\binom{n}{k} + \binom{n}{k+1})x^{k+1}y^{n+1-k}$. To finish the proof we just need to show that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!\cdot(n-k)!} + \frac{n!}{(k+1)!\cdot(n-(k+1))!}$$

$$\binom{n}{k} + \binom{n}{k+1} = n!\left(\frac{k+1}{(k+1)!\cdot(n-k)!} + \frac{n-k}{(k+1)!\cdot(n-k)!}\right)$$

$$\binom{n}{k} + \binom{n}{k+1} = n!\left(\frac{n+1}{(k+1)!\cdot((n+1)-(k+1))!}\right) = \binom{n+1}{k+1}$$

□

Exercise 2.1. *In how many ways can be n towers arranged in a $n\times n$ chess board such that there are not two towers in the same column or row?*

Exercise 2.2. *How many solutions have the equation $x_1 + x_2 + \cdots + x_k = n$ with x_i a positive integer for all $0 < i \leq k$? Two solutions $(x_1,\ldots,x_k)$ and $(y_1,\ldots,y_k)$ are equal if and only if $x_i = y_i$ for all $0 < i \leq k$.*

Exercise 2.3. *There is a tower in the position $(1,1)$ in a $n\times m$ chess board. In how many ways can the tower reach the position (n,m), if the tower is only allowed to increase the row or column in each move?*

Exercise 2.4. *How many words with eleven letters can be written with 4 a's, 4 b's, 2 c's, and one d?*

Exercise 2.5. *How many diagonals has a polygon with n sides?*

Exercise 2.6. *Give a different of proof to the identity:*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Exercise 2.7. *Prove the following identity:*

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

Theorem 2.7. *The following identity holds for all positive integers $0 < k \leq n$:*

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

Proof. It follows by the formula $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$. □

Exercise 2.8. *Give a different proof for Theorem 2.7.*

Exercise 2.9. *Determine a formula for $\sum_{k=0}^{n}k^3\binom{n}{k}$.*

Exercise 2.10. *How many words of length n can be written with the letters a_1, a_2, $\ldots$, a_k, if a_i appears n_i times in the word? (notices that $n_1 + n_2 + \cdots + n_k = n$.)*

Let n and k be positive integers, and let n_1, $n_2,\ldots$, n_k be positive integers such that $n_1 + n_2 + \cdots + n_k = n$. We define the multinomial coeficient by

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

3 Two more principles

Theorem 3.1 (Pigeonhole principle)**.** *Let A_1, A_2, ..., A_k finite pairwise disjoint sets. If $\mid A_1 \cup A_2 \cup \cdots \cup A_k \mid > k \cdot r$, then there is at least one set A_i such that $\mid A_i \mid > r$.*

Proof. We proceed by contradiction. Suppose for all $i \leq k$, $\mid A_i \mid \leq r$. By the addition principle,

$$\mid A_1 \cup A_2 \cup \cdots \cup A_k \mid = \sum_{i=1}^{k} \mid A_i \mid \leq k \cdot r.$$

This contradicts the assumption $\mid A_1 \cup A_2 \cup \cdots \cup A_k \mid > k \cdot r$. □

Example 3.1. *Consider the sequence $1, 3, 7, 15, \ldots$, where $a_n = 2^n - 1$. Let q be any odd integer, then there is $i \leq q$ such that a_i is a multiple of q.*

Solution. Suppose there is q an odd integer such that for all $i \leq q$, a_i is not a multiple of q. We can re-write the elements a_n, by $a_n = d_n q + r_n$ where r_n and d_n are integers satisfying $0 < r_n < q$ and $\lfloor a_n/q \rfloor$. By the Pigeonhole principle, there are $1 \leq s < t \leq q$ such that $r_s = r_t$. Notice that $a_s < a_t$, so $0 < a_t - a_s = (2^t - 1) - (2^s - 1)$. Replacing $a_t - a_s$ by $(d_t q + r_t) - (d_s q + r_s)$ we get

$$d_t q + r_t - d_s q - r_s = 2^t - 1 - 2^s + 1$$

$$(d_t - d_s)q = 2^s(2^{t-s} - 1)$$

$$(d_t - d_s)q = 2^s a_{t-s}.$$

Since $(d_t - d_s)q$ is a multiple of q, $2^s a_{t-s}$ is a multiple of q. But q is odd, so a_{t-s} is a multiple of q, a contradiction.

Exercise 3.1 (Strong form of the Pigeonhole principle)**.** *Let $q_1, q_2, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - n + 1$ objects are distributed into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, or the third box contains at least q_3 objects, or ..., or the n-box contains at least q_n objects.*

Exercise 3.2. *Show that if $n + 1$ distinct integers are chosen from the set $\{1, 2, \ldots, (k+1)n\}$ then there are always two which differ by at most k.*

Exercise 3.3. *Show that every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of $n^2 + 1$ real numbers contains either an increasing sequence of length $n + 1$, or a decreasing sequence of length $n + 1$.*

Exercise 3.4. *Show that if $A \subseteq \{1, 2, \ldots, 2n\}$ with $\mid A \mid = n + 1$, then there are $a, b, c, d \in A$ such that $a + b = 2n + 1$ and $d - c = n$.*

Exercise 3.5. *In a 4×82 chess board every square is coloured red, blue, or green. Show that no matter how the chess board is coloured, there are four squares with the same color, which make a rectangle.*

Theorem 3.2 (Inclusion-exclusion principle)**.** *Let S be a set and A_1, A_2, ..., and A_k be finite subsets of S. Then*

$$\mid A_1 \cup A_2 \cup \cdots \cup A_k \mid = \sum_{j=1}^{k} (-1)^{j-1} \left(\sum_{i_1 < i_2 < \cdots < i_j} \mid A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j} \mid \right) \tag{1}$$

where $(i_1, i_2, \ldots, i_j)$ range all j-element of $[n]$.

Proof. We will proof that if $x \in S$ is such that $x \in A_1 \cup A_2 \cup \cdots \cup A_k$, then x is counted only one time in the right side of (1) (no matter how complicated the right side looks). Without loss of generality, we can assume that x is an element of the sets A_1, , A_2, ..., and A_m only. x is counted m times in $\sum_{i_1} \mid A_{i_1} \mid$. x is counted $\binom{m}{2}$ times in $\sum_{i_1 < i_2} \mid A_{i_1} \cap A_{i_2} \mid$. In general x is counted $\binom{m}{j}$ times in $\sum_{i_1 < i_2 < \cdots < i_j} \mid A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j} \mid$. Therefore, x is counted $\sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j}$ times on the right side of (1). By Theorem 2.6 $\sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j} = -\sum_{j=1}^{m} (-1)^j \binom{m}{j} = -\sum_{j=1}^{m} (-1)^j \binom{m}{j} + 1 - 1 = -\sum_{j=0}^{m} (-1)^j \binom{m}{j} + 1$. By the binomial theorem, $-\sum_{j=0}^{m} (-1)^j \binom{m}{j} + 1 = -(1-1)^m + 1 = 1$. □

Theorem 3.3. *The number of permutations in S_n that have no fix points, i.e. $\pi(i) \neq i$, is*

$$n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right).$$

Proof. Let us denote by D_n the number of permutations in S_n that have no fix points, and for each $i \leq n$, let P_i be the set of permutations in S_n with i as a fix point, i.e. $\pi \in P_i \Leftrightarrow \pi(i) = i$. By the inclusion-exclusion principle,

$$D_n = n! - \mid P_1 \cup P_2 \cup \cdots \cup P_n \mid$$

by the inclusion-exclusion principle,

$$n! - \mid P_1 \cup P_2 \cup \cdots \cup P_n \mid = n! - \sum_{j=1}^{n} (-1)^{j-1} \left(\sum_{i_1 < i_2 < \cdots < i_j} \mid P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_j} \mid \right)$$

so

$$D_n = n! + \sum_{j=1}^{n} (-1)^{j} \left(\sum_{i_1 < i_2 < \cdots < i_j} \mid P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_j} \mid \right).$$

