Shifts and Singular Integrals

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Outline

1. Dyadic analysis and the boundedness of dyadic model operators.
2. Same kind of analysis in the X-valued setting, where $f : \mathbb{R} \to X$ takes values in a Banach space.
3. Connection to singular integral theory via representation theorems: bounds for dyadic operators imply bounds for singular integrals.
4. Bi-parameter analysis including the boundedness of bi-parameter model operators and singular integrals.

Notation

- We will work with functions $f\colon \mathbb{R} \to \mathbb{R}$ (one-parameter setup) or with with functions $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (bi-parameter setup). Everything would also work in $\mathbb{R}^d$ or $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.
- $\mathcal{D}_0 = \{2^{-k}([0,1) + m)\colon k \in \mathbb{Z}, m \in \mathbb{Z}\}$ is the standard dyadic grid. For each $\omega \in \Omega$, where $\Omega = \{0,1\}^{\mathbb{Z}}$, we define the lattice

  $$\mathcal{D}_\omega = \{I + \omega\colon I \in \mathcal{D}_0\},$$

  where

  $$I + \omega := I + \sum_{k\colon 2^{-k} < \ell(I)} \omega_k 2^{-k}.$$

  Here the side length of I is denoted by $\ell(I)$.
- Usually we work in some fixed $\mathcal{D} = \mathcal{D}_\omega$. We can induce randomness to $\omega \mapsto \mathcal{D}_\omega$ by equipping Ω with the natural probability product measure $\mathbb{P}$.

Notation

For a fixed $I \in \mathcal{D}$ and a locally integrable f we define as follows.

- If $k \in \mathbb{Z}$, $k \geq 0$, then $I^{(k)}$ denotes the unique interval $J \in \mathcal{D}$ for which $I \subset J$ and $\ell(I) = 2^{-k}\ell(J)$.
- The dyadic children of I are denoted by $\mathrm{ch}(I) = \{I' \in \mathcal{D} \colon (I')^{(1)} = I\} = \{I_-, I_+\}$.
- An average over I is $\langle f \rangle_I = \frac{1}{|I|}\int_I f$. We also write $E_I f = \langle f \rangle_I 1_I$ and $E_{2^{-k}} f = \sum_{I \colon \ell(I) = 2^{-k}} E_I f$.
- The martingale difference $\Delta_I f$ is defined by $\Delta_I f = \sum_{I' \in \mathrm{ch}(I)} E_{I'} f - E_I f$.
- For $k \in \mathbb{Z}$, $k \geq 0$, we define the martingale difference and average blocks

$$\Delta_I^k f = \sum_{\substack{J \in \mathcal{D} \\ J^{(k)} = I}} \Delta_J f \quad \text{and} \quad E_I^k f = \sum_{\substack{J \in \mathcal{D} \\ J^{(k)} = I}} E_J f.$$

An integral pairing is $\langle f, g \rangle = \int fg$.

Basic decompositions of functions

Let $f \colon \mathbb{R} \to \mathbb{R}$ be locally integrable and $I \in \mathcal{D}$. We can write the **martingale difference** on $I = I_- \cup I_+$ via

$$\begin{aligned}
\Delta_I f &:= \sum_{I \in \mathrm{ch}(I)} 1_{I'} \langle f \rangle_{I'} - 1_I \langle f \rangle_I \\
&= \frac{1_{I_-}}{|I_-|} \int_{I_-} f + \frac{1_{I_+}}{|I_+|} \int_{I_+} f - \frac{1_I}{|I|} \int_I f \\
&= \frac{2 \cdot 1_{I_-}}{|I|} \int_{I_-} f + \frac{2 \cdot 1_{I_+}}{|I|} \int_{I_+} f - \frac{(1_{I_-} + 1_{I_+})}{|I|} \Big(\int_{I_-} f + \int_{I_+} f \Big) \\
&= 1_{I_-} \Big(\frac{1}{|I|} \int_{I_-} f - \frac{1}{|I|} \int_{I_+} f \Big) - 1_{I_+} \Big(\frac{1}{|I|} \int_{I_-} f - \frac{1}{|I|} \int_{I_+} f \Big) \\
&= (1_{I_-} - 1_{I_+}) \frac{1}{|I|} \int (1_{I_-} - 1_{I_+}) f = \langle f, h_I \rangle h_I,
\end{aligned}$$

where h_I is the **Haar function** on I defined by

$$h_I := \frac{1}{|I|^{1/2}} (1_{I_-} - 1_{I_+}).$$

Basic decompositions of functions

As the following sum is telescoping, we have

$$\sum_{\substack{I \in \mathcal{D} \\ 2^{-k_1} < \ell(I) \le 2^{-k_2}}} \Delta_I f = E_{2^{-k_1}} f - E_{2^{-k_2}} f.$$

Therefore, we have both pointwise almost everywhere and in $L^p(\mathbb{R})$, $1 < p < \infty$, that

Fundamental decomposition

$$f = \lim_{\substack{k_1 \to \infty \\ k_2 \to -\infty}} \sum_{\substack{I \in \mathcal{D} \\ 2^{-k_1} < \ell(I) \le 2^{-k_2}}} \Delta_I f =: \sum_{I \in \mathcal{D}} \Delta_I f = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle h_I.$$

This uses Lebesgue's differentation theorem (to get $\lim_{k_1 \to \infty} E_{2^{-k_1}} f = f$) and the domination $|E_{2^{-k}} f| \le M_{\mathcal{D}} f$ (to get the L^p convergence). Here $M_{\mathcal{D}} f = \sup_{I \in \mathcal{D}} 1_I \langle |f| \rangle_I$.

Square Functions

Define the dyadic square function

$$S_{\mathcal{D}} f := \Big(\sum_{I \in \mathcal{D}} |\Delta_I f|^2 \Big)^{1/2} = \Big(\sum_{I \in \mathcal{D}} |\langle f, h_I \rangle|^2 \frac{1_I}{|I|} \Big)^{1/2}$$

$A \lesssim B$ means $A \le CB$; $A \sim B$ means $B \lesssim A \lesssim B$ (C constant).

Theorem

We have $\|f\|_{L^p} \sim \|S_{\mathcal{D}} f\|_{L^p}$, $1 < p < \infty$.

Proof.

Enough: $\|S_{\mathcal{D}} f\|_{L^p} \lesssim \|f\|_{L^p}$ (by duality). Here $p = 2$ follows by orthogonality and then $p \in (1,2)$ via the weak $(1,1)$ endpoint and interpolation. The case $p \in (2, \infty)$ uses Fefferman–Stein $\|g\|_{L^p} \lesssim \|M^{\sharp}_{\mathcal{D}} g\|_{L^p}$, where $M^{\sharp}_{\mathcal{D}} g = \sup_{I \in \mathcal{D}} 1_I \langle |g - \langle g \rangle_I| \rangle_I$:

$$\|S_{\mathcal{D}} f\|_{L^p} = \|(S_{\mathcal{D}} f)^2\|_{L^{p/2}}^{1/2} \lesssim \|M^{\sharp}_{\mathcal{D}}((S_{\mathcal{D}} f)^2)\|_{L^{p/2}}^{1/2} \lesssim \|M_{\mathcal{D}} f^2\|_{L^{p/2}}^{1/2}.$$

Here the last inequality was a simple pointwise estimate. $\square$

Consider a **martingale transform** (also called a **Haar multiplier**)

$$f = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle h_I \mapsto \sum_{I \in \mathcal{D}} \lambda_I \langle f, h_I \rangle h_I,$$

where $|\lambda_I| \le 1$ for every $I \in \mathcal{D}$.

As we have

$$\begin{aligned}\sum_{I \in \mathcal{D}} |\lambda_I| |\langle f, h_I \rangle| |\langle g, h_I \rangle| &\le \int \sum_{I \in \mathcal{D}} |\langle f, h_I \rangle| |\langle g, h_I \rangle| \frac{1_I}{|I|} \\ &\le \|S_{\mathcal{D}} f\|_{L^p} \|S_{\mathcal{D}} g\|_{L^{p'}} \lesssim \|f\|_{L^p} \|g\|_{L^{p'}},\end{aligned}$$

we have that martingale transforms are bounded $L^p \to L^p$, $1 < p < \infty$.

Dyadic Shifts

A **dyadic shift** is a simple generalisation of a martingale transform. It comes with the associated notion of complexity involving $i, j \in \{0, 1, 2, \ldots\}$. A martingale transform has $i = j = 0$.

Dyadic shifts

A dyadic shift has the form

$$S_{\mathcal{D}}^{i,j} f = \sum_{K \in \mathcal{D}} \sum_{\substack{I, J \in \mathcal{D} \\ I^{(i)} = J^{(j)} = K}} a_{KIJ} \langle f, h_I \rangle h_J, \qquad |a_{KIJ}| \leq \frac{|I|^{1/2} |J|^{1/2}}{|K|}.$$

Theorem

We have $\|S_{\mathcal{D}}^{i,j} f\|_{L^p} \lesssim \|f\|_{L^p}$, $1 < p < \infty$.

Proof:

$$
\begin{aligned}
&\sum_{K\in\mathcal{D}} \sum_{\substack{I,J\in\mathcal{D} \\ I^{(i)}=J^{(j)}=K}} |a_{KIJ}||\langle f, h_I\rangle||\langle g, h_J\rangle| \\
&\le \sum_{K\in\mathcal{D}} \frac{1}{|K|} \sum_{\substack{I,J\in\mathcal{D} \\ I^{(i)}=J^{(j)}=K}} \int_I |\Delta_K^i f| \int_J |\Delta_K^j g| \\
&= \int \sum_{K\in\mathcal{D}} \langle|\Delta_K^i f|\rangle_K \langle|\Delta_K^j g|\rangle_K 1_K \\
&\le \Big\| \Big(\sum_{K\in\mathcal{D}} \langle|\Delta_K^i f|\rangle_K^2 1_K \Big)^{1/2} \Big\|_{L^p} \Big\| \Big(\sum_{K\in\mathcal{D}} \langle|\Delta_K^j g|\rangle_K^2 1_K \Big)^{1/2} \Big\|_{L^{p'}}.
\end{aligned}
$$

The next step is to use Stein's inequality to remove the averages:

$$\Big\| \Big(\sum_{K \in \mathcal{D}} \langle |\Delta_K^i f| \rangle_K^2 1_K \Big)^{1/2} \Big\|_{L^p} \lesssim \Big\| \Big(\sum_{K \in \mathcal{D}} |\Delta_K^i f|^2 \Big)^{1/2} \Big\|_{L^p}.$$