Since in $P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_j}$ the elements $i_1, i_2, \cdots, i_j$ are fixed, we can know $\mid P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_j} \mid$ by counting the permutations of the other $n - j$ elements. By Proposition 2.2 we conclude that $\mid P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_j} \mid = (n-j)!$, therefore

$$D_n = n! + \sum_{j=1}^{n} (-1)^{j} \left(\sum_{i_1 < i_2 < \cdots < i_j} (n-j)! \right).$$

By Proposition 2.4 we know that there are $\binom{n}{j}$ subsets of $\{1, 2, \ldots, n\}$ with j elements, thus

$$D_n = n! + \sum_{j=1}^{n} (-1)^{j} \binom{n}{j} (n-j)! = \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} (n-j)!.$$

$$D_n = \sum_{j=0}^{n} (-1)^{j} \frac{n!}{j!}.$$

We conclude that

$$D_n = n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right).$$

□

Exercise 3.6. *Let $k \leq n$ be positive integers. Determine how many surjective functions are from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, k\}$.*

Exercise 3.7. *How many number of integers smaller or equal to 1000 are divisible by none 2, 3, 5?*

Exercise 3.8. *What is the sum of all integers from 1 to 100 that are multiples of 2, 3, or 5?*

Exercise 3.9. *Let $A = \{-n, -(n-1), \ldots, -2, -1, 1, 2, \ldots, n-1, n\}$ (notice that $0 \notin A$). In how many ways can the elements of A be arranged in such a way that k and $-k$ are not consecutive?*

Exercise 3.10. *For any positive integer n, let $\varphi(n)$ be the number of positive integers $k \leq n$ so that the greatest common divisor of k and n is 1.*

1. *Let $a = p_1 \cdot p_2 \cdot \cdots \cdot p_m$, where p_i is a prime number for every $i \leq m$. Show that $\varphi(a) = (p_1 - 1) \cdot (p_2 - 1) \cdot \cdots \cdot (p_m - 1)$.*

2. *Let n and m be relatively prime (their the greatest common divisor is 1). In a matrix $m \times \varphi(n)$ are written the numbers $k < nm$ that are relatively prime to n satisfying the following:*

   - $a_{ij} < a_{ir}$ *if and only if* $j < r$.
   - $a_{ij} \in \{in + r \mid 0 \leq s < n\}$.

   Show that the set $C_j = \{a_{ij} \mid a_{ij}$ is relatively prime with $m\}$ has $\varphi(m)$ elements.

3. *Show that if n and m be relatively prime, then $\varphi(nm) = \varphi(n)\varphi(m)$.*

4. *Show that $\varphi(p^d) = p^{d-1}(p-1)$ for all $0 < d$. Conclude a formula for $\varphi(n)$.*

4 Generating Functions

Generating function is a way to encode successions in power series. The power series use are allowed to diverge, the series is not necessarly a function. We will start by describing the idea of the method and then we will use the method to solve some problems. The method of generating functions for a succesion is described in [6] as follows (some cases are more complex than others):

1. Identify a recurrence formula that describes the succession. For example the formula $a_{i+2} = a_i + a_{i+1}$ describes the Fibonacci sequence if $a_0 = 0$ and $a_1 = 1$. The formula $a_{i+1} = a_i + 1$ describes the positive integers if $a_0 = 1$.
2. Define the generating function as a power series in terms of the succession. For example, given the succession $\{a_i\}_{i=0}^{\infty}$ we can define the generating function $G(x) = \sum_{i=0}^{\infty} a_i x^i$.
3. Multiply both sides of the recurrence by x^m (m depends on the recurrence formula) and sum over all the values of n for which the recurrence holds.
4. Express both sides of the resulting equation explicitly in terms of the generating function, $G(x)$.
5. Solve the resulting equation for $G(x)$.
6. For an exact formula of the sequence, expand $G(x)$ into a power series.

Ordinal generating functions

Definition 4.1. *Given a succession $\{a_i\}_{i=0}^{\infty}$, its ordinal generating function is*

$$G(x) = \sum_{i=0}^{\infty} a_i x^i.$$

We define the following operations between ordinal generating functions.

I. If $G(x) = \sum_{i=0}^{\infty} a_i x^i$ and $F(x) = \sum_{i=0}^{\infty} b_i x^i$ are the ordinal generating functions for the successions $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ respectively. Then we define $G + F$ by

$$(G + F)(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i,$$

$G + F$ is the ordinal generating function for the succession $c_i = a_i + b_i$.

II. If $G(x) = \sum_{i=0}^{\infty} a_i x^i$ and $F(x) = \sum_{i=0}^{\infty} b_i x^i$ are the ordinal generating functions for the successions $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ respectively. Then we define GF by

$$(GF)(x) = \sum_{i=0}^{\infty} (\sum_{j=0}^{i} a_j b_{i-j}) x^i,$$

GF is the ordinal generating function for the succession $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

III. If $F(x) = \sum_{i=0}^{\infty} a_i x^i$ is the ordinal generating function for the succession $\{a_i\}_{i=0}^{\infty}$. Then we define F' by

$$F'(x) = \sum_{i=1}^{\infty} (i a_i) x^{i-1},$$

F' is the ordinal generating function for the succession $c_{i-1} = i a_i$.

VI. If $F(x) = \sum_{i=0}^{\infty} a_i x^i$ is the ordinal generating function for the succession $\{a_i\}_{i=0}^{\infty}$. Then we define $x^t F$ by

$$x^t F(x) = \sum_{i=0}^{\infty} (a_i) x^{i+t},$$

$x^t F$ is the ordinal generating function for the succession $c_i = a_{i-t}$.

Example 4.1. *The Fibonacci sequence is defined as follows:*

- $F_0 = 0$.

- *$F_1 = 1$.*
- *For all $i \geq 0$, $F_{i+2} = F_i + F_{i+1}$.*

We will use the generating functions method described above to find a closed formula for the Fibonacci sequence. Let us start by finding the generating function of the Fibonacci sequence.