This has an easy proof, but one could also use the somewhat harder Fefferman–Stein inequality

$$\Big\| \Big(\sum_{K \in \mathcal{D}} [M_{\mathcal{D}} f_K]^2 \Big)^{1/2} \Big\|_{L^p} \lesssim \Big\| \Big(\sum_{K \in \mathcal{D}} |f_K|^2 \Big)^{1/2} \Big\|_{L^p}.$$

In any case, we are done with the boundedness of shifts as given K the intervals P for which $P^{(i)} = K$ are disjoint, and thus

$$\Big\| \Big(\sum_{K \in \mathcal{D}} |\Delta_K^i f|^2 \Big)^{1/2} \Big\|_{L^p} = \Big\| \Big(\sum_{I \in \mathcal{D}} |\Delta_I f|^2 \Big)^{1/2} \Big\|_{L^p} \sim \|f\|_{L^p}.$$

Vector-Valued Analysis: UMD Spaces

Definition

A Banach space X is said to be a UMD space if

$$\Big\| \sum_{i=1}^N \varepsilon_i d_i \Big\|_{L^p(\Omega;X)} \lesssim \Big\| \sum_{i=1}^N d_i \Big\|_{L^p(\Omega;X)}$$

for all X-valued L^p-martingale difference sequences $(d_i)_{i=1}^N$ (defined on some probability space Ω), and for all signs $\varepsilon_i \in \{-1+1\}$.

The spaces $X = \mathbb{R}$ and $X = \mathbb{C}$ are UMD. The UMD property is independent of the choice of the exponent $p \in (1, \infty)$. If X is UMD then so is X^* and $L^p(\mathbb{R}^d; X)$, $1 < p < \infty$. This is also automatically a two-sided estimate (apply to $\varepsilon_i d_i$):

$$\Big\| \sum_{i=1}^N \varepsilon_i d_i \Big\|_{L^p(\Omega;X)} \sim \Big\| \sum_{i=1}^N d_i \Big\|_{L^p(\Omega;X)}.$$

Remark

We do not want to carefully define what is a martingale difference. What is relevant for us is that for the martingale differences $\Delta_I f$, $I \in \mathcal{D}$, where $f \colon \mathbb{R} \to X$, we have

$$\Big\| \sum_{I \in \mathcal{D}'} \varepsilon_I \Delta_I f \Big\|_{L^p(X)} \sim \Big\| \sum_{I \in \mathcal{D}'} \Delta_I f \Big\|_{L^p(X)}, \qquad \varepsilon_I = \pm 1,$$

where $\mathcal{D}' \subset \mathcal{D}$ and $L^p(X) := L^p(\mathbb{R}; X)$.

Notice that $\Delta_I f$ has the exact same definition as in the scalar-valued case – the appearing integrals $\int_I f \in X$ are interpreted as standard Bochner integrals.

In particular, we have

$$\Big\| \sum_{I \in \mathcal{D}} \varepsilon_I \Delta_I f \Big\|_{L^p(X)} \sim \|f\|_{L^p(X)}.$$

Random signs

We say that $\{\varepsilon_k\}_k$ is a **collection of independent random signs**, if there exists a probability space $(\mathcal{M}, \mu)$ so that $\varepsilon_k \colon \mathcal{M} \to \{-1, 1\}$, $\{\varepsilon_k\}_k$ is independent and

$$\mu(\{\varepsilon_k = 1\}) = \mu(\{\varepsilon_k = -1\}) = 1/2.$$

In X-valued analysis we often average over independent random signs (ε_I) as in

$$\mathbb{E}\Big\| \sum_{I \in \mathcal{D}} \varepsilon_I \Delta_I f \Big\|_{L^p(X)} \sim \|f\|_{L^p(X)}.$$

This is the replacement of square function estimates in the scalar-valued setting! Indeed, in the scalar-valued setting

$$\mathbb{E}\Big\| \sum_{I \in \mathcal{D}} \varepsilon_I \Delta_I f \Big\|_{L^p} \sim \Big\| \Big(\sum_{I \in \mathcal{D}} |\Delta_I f|^2 \Big)^{1/2} \Big\|_{L^p}.$$

The Kahane–Khintchine inequality says that

$$\Big(\mathbb{E}\Big|\sum_{i=1}^N \varepsilon_i x_i\Big|_X^q\Big)^{1/q} \sim_q \Big(\mathbb{E}\Big|\sum_{i=1}^N \varepsilon_i x_i\Big|_X^2\Big)^{1/2}$$

for all $1 \le q < \infty$, Banach spaces X and $x_i \in X$.

The previous connection to square functions follows by using Kahane–Khintchine a few times and noticing that in the scalar case

$$\begin{aligned}\mathbb{E}\Big|\sum_I \varepsilon_I \Delta_I f(x)\Big|^2 &= \sum_{I,J} \mathbb{E}(\varepsilon_I \varepsilon_J)\Delta_I f(x)\Delta_J f(x) \\ &= \sum_{I,J} \delta_{I,J}\Delta_I f(x)\Delta_J f(x) = \sum_I |\Delta_I f(x)|^2.\end{aligned}$$

Kahane Contraction Principle

The Kahane contraction principle says that if $(a_m)_{m=1}^M$ is a sequence of scalars and $p \in (0, \infty]$, then

$$\Big(\mathbb{E}\Big|\sum_{m=1}^{M} \varepsilon_m a_m x_m\Big|_X^p\Big)^{1/p} \lesssim \max|a_m| \Big(\mathbb{E}\Big|\sum_{m=1}^{M} \varepsilon_m x_m\Big|_X^p\Big)^{1/p}.$$

In the scalar-valued, square function setting, estimates like

$$\sum_{I\in\mathcal{D}} |a_I|^2 |\Delta_I f|^2 \le \sum_{I\in\mathcal{D}} |\Delta_I f|^2, \qquad |a_I| \le 1,$$

are more than obvious. Kahane's result simply says that in random sums we can do similar things.

Suppose $f\colon \mathbb{R} \to X$, where X is UMD, and that $|\lambda_I| \le 1$. Then we have

$$\begin{aligned}
\Big\| \sum_{I \in \mathcal{D}} \lambda_I \langle f, h_I \rangle h_I \Big\|_{L^p(X)} &\sim \mathbb{E} \Big\| \sum_{I \in \mathcal{D}} \varepsilon_I \lambda_I \langle f, h_I \rangle h_I \Big\|_{L^p(X)} \\
&\sim \Big(\mathbb{E} \Big\| \sum_{I \in \mathcal{D}} \varepsilon_I \lambda_I \langle f, h_I \rangle h_I \Big\|_{L^p(X)}^p \Big)^{1/p} \\
&\lesssim \Big(\mathbb{E} \Big\| \sum_{I \in \mathcal{D}} \varepsilon_I \langle f, h_I \rangle h_I \Big\|_{L^p(X)}^p \Big)^{1/p} \\
&\sim \mathbb{E} \Big\| \sum_{I \in \mathcal{D}} \varepsilon_I \Delta_I f \Big\|_{L^p(X)} \sim \|f\|_{L^p(X)},
\end{aligned}$$

where we used the UMD property to introduce and to remove the random signs, Kahane–Khintchine inequality repeatedly and Kahane contraction principle once (to remove λ_I).

Decoupling inequality

The $L^p(X) \to L^p(X)$ boundedness of Martingale Transforms – that is, complexity zero shifts – was an application of the most fundamental X-valued tools. The case of a general dyadic shift is surprisingly more involved due to the complexity. For this, we need one more tool: the decoupling inequality.

Decoupling notation

For $I \in \mathcal{D}$ let $\mathcal{V}_I$ be the probability measure space

$$\mathcal{V}_I = (I, \mathrm{Leb}(I), |I|^{-1}\,\mathrm{d}x\lfloor I).$$

Define the product probability space

$$\mathcal{V} = \mathcal{V}_{\mathcal{D}} = \prod_{I \in \mathcal{D}} \mathcal{V}_I,$$

and let ν be the related product measure. If $y \in \mathcal{V}$, we denote the coordinate related to $I \in \mathcal{D}$ by y_I.

Decoupling inequality

Let $k \in \{0, 1, 2, \dots\}$ and $j \in \{0, \dots, k\}$. Define the sublattice $\mathcal{D}_{j,k} \subset \mathcal{D}$ by

$$\mathcal{D}_{j,k} = \{Q \in \mathcal{D} \colon \ell(Q) = 2^{m(k+1)+j} \text{ for some } m \in \mathbb{Z}\}.$$

If $I, I' \in \mathcal{D}_{j,k}$ such that $I' \subsetneq I$, then $\ell(I') < 2^{-k}\ell(I)$.

Proposition (Decoupling (McConnell, Hytönen, Hytönen–Hänninen))

If X is UMD and $p \in (1, \infty)$ then

$$\int_{\mathbb{R}} \Big| \sum_{I \in \mathcal{D}_{j,k}} \Delta_I^u f(x) \Big|_X^p \,\mathrm{d}x$$
$$\sim \mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \Big| \sum_{I \in \mathcal{D}_{j,k}} \varepsilon_I 1_I(x) \Delta_I^u f(y_I) \Big|_X^p \,\mathrm{d}\nu(y)\,\mathrm{d}x$$

for any $u \in \{0, 1, \dots, k\}$.

Lets fix the dyadic shift

$$S^{i,j}_{\mathcal{D}} f = \sum_{K \in \mathcal{D}} \sum_{\substack{I,J \in \mathcal{D} \\ I^{(i)} = J^{(j)} = K}} a_{KIJ} \langle f, h_I \rangle h_J, \qquad |a_{KIJ}| \le \frac{|I|^{1/2}|J|^{1/2}}{|K|}.$$

We begin with the following consequence of the UMD property and Kahane–Khintchine inequality

$$\begin{aligned} \|S^{i,j}_{\mathcal{D}} f\|_{L^p(X)} &\sim \mathbb{E}\Big\| \sum_{P \in \mathcal{D}} \varepsilon_P \Delta^j_P S^{i,j}_{\mathcal{D}} f \Big\|_{L^p(X)} \\ &= \mathbb{E}\Big\| \sum_{K \in \mathcal{D}} \epsilon_K \sum_{\substack{I,J \in \mathcal{D} \\ I^{(i)} = J^{(j)} = K}} a_{KIJ} \langle \Delta^i_K f, h_I \rangle h_J \Big\|_{L^p(X)} \\ &\sim \Big(\mathbb{E}\Big\| \sum_{K \in \mathcal{D}} \epsilon_K \sum_{\substack{I,J \in \mathcal{D} \\ I^{(i)} = J^{(j)} = K}} a_{KIJ} \langle \Delta^i_K f, h_I \rangle h_J \Big\|^p_{L^p(X)} \Big)^{1/p}. \end{aligned}$$