1. *The first step is to identify a recurrence formula, this one is $F_{i+2} = F_i + F_{i+1}$.*
2. *Now we define $G(x) = \sum_{i=0}^{\infty} F_i x^i$ as the ordinal generating function of the Fibonacci sequence.*
3. *The next step is to multiply by x^m, because of the recurrence formula, we choose $m = i+2$. By the recursive formula of Fibonacci (item 3), we know that*
$$F_{i+2}x^{i+2} = F_i x^{i+2} + F_{i+1}x^{i+2}$$
holds for all $i \geq 0$, therefore
$$F_{i+2}x^{i+2} = x^2 F_i x^i + x F_{i+1} x^{i+1}. \tag{2}$$
By (2) we know that
$$\sum_{i=0}^{\infty} F_{i+2}x^{i+2} = \sum_{i=0}^{\infty} x^2 F_i x^i + \sum_{i=0}^{\infty} x F_{i+1} x^{i+1}$$
therefore
$$\sum_{i=2}^{\infty} F_i x^i = x^2 \sum_{i=0}^{\infty} F_i x^i + x \sum_{i=1}^{\infty} F_i x^i.$$
4. *Since $F_0 = 0$ and $F_1 = 1$,*
$$G(x) - x = x^2 G(x) + xG(x).$$
5. *We conclude that*
$$G(x) = \frac{x}{1-x-x^2}.$$
6. *Now we will find a formula for F_i from the generating function $G(x)$ by writing $\frac{x}{1-x-x^2}$ as a power series. Since $r_1 = -\frac{1+\sqrt{5}}{2}$ and $r_2 = -\frac{1-\sqrt{5}}{2}$ are the zeros of $1-x-x^2$, we can write $\frac{x}{1-x-x^2}$ as $\frac{-x}{(x+\frac{1+\sqrt{5}}{2})(x+\frac{1-\sqrt{5}}{2})}$. Now we determined A and B such that*
$$\frac{A}{x+\frac{1+\sqrt{5}}{2}} + \frac{B}{x+\frac{1-\sqrt{5}}{2}} = \frac{1}{(x+\frac{1+\sqrt{5}}{2})(x+\frac{1-\sqrt{5}}{2})}$$
holds. By solving the system
$$A + B = 0$$
$$B(\frac{1+\sqrt{5}}{2}) + A(\frac{1-\sqrt{5}}{2}) = 1$$
we obtain that $A = \frac{-1}{\sqrt{5}}$ and $B = \frac{1}{\sqrt{5}}$. We conclude that $G(x) = \frac{-x}{\sqrt{5}}\left(\frac{-1}{x-r_1} + \frac{1}{x-r_2}\right)$ therefore
$$G(x) = \frac{x}{\sqrt{5}}\left(\frac{r_2}{r_2}\cdot\frac{1}{x-r_1} + \frac{r_1}{r_1}\cdot\frac{-1}{x-r_2}\right) = \frac{x}{\sqrt{5}}\left(\frac{r_2}{r_2 x - r_2\cdot r_1} + \frac{-r_1}{r_1 x - r_1\cdot r_2}\right).$$
Since $r_1 \cdot r_2 = -1$
$$G(x) = \frac{x}{\sqrt{5}}\left(\frac{-\frac{1-\sqrt{5}}{2}}{1-\frac{1-\sqrt{5}}{2}x} + \frac{\frac{1+\sqrt{5}}{2}}{1-\frac{1+\sqrt{5}}{2}x}\right).$$
Since $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$,
$$G(x) = \frac{x}{\sqrt{5}}\left(-\frac{1-\sqrt{5}}{2}\sum_{n=0}^{\infty}(\frac{1-\sqrt{5}}{2}x)^n + \frac{1+\sqrt{5}}{2}\sum_{n=0}^{\infty}(\frac{1+\sqrt{5}}{2}x)^n\right).$$
Therefore
$$G(x) = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)x^{n+1}\right).$$
Finally we conclude that $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

Exercise 4.1. *Show that* $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ *satisfies* $F_{i+2} = F_i + F_{i+1}$.

Exercise 4.2. *The sequence* $\{a_i\}_{i=0}^{\infty}$ *is defined by* $a_n = n$. *Determined the ordinal generating function of* $\{a_i\}_{i=0}^{\infty}$.

Exercise 4.3. *The sequence* $\{a_i\}_{i=0}^{\infty}$ *is defined by* $a_n = br^n$ *for some constants* b *and* r. *Determined the ordinal generating function of* $\{a_i\}_{i=0}^{\infty}$.

Exercise 4.4. *Let* $a_n = \sum_{m=0}^{n} m$. *Find the ordinal generating function of* $\{a_i\}_{i=0}^{\infty}$.

Exercise 4.5. *The sequence* $\{a_i\}_{i=0}^{\infty}$ *is defined by* $a_{n+1} = 2a_n + n$ *for all* $n \geq 0$ *and* $a_0 = 1$.

- *Determine the ordinal generating function of* $\{a_i\}_{i=0}^{\infty}$.
- *Determine a formula for the sequence.*

Exercise 4.6. *The sequence* $\{a_i\}_{i=0}^{\infty}$ *is defined by* $a_{n+2} = 2a_{n+1} - a_n$ *for all* $n \geq 0$, $a_0 = 0$, *and* $a_1 = 1$.

- *Determine the ordinal generating function of* $\{a_i\}_{i=0}^{\infty}$.
- *Determine a formula for the sequence.*

Exercise 4.7. *Let* $F(x)$ *be the ordinal generating function of the sequence* $\{a_i\}_{i=0}^{\infty}$. *Find the ordinal generating function of* $\{b_i\}_{i=0}^{\infty}$ *in terms of* $F(x)$, *where* $b_n = \sum_{m=0}^{n} a_m$.

Exponential generating functions

The exponential function $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ allows us to work with another kind of generation functions different from the ordinal generation functions.

Definition 4.2. *Given a succession* $\{a_i\}_{i=0}^{\infty}$, *its exponential generating function is*

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

We define the following operations between exponential generating functions.

I. If $G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ and $F(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} x^k$ are the exponential generating functions for the successions $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ respectively. Then we define $G + F$ by

$$(G + F)(x) = \sum_{k=0}^{\infty} \frac{a_k + b_k}{k!} x^k,$$

$G + F$ is the exponential generating function for the succession $c_i = a_i + b_i$.

II. If $G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ and $F(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} x^k$ are the exponential generating functions for the successions $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ respectively. Then we define GF by

$$(GF)(x) = \sum_{k=0}^{\infty} \frac{(\sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j})}{k!} x^k,$$

GF is the exponential generating function for the succession $c_k = \sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j}$.

III. If $F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is the exponential generating function for the succession $\{a_k\}_{k=0}^{\infty}$. Then we define F' by

$$F'(x) = \sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} x^{k-1},$$

F' is the exponential generating function for the succession $c_k = a_{k+1}$.

VI. If $F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is the exponential generating function for the succession $\{a_k\}_{k=0}^{\infty}$. Then we define $x^t F$ by

$$x^t F(x) = \sum_{k=0}^{\infty} \frac{a_k (k+1)(k+2)\cdots(k+t)}{(k+t)!} x^{k+t},$$

$x^t F$ is the exponential generating function for the succession $c_k = a_{k-t}(k+1-t)(k+2-t)\cdots(k)$ for $k \geq t$, and $c_k = 0$ for $k < t$.

Example 4.2. *We will show a proof of the binomial theorem by the use of generation functions. The key for this property is the property* $e^a e^b = e^{a+b}$.

$$e^{xt}e^{yt} = e^{(x+y)t}$$

$$\left(\sum_{n=0}^{\infty} \frac{x^n t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n t^n}{n!}\right) = \left(\sum_{n=0}^{\infty} \frac{(x+y)^n t^n}{n!}\right)$$

$$\sum_{n=0}^{\infty} \frac{\left(\sum_{j=0}^{n} \binom{n}{k} x^k y^{n-k}\right)}{n!} t^n = \left(\sum_{n=0}^{\infty} \frac{(x+y)^n t^n}{n!}\right).$$

Therefore $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$.

Let us generalize the Binomial theorem.

Definition 4.3. *For every positive integer k and real number r, let us define* $\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$. *For a real number r let* $\binom{r}{0} = 1$.

Theorem 4.4 (Generalized binomial theorem)**.** *If r is a real number and* $\mid x \mid < \mid y \mid$, *then*

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k y^{r-k}.$$

Proof. Let us start by proving the theorem for $y = 1$. By the Taylor series, the function $f(x) = (1+x)^r$ is equal to $\sum_{n\geq 0} f^{(n)}(0)\frac{x^n}{n!}$ in a neighborhood of 0.