Next, we define the kernel

$$a_K(x,y) = |K| \sum_{\substack{I,J\in\mathcal{D} \\ I^{(i)}=J^{(j)}=K}} a_{KIJ} h_I(y) h_J(x),$$

and notice that $|a_K(x,y)| \le 1$. We can now write

$$\sum_{\substack{I,J\in\mathcal{D} \\ I^{(i)}=J^{(j)}=K}} a_{KIJ} \langle \Delta_K^i f, h_I \rangle h_J(x) = \frac{1}{|K|} \int_K a_K(x,y) \Delta_K^i f(y) \, \mathrm{d}y.$$

The decoupling space allows us to further write this in the convenient form:

$$\frac{1}{|K|} \int_K a_K(x,y) \Delta_K^i f(y) \, \mathrm{d}y = \int_{\mathcal{V}} a_K(x, y_K) \Delta_K^i f(y_K) \, \mathrm{d}\nu(y).$$

Boundedness of Dyadic Shifts: $L^p(X) \to L^p(X)$

The idea is that we can now take the $\int_{\mathcal{V}}$ integral outside the K summation and use Hölder's inequality ($\int_{\mathcal{V}} |g|_X \le \big(\int_{\mathcal{V}} |g|_X^p \big)^{1/p}$):

$$\Big(\mathbb{E} \Big\| \int_{\mathcal{V}} \sum_{K \in \mathcal{D}} \epsilon_K a_K(x, y_K) \Delta_K^i f(y_K) \, \mathrm{d}\nu(y) \Big\|_{L_x^p(X)}^p \Big)^{1/p}$$

$$\le \Big(\mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \Big| \sum_{K \in \mathcal{D}} \epsilon_K a_K(x, y_K) \Delta_K^i f(y_K) \Big|_X^p \, \mathrm{d}\nu(y) \, \mathrm{d}x \Big)^{1/p}.$$

We are finally in the position to use $|a_K(x, y)| \le 1$ and the Kahane contraction principle – after this we are left with

$$\Big(\mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \Big| \sum_{K \in \mathcal{D}} \epsilon_K 1_K(x) \Delta_K^i f(y_K) \Big|_X^p \, \mathrm{d}\nu(y) \, \mathrm{d}x \Big)^{1/p}.$$

Boundedness of Dyadic Shifts: $L^p(X) \to L^p(X)$

We have arrived at the term from the decoupling inequality – justifying its a priori weird form. A technical detail is that we have the full grid $\mathcal{D}$ here – we can simply fix this by writing in the beginning

$$\mathcal{D} = \bigcup_{v=0}^{i} \mathcal{D}_{v,i}$$

and doing the previous estimate with each piece $\mathcal{D}_{u,i}$ separately.

With such a fixed v we get

$$\Big(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\Big|\sum_{K\in\mathcal{D}_{v,i}} \epsilon_K 1_K(x)\Delta_K^i f(y_K)\Big|_X^p \,\mathrm{d}\nu(y)\,\mathrm{d}x\Big)^{1/p}$$

$$\sim \Big(\int_{\mathbb{R}}\Big|\sum_{K\in\mathcal{D}_{v,i}} \Delta_K^i f(x)\Big|_X^p \,\mathrm{d}x\Big)^{1/p} \lesssim \|f\|_{L^p(X)}.$$

To see the last inequality you need to again introduce random signs (by UMD) and remove the summing restriction by contraction.

We have proved the following:

Theorem

If X is UMD and $p \in (1, \infty)$ then

$$\|S_{\mathcal{D}}^{i,j} f\|_{L^p(X)} \lesssim (1+i)\|f\|_{L^p(X)}.$$

With duality it is possible to get the constant $1 + \min(i, j)$ here. We will soon see that when we apply the theory of shifts to prove results for singular integrals, any polynomial dependence will be OK. Thus, we do not care too much.

Standard Kernels

Let $K\colon \mathbb{R}\times\mathbb{R}\setminus\Delta\to\mathbb{R}$, where $\Delta := \{(x,y)\in\mathbb{R}\times\mathbb{R}\colon x=y\}$, satisfy the **size estimate**

$$|K(x,y)| \lesssim \frac{1}{|x-y|}$$

and, for some $\alpha\in(0,1]$, the Hölder estimates

$$|K(x,y)-K(x',y)| \lesssim \frac{|x-x'|^{\alpha}}{|x-y|^{1+\alpha}}, \qquad \text{whenever} \quad |x-x'|\le|x-y|/2$$

and

$$|K(x,y)-K(x,y'| \lesssim \frac{|y-y'|^{\alpha}}{|x-y|^{1+\alpha}}, \qquad \text{whenever} \quad |y-y'|\le|x-y|/2.$$

Example: $K(x,y)=1/(x-y)$. Such a K is called a **standard singular integral kernel**. We denote the best kernel constant by $\|K\|_{CZ_\alpha}$.

A linear operator T – a priori defined on linear combinations of indicators of intervals – is called a **singular integral operator (SIO)** if there exists a standard kernel K so that, whenever $\operatorname{spt} f \cap \operatorname{spt} g = \emptyset$, we have

$$\langle Tf, g \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} K(x,y) f(y) g(x) \, \mathrm{d}x \, \mathrm{d}y.$$

This kernel structure alone is not enough for boundedness properties. An SIO T is called a **Calderón–Zygmund operator (CZO)** if for all intervals $I \subset \mathbb{R}$ we have

$$\int_I |T1_I| \lesssim |I| \qquad \text{and} \qquad \int_I |T^*1_I| \lesssim |I|.$$

If an SIO T is L^p, $p \in (1, \infty)$, bounded, then T is clearly a CZO.

$T1 \in \mathrm{BMO}$

The completely formal object $T1(x) = \int K(x,y)\,\mathrm{d}y$ can be defined in the sense that all the following pairings are well-defined:

$$\langle T1, \varphi_I \rangle := \langle T1_{3I}, \varphi_I \rangle + \int_{(3I)^c} \int_I [K(x,y) - K(c_I, y)] \varphi_I(y)\,\mathrm{d}y\,\mathrm{d}x$$

for all φ_I supported on an interval I with $\int \varphi_I = 0$ and $\|\varphi_I\|_{L^\infty} \le 1$. By the Hölder estimate of K the second term is a well-defined absolutely convergent integral dominated by $|I|$.

We say $T1 \in \mathrm{BMO}$ if for all intervals I and φ_I like above we have

$$|\langle T1, \varphi_I \rangle| \lesssim |I|.$$

Best constant is denoted $\|T1\|_{\mathrm{BMO}}$. As observed above this follows from

$$|\langle T1_{3I}, \varphi_I \rangle| \lesssim |I|.$$

CZO Reformulation

If

$$|\langle T1_I, 1_I \rangle| \lesssim |I|$$

for all intervals $I \subset \mathbb{R}$, T is said to satisfy the **weak boundedness property (WBP)** – best constant is denoted by $\|T\|_{\mathrm{WBP}}$.

Lemma

An SIO T is a CZO if and only if $T1 \in \mathrm{BMO}$, $T^*1 \in \mathrm{BMO}$ and the WBP holds.

Proof: If T is a CZO, then the desired conditions hold trivially (for the BMO recall that it is enough to control $\langle T1_{3I}, \varphi_I \rangle$).

Suppose conversely that $T1 \in \mathrm{BMO}$ and the WBP holds. Then

$$\int_I |T1_I| = \sup \Big| \int T(1_I)1_I g \Big|,$$

where $\|g\|_{L^\infty} \leq 1$.Write $1_I g = 1_I(g - \langle g \rangle_I) + 1_I \langle g \rangle_I$, and control the first term by $T1 \in \mathrm{BMO}$ and the second by the WBP.

The Representation Theorem

Theorem (Hytönen)

Suppose T is a CZO. Then

$$\langle Tf, g\rangle = C(\|K\|_{\mathrm{CZ}_\alpha} + \|T\|_{\mathrm{WBP}})\mathbb{E}_\omega \sum_{i,j=0}^{\infty} 2^{-\alpha \max(i,j)/2} \langle S_\omega^{i,j} f, g\rangle$$

$$+ C\|T1\|_{\mathrm{BMO}} \mathbb{E}_\omega \Big\langle \frac{\pi_{\omega, T1} f}{C\|T1\|_{\mathrm{BMO}}}, g \Big\rangle$$

$$+ C\|T^*1\|_{\mathrm{BMO}} \mathbb{E}_\omega \Big\langle \frac{\pi^*_{\omega, T^*1} f}{C\|T^*1\|_{\mathrm{BMO}}}, g \Big\rangle.$$

Here $C = C(\alpha) < \infty$, $S_\omega^{i,j}$ is a dyadic shift in the grid $\mathcal{D}_\omega$ and

$$\pi_{\omega,b} f := \sum_{I \in \mathcal{D}_\omega} \langle b, h_I\rangle \langle f\rangle_I h_I$$

is a dyadic paraproduct in the grid $\mathcal{D}_\omega$.

Boundedness of CZOs

If X is a Banach space and T is a CZO, we can hit simple functions $f = \sum_{i=1}^N f_i x_i$, where f_i are scalar-valued and $x_i \in X$, by $Tf = \sum_{i=1}^N (Tf_i) x_i$. These are dense in $L^p(X)$.

Corollary

Let T be a CZO, X be a UMD space and $p \in (1, \infty)$. Then we have

$$\|Tf\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}.$$

For now, we only know this result for those SIOs T satisfying $T1 = T^*1 = 0$ and the WBP, as we have only proved results for the dyadic shifts. Convolution form SIOs ($K(x,y) = K(x-y)$) satisfy $T1 = T^*1 = 0$, so this is already a very reasonable class. In particular, the Hilbert transform H for which $K(x,y) = 1/(x-y)$ maps $L^p(X)$ to $L^p(X)$ if X is UMD. In fact, X is UMD if and only if this happens (Burkholder, Bourgain).

Next, we will still prove the $L^p(X)$ boundedness of the dyadic paraproducts. We need some additional important tools for this.