Claim 4.5. $f^{(n)}(x) = r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n}$.

Proof. We proceed by induction. It is clear for $n = 1$. Suppose n is such that $f^{(n)}(x) = r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n}$. Then $f^{(n+1)}(x) = r(r-1)(r-2)\cdots(r-n+1)(r-n)(1+x)^{r-n-1}$. □

We conclude that $(1+x)^r = \sum_{n\geq 0} r(r-1)(r-2)\cdots(r-n+1)\frac{x^n}{n!} = \sum_{k=0}^{\infty} \binom{r}{k} x^k$.

To complete the proof, notice that since $\mid x \mid < \mid y \mid$, then we can replace x by $\frac{x}{y}$ in the previous case. We obtain

$$(1+\frac{x}{y})^r = \sum_{k=0}^{\infty} \binom{r}{k} \frac{x^k}{y^k}.$$

By multiplying by y^r to both sides, we obtain

$$(y+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k y^{r-k}.$$

□

Example 4.3. *Let us use generating functions to find the formula in Theorem 3.3, So far we have defined D_n only for $n \geq 1$. By convention, we define $D_0 = 1$. As it was shown in the lecture, if $D_0 = 0$ we obtain the same result as in Theorem 3.3 (for $n \geq 1$) but the calculations are a bit more complicated. We will start by showing the following claim.*

Claim 4.6. *The following identities hold for all $n \geq 3$.*

1. $D_n = (n-1)(D_{n-1} + D_{n-2})$.
2. $D_n = nD_{n-1} + (-1)^n$.

Proof. 1. Let $\pi \in S_n$ be a permutation with no fix points and $\pi(1) = k$. We have the following two cases for π:

- $\pi(k) = 1$.
- $\pi(k) \neq 1$.

For the first case, it is easy to see that there are $(n-1)D_{n-2}$ of these permutations. For the second case, let π' be a permutation of the set $\{2, 3, \ldots, n\}$ with no fix points. Define $\mathcal{F} : \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, n\}$ by

$$\mathcal{F}(i) = \begin{cases} k & \text{if } i = 1 \\ \pi'(i) & \text{if } i \neq 1 \text{ and } \pi'(i) \neq k \\ 1 & \text{if } \pi'(i) = k \end{cases}$$

It is clear that from every permutation of the second case ($\pi(k) \neq 1$) we can obtain a permutation π' of the set $\{2, 3, \ldots, n\}$ with no fix points by using $\mathcal{F}$. In the same way every permutation of the second case can be obtain from a permutation π' of the set $\{2, 3, \ldots, n\}$ with no fix points by using $\mathcal{F}$. We conclude that there are $(n-1)D_{n-1}$ permutation of in the second case.

Therefore by the addition principle, $D_n = (n-1)(D_{n-1} + D_{n-2})$.

2. We will prove this item by induction. If $n = 3$, we know that $D_3 = 2$ and $D_2 = 1$. It is easy to check that $D_3 = 3D_2 + (-1)^3$ holds. Let us suppose that $n \geq 3$ is such that $D_n = nD_{n-1} + (-1)^n$ holds. By adding $nD_n + (-1)^{n+1}$ to both sides we obtain:

$$D_n + nD_n + (-1)^{n+1} = nD_{n-1} + (-1)^n + nD_n + (-1)^{n+1}$$

$$(n+1)D_n + (-1)^{n+1} = n(D_{n-1} + D_n).$$

By the prviuos item,

$$(n+1)D_n + (-1)^{n+1} = D_{n+1}.$$

□

Now we can use the generating function method.

1. *The first step is to identify a recurrence formula, this one is $D_n = nD_{n-1} + (-1)^n$. Notice that by the previous Claim this formula holds for $n \geq 3$, but it is easy to check that this formula also holds for $n = 1, 2$.*

2. *Now we define $D(x) = \sum_{k=0}^{\infty} D_k \frac{x^k}{k!}$ as the exponential generating function of the sequence $\{D_i\}_{0 \leq i}$.*

3. *The next step is to multiply by $\frac{x^m}{m!}$, because of the recurrence formula, we choose $m = k$. By the recursive formula (item 1), we know that*

$$D_k \frac{x^k}{k!} = xD_{k-1} \frac{x^{k-1}}{(k-1)!} + (-1)^k \frac{x^k}{k!}$$

holds for all $k \geq 1$, therefore

$$\sum_{k=1}^{\infty} D_k \frac{x^k}{k!} = \sum_{k=1}^{\infty} xD_{k-1} \frac{x^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}$$

therefore

$$\sum_{k=1}^{\infty} D_k \frac{x^k}{k!} = x \sum_{k=0}^{\infty} D_k \frac{x^k}{(k)!} + \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}$$

4. *Since $D_0 = 1$, By adding $D_0 \frac{x^0}{0!}$ to both sides, we obtain*

$$\sum_{k=0}^{\infty} D_k \frac{x^k}{k!} = x \sum_{k=0}^{\infty} D_k \frac{x^k}{(k)!} + \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!} + 1.$$

Finally since $(-1)^0 \frac{x^0}{0!} = 1$, we conclude

$$D(x) = xD(x) + e^{-x}$$

5. *We conclude that*

$$D(x) = \frac{e^{-x}}{1-x}.$$

6. *Now we will find a formula for D_k from the generating function $D(x)$ by writing e^{-x} and $\frac{1}{1-x}$ as a power series.*

$$D(x) = \left(\sum_{k=0}^{\infty}(-1)^k \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} x^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{(-1)^j}{j!}\right)x^k.$$

We compare the coefficients of both sides to obtain $\frac{D_k}{k!}x^k = \left(\sum_{j=0}^{k}\frac{(-1)^j}{j!}\right)x^k$, we conclude $D_k = k!\sum_{j=0}^{k}\frac{(-1)^j}{j!}$. Notice that $D_0 = 0!\frac{(-1)^0}{0!} = 1$.

Exercise 4.8. *Let $p_n(k)$ be the number of permutations in S_n that have exactly k fix points. Show that $p_n(k) = \binom{n}{k}D_{n-k}$, for all $n \geq 1$.*

Example 4.4. *We will show that*

$$\sum_{k=0}^{n} kp_n(k) = n!$$

holds for all $n \geq 1$. Let us define the following generating functions:

$$F(x) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty}(n+1)\frac{x^{n+1}}{(n+1)!} = xe^x$$

$$D(x) = \sum_{k=0}^{\infty} D_k\frac{x^k}{k!} = \frac{e^{-x}}{1-x}.$$

By multiplying these two generating functions, we obtain

$$FD(x) = \sum_{n=0}^{\infty}\frac{\left(\sum_{k=0}^{n}\binom{n}{k}kD_{n-k}\right)}{n!}x^n = \frac{x}{1-x}$$

$$\frac{x}{1-x} = x\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}x^{n+1} = \sum_{n=1}^{\infty}x^n.$$

We compare the coefficients of both sides to obtain $x^n = \frac{\left(\sum_{k=0}^{n}\binom{n}{k}kD_{n-k}\right)}{n!}x^n$, thus $n! = \sum_{k=0}^{n}\binom{n}{k}kD_{n-k}$. By Exercise 4.8 we conclude $n! = \sum_{k=0}^{n} kp_n(k)$.

5 More on generating functions

Example 5.1. *Catalan numbers appears very often in combinatorial problems, in many counting problems the solution is given by the Catalan numbers. Because of this, there are many equivalent ways to define the Catalan numbers. These numbers satisfy the following recursion formula.*

- $C_0 = 1$.
- $C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$ *for all* $n \geq 1$.

We can use the generating functions method to find a formula for the Catalan numbers. Let us find the ordinal generating function of the sequence of Catalan numbers and the formula from the generating function.

1. *The first step is to identify a recurrence formula, this one is $C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$.*
2. *Now we define $G(x) = \sum_{i=0}^{\infty} C_i x^i$ as the generating function of the Catalan sequence.*
3. *The next step is to multiply by x^m, because of the recurrence formula, we choose $m = n$. Now we sum over all possible values of n to obtain*

$$\sum_{n=1}^{\infty} C_n x^n = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n.$$

4. *Since $C_0 = 1$,*

$$G(x) - 1 = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n. \tag{3}$$

We can see that the expression $\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$ *looks almost the same as the expression* $\sum_{i=0}^{\infty}(\sum_{j=0}^{i} a_j b_{i-j})x^i$ *from II. This strongly suggests us to write* $\sum_{n=0}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$ *as the multiplication of two generating functions. In II the generation functions used are the ones defined from* $\{a_i\}_{i=0}^{\infty}$ *and* $\{b_i\}_{i=0}^{\infty}$, *so the generation functions we need to express* $\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$ *as the multiplication of generating function should be the ones defined from* $\{C_i\}_{i=0}^{\infty}$, $G(x)$. *By doing this, we obtain*

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n = x\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^{n-1} = xG(x)G(x),$$

now we can write (3) as

$$G(x) - 1 = xG(x)G(x) \tag{4}$$

5. *We conclude that*

$$xG(x)^2 - G(x) + 1 = 0$$

solving this equation we know that there are two possibilities for $G(x)$,

$$G(x) = \frac{1-\sqrt{1-4x}}{2x}$$

or

$$G(x) = \frac{1+\sqrt{1-4x}}{2x}.$$

Notice that by L'Hopital rule, $\lim_{x=0}\frac{1-\sqrt{1-4x}}{2x} = 1$. *Clearly* $G(x) = \frac{1+\sqrt{1-4x}}{2x}$ *has a pole at* $x = 0$. *Since* $G(x)$ *determines the Catalan numbers, by the way it was defined,* $G(0) = C_0 = 1$, *therefore*

$$G(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{5}$$

6. *We know that* $G(x) = \frac{1-\sqrt{1-4x}}{2x}$ *is the generating function for the Catalan numbers. Let us expand* $\frac{1-\sqrt{1-4x}}{2x}$ *as a power series. By the binomial theorem we know that*

$$(1-4x)^{1/2} = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-4x)^k. \tag{6}$$

First of all we have to find an expression for $\binom{\frac{1}{2}}{k}$, *for* $k = 0$ *we know that* $\binom{\frac{1}{2}}{k} = 1$. *For* $k \geq 1$, *by definition*

$$\binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!}$$

$$\binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})\cdots(\frac{1-2(k-1)}{2})}{k!}$$

$$\binom{\frac{1}{2}}{k} = (-1)^{k-1}\frac{\frac{1}{2}(\frac{1}{2})(\frac{3}{2})\cdots(\frac{2(k-1)-1}{2})}{k!}$$

$$\binom{\frac{1}{2}}{k} = (-1)^{k-1}\frac{1}{2^k}\cdot\frac{1\cdot 1\cdot 3\cdot\cdots\cdot(2(k-1)-1)}{k!}.$$

Notice that $2^k k! = 2\cdot 4\cdot 6\cdot\cdots\cdot 2k$, *so multiplying by* $\frac{2^k k!}{2^k k!}$ *in the right side of the previous equation we obtain:*

$$\binom{\frac{1}{2}}{k} = (-1)^{k-1}\frac{1}{2^k}\cdot\frac{1\cdot 3\cdot\cdots\cdot(2(k-1)-1)}{k!}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}{2^k k!}$$

we multiply by $\frac{2k-1}{2k-1}$ *on the right side to obtain:*

$$\binom{\frac{1}{2}}{k} = (-1)^{k-1}\frac{1}{2^k}\cdot\frac{1\cdot 3\cdot\cdots\cdot(2(k-1)-1)}{k!}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}{2^k k!}\cdot\frac{2k-1}{2k-1}$$

$$\binom{\frac{1}{2}}{k} = (-1)^{k-1}\frac{1}{4^k(2k-1)}\binom{2k}{k}.$$

Replacing $\binom{\frac{1}{2}}{k}$ in (6),

$$(1-4x)^{1/2} = 1+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{4^k(2k-1)}\binom{2k}{k}(-4x)^k = 1-\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\binom{2k}{k}x^k. \tag{7}$$

From (5) and (7), we know that

$$G(x) = \frac{1-(1-\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\binom{2k}{k}x^k)}{2x} = \frac{1}{2x}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\binom{2k}{k}x^k$$

$$G(x) = \sum_{k=0}^{\infty}\frac{1}{2(2k+1)}\binom{2k+2}{k+1}x^k.$$

From this we can conclude that $C_k = \frac{1}{2(2k+1)}\binom{2k+2}{k+1}$, we can write this in a better way,

$$C_k = \frac{1}{2(2k+1)}\binom{2k+2}{k+1} = \frac{(2k+2)!}{2(2k+1)(k+1)!(k+1)!}$$

$$C_k = \frac{(2k+2)(2k+1)(2k)!}{2(2k+1)(k+1)k!(k+1)k!} = \frac{1}{k+1}\binom{2k}{k}.$$

Example 5.2 (Vandermonde's formula)**.** *Let us show the following identity: For all non-negative integers n, m, and r,*

$$\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}.$$

By the binomial theorem we know that $(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}x^r$ and $(1+x)^m = \sum_{r=0}^{m}\binom{m}{r}x^r$. By multiplying these two equations, we obtain $(1+x)^{n+m} = \sum_{r=0}^{n+m}\left(\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}\right)x^r$. Comparing the coefficient of both sides we conclude $\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}$.

Example 5.3. *Let us show the following identity: For all non-negative integer n,*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} = 4^n.$$

Let us define the following ordinal generating functions:

$$F(x) = \sum_{k=0}^{\infty}\binom{2k}{k}x^k$$

$$G(x) = \sum_{k=0}^{\infty}4^k x^k.$$

We know that $F(x)^2 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\right)x^n$, so we need to show that $F(x)^2 = G(x)$. By a previous exercise, $G(x) = \frac{1}{1-4x}$. By the binomial theorem,

$$(1-4x)^{-1/2} = \sum_{k=0}^{\infty}\binom{-1/2}{k}(-4)^k x^k$$

Let us find $\binom{-1/2}{k}(-4)^k$. By definition $\binom{-1/2}{k}(-4)^k = \frac{(-1/2)(-1/2\ -1)\cdots(-1/2\ -k+1)}{k!}(-4)^k$

$$\binom{-1/2}{k}(-4)^k = \frac{(1/2)(3/2)\cdots((2k-1)/2)}{k!}4^k$$

$$\binom{-1/2}{k}(-4)^k = \frac{(1)(3)\cdots((2k-1))}{k!}2^k$$

$$\binom{-1/2}{k}(-4)^k = \frac{(2k)!}{k!k!} = \binom{2k}{k}.$$

We conclude $G(x)^{1/2} = \sum_{k=0}^{\infty}\binom{2k}{k}x^k$, the proof follows.