It is trivial that

$$\Big\| \sum_{S \in \mathcal{S}} f_S \Big\|_{L^p(X)} = \Big(\sum_{S \in \mathcal{S}} \|f_S\|_{L^p(X)}^p \Big)^{1/p}$$

if $\mathcal{S} \subset \mathcal{D}$ is a collection of disjoint cubes and spt $f_S \subset S$.

This holds as a $\sim$ if $\mathcal{S}$ is **sparse** and the functions f_S satisfy some additional assumptions. The collection $\mathcal{S}$ is sparse if for all $S \in \mathcal{S}$ there is $E_S \subset S$ so that the sets E_S are mutually disjoint and $|E_S| \gtrsim |S|$.

Pythagoras' Theorem and Sparse Collections

Lemma

Let X be a Banach space and $p \in (1, \infty)$. Let $\mathcal{S} \subset \mathcal{D}$ be a sparse collection of dyadic cubes, and assume that for each $S \in \mathcal{S}$ we have a function f_S that satisfies:

- *$\operatorname{spt} f_S \subset S$;*
- *$\int f_S = 0$;*
- *f_S is constant on the maximal $S' \in \mathcal{S}$ satisfying $S' \subsetneq S$ (the collection of such S' is denoted by $\operatorname{ch}_{\mathcal{S}}(S)$).*

Then we have

$$\Big\| \sum_{S \in \mathcal{S}} f_S \Big\|_{L^p(X)} \sim \Big(\sum_{S \in \mathcal{S}} \|f_S\|_{L^p(X)}^p \Big)^{1/p}.$$

Estimate for $(\langle T1, h_I \rangle)_{I \in \mathcal{D}}$

Define the scalars $a_I = \langle T1, h_I \rangle / C \| T1 \|_{\mathrm{BMO}}$, where T is a CZO. Define the function $b = \sum_I a_I h_I$. Then it is easy to see that $b \in \mathrm{BMO}_1$ in the usual sense, and so by square function estimates and John–Nirenberg inequality we have for $p \in (1, \infty)$ that

$$\sup_{I_0 \in \mathcal{D}} \frac{1}{|I_0|^{1/p}} \Big\| \Big(\sum_{I \subset I_0} |a_I|^2 \frac{1_I}{|I|} \Big)^{1/2} \Big\|_{L^p}$$

$$= \sup_{I_0 \in \mathcal{D}} \frac{1}{|I_0|^{1/p}} \Big\| \Big(\sum_{I \subset I_0} |\langle b, h_I \rangle|^2 \frac{1_I}{|I|} \Big)^{1/2} \Big\|_{L^p}$$

$$\sim \sup_{I_0 \in \mathcal{D}} \frac{1}{|I_0|^{1/p}} \Big\| \sum_{I \subset I_0} \langle b, h_I \rangle h_I \Big\|_{L^p}$$

$$= \sup_{I_0 \in \mathcal{D}} \frac{1}{|I_0|^{1/p}} \| 1_{I_0} (b - \langle b \rangle_{I_0}) \|_{L^p} \sim \sup_{I_0 \in \mathcal{D}} \frac{1}{|I_0|} \| 1_{I_0} (b - \langle b \rangle_{I_0}) \|_{L^1} < \infty.$$

Thus

$$\sup_{I_0 \in \mathcal{D}} \frac{1}{|I_0|^{1/p}} \Big\| \Big(\sum_{I \subset I_0} |a_I|^2 \frac{1_I}{|I|} \Big)^{1/2} \Big\|_{L^p} \lesssim 1.$$

Stopping Construction: Principal Intervals

Fix a function $f \colon \mathbb{R} \to X$. Given an intercal $I_0 \in \mathcal{D}$ let $\operatorname{Stop}(I_0)$ denote the maximal $I \subset I_0$ such that $\langle |f|_X \rangle_I > 2\langle |f|_X \rangle_{I_0}$. With $I_0 \in \mathcal{D}$ fixed we define $\mathcal{S}_0(I_0) = \{I_0\}$ and

$$\mathcal{S}_{j+1}(I_0) = \bigcup_{I \in \mathcal{S}_j(I_0)} \operatorname{Stop}(I), \qquad j \geq 0.$$

We define the sparse collection of stopping intervals

$$\mathcal{S} = \mathcal{S}(I_0) = \bigcup_{j=0}^{\infty} \mathcal{S}_j(I_0),$$

for each $S \in \mathcal{S}$ we set

$$E_S = S \setminus \bigcup_{S' \in \operatorname{Stop}(S)} S',$$

and for each $I \subset I_0$ we denote by $\pi I = \pi_{\mathcal{S}} I$ the smallest $S \in \mathcal{S}$ such that $I \subset S$. Notice also $\operatorname{ch}_{\mathcal{S}}(S) = \operatorname{Stop}(S)$.