Example 5.4. *For all n and k positive integers it holds that*

$$\sum_{c_1+c_2+\cdots+c_k=n,\ c_1,\ldots,c_k\in\mathbb{Z}^+} c_1c_2\cdots c_k = \binom{n+k-1}{n-k}.$$

The case $n=k$ is clear, let us assume $n>k$. Let us define the ordinal generating function $f(x)=\sum_{n=1}^{\infty} nx^{n-1}$. It is clear that

$$f(x)^k = \sum_{n=k}^{\infty}\left(\sum_{c_1+c_2+\cdots+c_k=n,\ c_1,\ldots,c_k\in\mathbb{Z}^+} c_1c_2\cdots c_k\right)x^{n-k}.$$

On the other hand we know that $f(x)=g'(x)$, where $g(x)=\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$. So, $f(x)=\frac{1}{(1-x)^2}$ and $f(x)^k=\frac{1}{(1-x)^{2k}}$. By the binomial theorem,

$$\frac{1}{(1-x)^{2k}} = \sum_{m=0}^{\infty}\binom{-2k}{m}(-x)^m = \sum_{n=k}^{\infty}\binom{-2k}{n-k}(-1)^{n-k}x^{n-k}.$$

It is enough to show that $\binom{-2k}{n-k}(-1)^{n-k}=\binom{n+k-1}{n-k}$, by definition

$$\binom{-2k}{n-k}(-1)^{n-k} = (-1)^{n-k}\frac{(-2k)(-2k-1)\cdots(-k-n+1)}{(n-k)!}$$

$$\binom{-2k}{n-k}(-1)^{n-k} = \frac{(2k)(2k+1)\cdots(k+n-1)}{(n-k)!} = \binom{n+k-1}{n-k}.$$

Exercise 5.1. *Find $\sum_{k=0}^{n}\binom{n}{k}^2$.*

Exercise 5.2. *Let u_n be the number of non-negative solutions of the equation $3a+7b=n$. Find a recursive formula for u_n.*

Exercise 5.3. *For every n determine how many numbers of n digits are such that it only has odd digits and the digit 1 and 3 appear an even number of times. (11533 is a number of the kind we are counting, but the numbers 11433 and 11333 are not).*

6 Catalan numbers

Catalan numbers are the solution too many combinatorial problems. We will start by studying some of these problems and sow that they have the same solution. Later we will show that the solution is the Catalan number.

1. How many sequences of n 1's and n -1's are such that the sum of the first k numbers of the sequence is a non-negative integer, for all $k>0$?.
2. How many expressions with n pairs of parenthesis exist such that the parenthesis are well matched?.
3. A monotonic path on a grid $n\times n$, is a path that starts in the lower left corner, finishes in the upper right corner, only moving to the right or up. How many monotonic path on a grid $n\times n$ that don't cross the diagonal exist?
4. In how many ways the sum of $n+1$ 1's can be associated? (i.e. for $n=2$ the associations $(1+(1+1))$ and $((1+1)+1)$ are different).
5. A binary tree is a tree with a vertex designated as the root (the root has two children) and each vertex that is not the root has at most two children, and the children of each vertex are ordered. How many binary trees are with $2n+1$ vertices?
6. In how many ways can a convex polygon with $n+2$ sides can be divided into n triangles with $n-1$ non-crossing diagonals?

Let us show that for a fix n all the previous problems have the same answer.

1 and 2. Define the following bijection. Given a sequences $a_1a_2\ldots a_{2n}$ of n 1's and n -1's we can change every "1" by "(" and every "-1" by ")" to obtain the sequence $x_1x_2\ldots x_{2n}$. So, $\sum_{i=1}^{k}a_i\geq 0$ holds for all $k>0$, if and only if

$$|\ \{x_i \mid i\leq j \wedge x_i = (\}\ | \geq |\ \{x_i \mid i\leq j \wedge x_i =)\}\ |$$

holds for all $j \geq 1$.

1 and 3. Define the following bijection. Given a sequences $a_1 a_2 \dots a_{2n}$ of n 1's and n -1's we can change every "1" by "$\rightarrow$" and every "-1" by "$\uparrow$" to obtain the sequence $x_1 x_2 \dots x_{2n}$. So, $\sum_{i=1}^{k} a_i \geq 0$ holds for all $k > 0$, if and only if

$$\mid \{x_i \mid i \leq j \wedge x_i = \rightarrow\} \mid \geq \mid \{x_i \mid i \leq j \wedge x_i = \uparrow\} \mid$$

holds for all $j \geq 1$. Therefore $\sum_{i=1}^{k} a_i \geq 0$ holds for all $k > 0$ if and only if the path $x_1 x_2 \dots x_{2n}$ doesn't cross the diagonal.

2 and 4. Define the following bijection. Given an association τ, change every "(" by a blank space, every "1" by a blank space, every "+" by "(", and every ")" by ")". This gives us an expressions with n pairs of well matches parenthesis.

4 and 5. For every association τ, we construct a tree T in the following inductive way. We define the expression 1 as an association of one element, to this association we assign the binary tree that only has one vertex. If τ is n association different than the association 1, then we know that τ is an association of the form $(\tau_0 + \tau_1)$ where τ_0 and τ_1 are associations. Let T_0 and T_1 be the binary trees constructed from τ_0 and τ_1, respectively. Let r_0 be the root of T_0 and r_1 the root of T_1. Construct T by:

- The set of vertices. $V(T) = \{r\} \cup V(T_0) \cup V(T_1)$.
- The set of edges. $E(T) = \{(r, r_0), (r_0, r), (r, r_1), (r_1, r)\} \cup E(T_0) \cup E(T_1)$.

5 and 6. Let us construct a bijection using the following method. Let $\mathcal{G}$ be a convex polygon with $n+2$ sides and vertices $\{1, 2, 3, \dots, n+2\}$. Let $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n\}$ be a set in which for every $0 \leq i \leq n$, $\mathcal{G}_i$ is a triangle and its vertices are vertices of the polygon, and $\mathcal{G}$ can be divided into these n triangles with $n-1$ non-crossing diagonals. Construct the tree T as follows, $V(T) = \{g_i \mid 0 \leq i \leq n\} \cup \{l_{i,j} \mid (i+1 = j \wedge 1 < i < n+2) \vee (i = n+2 \wedge j = 1)\}$, $E(T) = \{(l_{ij}, g_k), (g_k, l_{ij}) \mid \text{ if } i, j \text{ are vertices of the triangle } \mathcal{G}_k\} \cup \{(g_k, g_m) \mid \text{ if } \mathcal{G}_k \text{ and } \mathcal{G}_m \text{ have two vertices in common}\}$.

Exercise 6.1. *Show that the previous constructions are indeed bijections.*

Let us show that the solution to these problems is the Catalan numbers. We will use the first problem to do this.

Let A be the set of sequences of n 1's and n -1's in which the sum of the first k numbers of the sequence is a non-negative integer, for all $k > 0$. Let B be the set of sequences of n 1's and n -1's in which there is k such that sum of the first k numbers is a negative number. It is clear that $\mid A \mid + \mid B \mid = \binom{2n}{n}$. Let us construct a bijection between B and the set of sequences of $n-1$ 1's and $n+1$ -1's.

Let $(x_1, x_2, \dots, x_{2n}$ be a sequence of B, and let k be the least number such that $\sum_{i=1}^{2k+1} x_i < 0$. From $\{x_i\}_{1 \leq i \leq 2n}$ we can construct the following sequence, $(x_1, x_2, \dots, x_{2k+1}, -x_{2k+2}, -x_{2k+3}, \dots, -x_{2n})$. Clearly $(x_1, x_2, \dots, x_{2k+1}, -x_{2k+2}, -x_{2k+3}, \dots, -x_{2n})$ is a sequence with $n-1$ 1's and $n+1$ -1's and it is a bijection. Therefore $\mid B \mid = \binom{2n}{n-1}$. So $\mid A \mid = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n} = C_n$.

Let D_n be the solution of the first problem. Now we will show that $D_n = \sum_{k=0}^{n-1} D_k D_{n-1-k}$, this is the recursive formula used in the previous section to calculate the formula for the Catalan numbers.

Let A be the set of sequences of n 1's and n -1's in which the sum of the first k numbers of the sequence is a non-negative integer, for all $k > 0$. For every k, let A_k be the subset of A in which $(x_1, x_2, \dots, x_{2n}) \in A_k$ if and only if $\sum_{i=1}^{2k+2} x_i = 0$ and $\sum_{i=1}^{j} x_i > 0$ holds for all $j < 2k+2$. Clearly $A = \bigcup_{k=1}^{n} A_k$, $\mid A \mid = \sum_{k=0}^{n-1} \mid A_k \mid$. We need to show that $\mid A_k \mid = D_k D_{n-1-k}$, to do this, notice that $x_1 = 1$ and $x_{2k+2} = -1$, so the sequence $(x_2, \dots, x_{2k+1})$ has k 1's and k -1's and the sum of the first j numbers of the sequence is a non-negative integer, for all $j > 0$. Also notice that the sequence $(x_{2k+3}, \dots, x_{2n})$ has $n-1-k$ 1's and $n-1-k$ -1's and the sum of the first j numbers of the sequence is a non-negative integer, for all $j > 0$. We conclude that $\mid A_k \mid = D_k D_{n-1-k}$.

Exercise 6.2. *In how many ways the numbers from 1 to $2n$ can be placed on a $2 \times n$ board, in such a way that in every row and column the numbers appear in an increasing order.*

Exercise 6.3. *Let $\{v_1, v_2, \dots, v_{2n}\}$ be the vertices of a $2n$-regular convex polygon. Let $\{l_1, l_2, \dots, l_{2n}\}$ the set of sides of the polygon and $\{d_1, d_2, \dots, d_{2n}\}$ the set of diagonals. How many sets $B \subset \{l_1, l_2, \dots, l_{2n}\} \cup \{d_1, d_2, \dots, d_{2n}\}$ are such that $\mid B \mid = n$ and for all $X, Y \in B$ X and Y don't intersect each other.*

Exercise 6.4. *Let $x_1, x_2, \dots, x_{2n}$ be $2n$ points on a line $\mathcal{L}$. In how many ways can n circles be drawn such that for every circle $\mathcal{O}$, there are $1 \leq i, j \leq 2n$ such that $\bar{x_i x_j}$ is a diameter of $\mathcal{O}$ and doesn't intersect with any of the other circles.*

Exercise 6.5. *A plane tree is a tree with a vertex designated as the root and the children of each vertex are ordered. How many ordered trees are with n edges.*

7 Stirling number

Let us define some notation for a permutations. If $\pi \in S_4$:

$$\pi(i) = \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \\ 4 & \text{if } i = 3 \\ 3 & \text{if } i = 4, \end{cases}$$

then we will denote π by $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$. Notice that 1 and 2 have the following property in the previous example: $\pi(1) = 2$, $\pi(\pi(1)) = \pi(2) = 1$.

Given a permutation $\pi \in S_n$ we say that the numbers $a_1, a_2, \ldots, a_k$ form a cycle if $\pi(a_i) = a_{i+1}$ for $1 \leq i < k$ and $\pi(a_k) = a_1$, we will denote the cycle by $(a_1 a_2 \ldots a_k)$ where $a_1 > a_j$ for all $j > 1$.

Example 7.1. *The permutation* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 7 & 8 & 4 & 6 & 3 & 1 & 5 \end{pmatrix}$ *has the cycles* (4), (712), *and* (8563).

We can describe a permutation by describing the cycles, we will denote a permutation by listing the cycles in increasing order of the largest element of the cycle.

Example 7.2. *Using the cycle notation, we write the permutation* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 7 & 8 & 4 & 6 & 3 & 1 & 5 \end{pmatrix}$ *as* $(4)(712)(8563)$.

Let us denote by $c(n,k)$ the number of permutations in S_n with k cycles. The Stirling numbers of first kind are the numbers $s(n,k) = (-1)^{n-k} c(n,k)$.

Theorem 7.1.

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1).$$

Proof. Every permutation $\pi \in S_n$ satisfies one of the following conditions.

1. $\pi(n) = n$.
2. $\pi(n) \neq n$.

If π satisfies the first condition, then $\pi' : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n-1\}$ given by $\pi'(i) = \pi(i)$, is a permutation of S_{n-1} with $k-1$ cycles. Therefore, there are $c(n-1,k-1)$ permutation in S_n with k cycles that satisfies the first item.

By deleting n in the cycle notation, we obtain a permutation in S_{n-1} with k cycles, since n could have been in any cycle and in between any two elements of the cycle (for example 5 could be added to any cycle of the permutation (3)(412), given the possible permutations (3)(5412), (3)(5124), (3)(5241), (412)(53)), there are $(n-1)c(n-1,k)$ permutation in S_n with k cycles that satisfies the second item. □

Theorem 7.2.

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1)$$

holds for $n > 0$, *where* $c(n,0) = 0$ *for all* n.

Proof. We will proceed by induction over n. If $n = 1$, then $c(1,0) + c(1,1)x = 0 + 1 \cdot x = x$. Let us suppose that n is such that

$$\sum_{k=0}^{n-1} c(n-1,k)x^k = x(x+1)(x+2)\cdots(x+n-2).$$

By the induction hypothesis

$$x(x+1)(x+2)\cdots(x+n-1) = (x+n-1)\sum_{k=0}^{n-1} c(n-1,k)x^k$$

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n-1} c(n-1,k)x^{k+1} + \sum_{k=0}^{n-1} (n-1)c(n-1,k)x^k$$

$$x(x+1)(x+2)\cdots(x+n-1) = (n-1)c(n-1,0)x^0+\sum_{k=1}^{n-1}\left(c(n-1,k-1)+(n-1)c(n-1,k)\right)x^k+c(n-1,n-1)x^n.$$

By the definition of $c(n,0)$ and $c(n,n)$, and Theorem 7.1,

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=1}^{n-1} c(n,k)x^k + c(n,n)x^n = \sum_{k=0}^{n} c(n,k)x^k.$$

$\square$

Exercise 7.1. *We say that $\{A_1, A_2, \ldots, A_k\}$ is a partition of A into k non-empty subsets if $A_i \cap A_j$ is empty for all $1 \leq i \neq j \leq k$, A_i is non-empty for all $1 \leq i \leq k$, and $\bigcup_{i=1}^{k} A_i = A$.*

A Stirling number of the second kind is the number of ways to partition a set of n elements into k non-empty subsets, it is denoted by $S(m,k)$. Prove the following properties of the Stirling numbers of second kind.

1. $S(m,1) = S(m,m) = 1$.
2. $S(m,2) = 2^{m-1} - 1$.
3. $S(m,k) = S(m-1,k-1) + kS(m-1,k)$.

8 Five exercises

Exercise 8.1. *Prove that $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$.*

Exercise 8.2. *In a 4×19 chess board every square is colored red, blue, or green. Show that no matter how the chess board is colored, there are four squares with the same color, which make a rectangle.*

Exercise 8.3. *The sequence $\{a_i\}_{i=0}^{\infty}$ is defined by $a_n = \binom{n}{2}$. Determined the ordinal generating function of $\{a_i\}_{i=0}^{\infty}$.*

Exercise 8.4. *In how many ways can the set $\{1, 2, \ldots, 2n\}$ be arranged such that:*

- *The odd numbers are in an increasing order.*
- *The even numbers are in an increasing order.*
- $2k+1$ *is before* $2k$ *in the arrangement.*

Exercise 8.5. *How many permutations $\pi \in S_n$ exists such that $\pi(i) - \pi(i+1) \leq 1$ holds for all $1 \leq i < n$?*

9 Appendix: Partial Fractions

Partial fraction is an useful method when we wantto write an ordinal generating function as a power series. This can be seen in Example 4.1 when we obtain the formula for the Fibonacci's numbers from the generating function $G(x) = \frac{x}{1-x-x^2}$. Let us explain a bit this method.

We know how to add fractions, for example $\frac{2}{x} + \frac{3}{x-1} = \frac{2(x-1)+3x}{x(x-1)} = \frac{5x-2}{x^2-x}$. The question is: "How to obtain $\frac{2}{x} + \frac{3}{x-1}$ from $\frac{5x-2}{x^2-x}$?". To answer this let us go backwards, from $\frac{5x-2}{x^2-x}$ we obtain $\frac{2(x-1)+3x}{x(x-1)}$ and from this we obtain $\frac{2}{x} + \frac{3}{x-1}$. We can argue that from the polynomial $x^2 - x$ we first find its roots, these are 0 and 1, so $x^2 - x = x(x-1)$. From these, we look for two numbers A and B such that $\frac{5x-2}{x^2-x} = \frac{A}{x} + \frac{B}{x-1}$. To find these numbers we just expand the right side and solve the equation, $\frac{5x-2}{x^2-x} = \frac{A(x-1)+Bx}{x(x-1)}$, multiplying by $x^2 - x$ to both sides, we obtain $5x - 2 = Ax - A + Bx$, so $5x - 2 = (A+B)x - A$. From here we can deduce the system

$$\begin{array}{rcl} 5x & = & (A+B)x \\ -2 & = & -A \end{array}$$

By solving this system, we obtain $A = 2$ and $B = 3$. The partial fraction method allows us to write fraction of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is smaller than the degree of $Q(x)$, as a sum of fractions of the form $\frac{p_i(x)}{q_i(x)}$, where $q_i(x)$ is a polynimial with degree smaller or equal to the degree of $Q(x)$, and $p_i(x)$ is a polynomial with degree smaller than the degree of $q_i(x)$. In this appendix we will focus only on the case when $Q(x)$ has real roots, $Q(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$ where r_i is a real number for all $0 < i \leq n$. From now on $Q(x)$ is a polynomial with real roots.

1. Find the roots of $Q(x)$ ($r_1, r_2, \ldots, r_n$, where n is the degree of $Q(x)$), and write $Q(x)$ as $(x-r_1)(x-r_2)\cdots(x-r_n)$. Notice that $Q(x)$ could have a root with multiplicity bigger than 1.

2. Simplify the polynopmial $(x-r_1)(x-r_2)\cdots(x-r_n)$ by the multiplicity of each root. For example the polynomial $(x-0)(x-1)(x-3)(x-3)(x-5)(x-5)(x-5)$ is the polynomial $(x-0)(x-1)(x-3)^2(x-5)^3$.

3. For each factor $(x-r)$ use the following rule to write the fraction descomposition.

$$(x-r)^k \longmapsto \frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_k}{(x-r)^k}.$$

   For example for the polynomial $(x-0)(x-1)(x-3)^2(x-5)^3$ by using the previous rule, we write $\frac{A}{x} + \frac{B}{x-1} + \frac{C_1}{x-3} + \frac{C_2}{(x-3)^2} + \frac{D_1}{x-5} + \frac{D_2}{(x-5)^2} + \frac{D_3}{(x-5)^3}$ in the fraction descomposition.

4. Expand the fraction descomposition into $\frac{p_1(x)+p_2(x)+p_3(x)+p_4(x)+p_5(x)+p_6(x)+\cdots+p_n(x)}{Q(x)}$. For example for the fraction descomposition $\frac{A}{x} + \frac{B}{x-1} + \frac{C_1}{x-3} + \frac{C_2}{(x-3)^2} + \frac{D_1}{x-5} + \frac{D_2}{(x-5)^2} + \frac{D_3}{(x-5)^3}$ is expanded to

$$\frac{p_1(x)+p_2(x)+p_3(x)+p_4(x)+p_5(x)+p_6(x)+p_7(x)}{(x)(x-1)(x-3)^2(x-5)^3}$$

   where

$$\begin{array}{rclcc} p_1(x) & = & A(x-1)(x-3)^2(x-5)^3 & = & A(1125-2550x+2135x^2-884x^3+195x^4-22x^5+x^6) \\ p_2(x) & = & B(x)(x-3)^2(x-5)^3 & = & B(-1125x+1425x^2-710x^3+174x^4-21x^5+x^6) \\ p_3(x) & = & C_1(x)(x-1)(x-3)(x-5)^3 & = & C_1(-375x+725x^2-470x^3+138x^4-19x^5+x^6) \\ p_4(x) & = & C_2(x)(x-1)(x-5)^3 & = & C_2(125x-200x^2+90x^3-16x^4+x^5) \\ p_5(x) & = & D_1(x)(x-1)(x-3)^2(x-5)^2 & = & D_1(-225x+465x^2-334x^3+110x^4-17x^5+x^6) \\ p_6(x) & = & D_2(x)(x-1)(x-3)^2(x-5) & = & D_2(45x-84x^2+50x^3-12x^4+x^5) \\ p_7(x) & = & D_3(x)(x-1)(x-3)^2 & = & D_3(-9x+15x^2-7x^3+x^4) \end{array}$$

5. Now solve the equation $P(x) = p_1(x)+p_2(x)+p_3(x)+p_4(x)+p_5(x)+p_6(x)+\cdots+p_n(x)$ for the reals A_i, B_i, C_i, etc, chosen in the step 3. Replace the values of A_i, B_i, C_i, etc, in the fraction descomposition obtained in the step 3.

Example 9.1. *Let $P(x) = 2x+3$ and $Q(x) = x^3+2x^2+x$, and the fraction we want to work on is $\frac{2x+3}{x^3+2x^2+x}$.*

1. *The roots of $Q(x)$ are $r_1 = 0$, $r_2 = -1$, $r_3 = -1$, and $Q(x) = x(x+1)(x+1)$.*

2. *$Q(x) = x(x+1)^2$.*

3.
$$x \longmapsto \frac{A}{x}$$
$$(x+1)^2 \longmapsto \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2}$$

   $\frac{A}{x} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2}$ is the fraction descomposition.

4.
$$\frac{A(x^2+2x+1)+B_1(x^2+x)+B_2x}{x(x+1)^2}$$

   is the expansion of the fraction descomposition.

5. *From $2x+3 = A(x^2+2x+1)+B_1(x^2+x)+B_2x$ we obtain the following system:*

$$\begin{array}{rcl} 0 & = & A+B_1 \\ 2 & = & 2A+B_1+B_2 \\ 3 & = & A. \end{array}$$

   Solving this system we obtain $A = 3$, $B_1 = -3$, and $B_2 = -1$. By replacing the values of A, B_1, and B_2, we conclude that $\frac{2x+3}{x^3+2x^2+x} = \frac{3}{x} + \frac{-3}{x+1} + \frac{-1}{(x+1)^2}$.

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