Fix a UMD space X, $p \in (1, \infty)$, a function $f : \mathbb{R} \to X$ and a dyadic paraproduct

$$\pi_{\mathcal{D}} f = \sum_{I \in \mathcal{D}} a_I \langle f \rangle_I h_I, \qquad \sup_{I_0 \in \mathcal{D}} \frac{1}{|I_0|^{1/p}} \Big\| \Big(\sum_{I \subset I_0} |a_I|^2 \frac{1_I}{|I|} \Big)^{1/2} \Big\|_{L^p} \le 1.$$

Fix an arbitrary $I_0 \in \mathcal{D}$ and notice that it is enough to bound

$$\sum_{I \subset I_0} a_I \langle f \rangle_I h_I = \sum_{S \in \mathcal{S}} \sum_{\pi I = S} a_I \langle f \rangle_I h_I.$$

By Pythagoras' we have

$$\Big\| \sum_{I \subset I_0} a_I \langle f \rangle_I h_I \Big\|_{L^p(X)} \sim \Big(\sum_{S \in \mathcal{S}} \Big\| \sum_{\pi I = S} a_I \langle f \rangle_I h_I \Big\|_{L^p(X)}^p \Big)^{1/p}.$$

Given $S \in \mathcal{S}$ we can now replace f with

$$f_S = f 1_{E(S)} + \sum_{S' \in \mathrm{Stop}(S)} \langle f \rangle_{S'} 1_{S'}$$

as $\langle f \rangle_I = \langle f_S \rangle_I$ if $\pi I = S$. Key property: $\|f_S\|_{L^\infty(X)} \lesssim \langle |f|_X \rangle_S$.

By UMD we have

$$\Big\| \sum_{\pi I = S} a_I \langle f_S \rangle_I h_I \Big\|_{L^p(X)} \sim \mathbb{E} \Big\| \sum_{\pi I = S} \epsilon_I a_I \langle f_S \rangle_I \frac{1_I}{|I|^{1/2}} \Big\|_{L^p(X)}.$$

By UMD-valued Stein's inequality (by Bourgain) we can remove the averages and have

$$\mathbb{E} \Big\| \sum_{\pi I = S} \epsilon_I a_I \langle f_S \rangle_I \frac{1_I}{|I|^{1/2}} \Big\|_{L^p(X)} \lesssim \mathbb{E} \Big\| \sum_{\pi I = S} \epsilon_I a_I f_S \frac{1_I}{|I|^{1/2}} \Big\|_{L^p(X)}.$$

Next, we have

$$\mathbb{E}\Big\| \sum_{\pi I = S} \epsilon_I a_I f_S \frac{1_I}{|I|^{1/2}} \Big\|_{L^p(X)} \le \|f_S\|_{L^\infty(X)} \mathbb{E}\Big\| \sum_{I \subset S} \epsilon_I a_I \frac{1_I}{|I|^{1/2}} \Big\|_{L^p}$$
$$\sim \|f_S\|_{L^\infty(X)} \Big\| \Big(\sum_{I \subset S} |a_I|^2 \frac{1_I}{|I|} \Big)^{1/2} \Big\|_{L^p}.$$

Recalling

$$\|f_S\|_{L^\infty(X)} \lesssim \langle |f|_X \rangle_S$$

and

$$\Big\| \Big(\sum_{I \subset S} |a_I|^2 \frac{1_I}{|I|} \Big)^{1/2} \Big\|_{L^p} \le |S|^{1/p},$$

we have

$$\sum_{S \in \mathcal{S}} \Big\| \sum_{\pi I = S} a_I \langle f \rangle_I h_I \Big\|_{L^p(X)}^p \lesssim \sum_{S \in \mathcal{S}} \langle |f|_X \rangle_S^p |S| \lesssim \|f\|_{L^p(X)}^p.$$

The last estimate is a simple consequence of the sparseness of $\mathcal{S}$. We are done with the $L^p(X)$ boundedness of paraproducts.

Multi-Parameter Analysis

Classical *one-parameter* kernels are "singular" (involve "division by zero") exactly when $x = y$. In contrast, the *multi-parameter* theory is concerned with kernels whose singularity is spread over the union of all hyperplanes of the form $x_i = y_i$, where $x, y \in \mathbb{R}^d$ are written as

$$x = (x_i)_{i=1}^t \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_t}$$

for a fixed partition $d = d_1 + \ldots + d_t$. The bi-parameter case $d = d_1 + d_2$ is already representative of many of the challenges arising in this context. The prototype example is

$$1/[(x_1 - y_1)(x_2 - y_2)],$$

the product of Hilbert kernels in both coordinate directions of $\mathbb{R}^2$, but general two-parameter kernels are neither assumed to be of the product nor of the convolution form.

Bi-Parameter Shifts

We work in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, fix two dyadic grids $\mathcal{D}_k$ in $\mathbb{R}$, $k = 1, 2$, and write $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ for the related dyadic rectangles. If $I = I_1 \times I_2 \in \mathcal{D}$ and $\mathbf{i} = (i_1, i_2)$, then $I^{(\mathbf{i})} := I_1^{(i_1)} \times I_2^{(i_2)}$. Moreover, we define $h_I := h_{I_1} \otimes h_{I_2}$.

A bi-parameter shift has the form

$$S_{\mathcal{D}}^{\mathbf{i},\mathbf{j}} f = \sum_{K \in \mathcal{D}} \sum_{\substack{I, J \in \mathcal{D} \\ I^{(\mathbf{i})} = J^{(\mathbf{j})} = K}} a_{KIJ} \langle f, h_I \rangle h_J,$$

where f is a function defined in $\mathbb{R}^2$ and

$$|a_{KIJ}| \le \frac{|I|^{1/2} |J|^{1/2}}{|K|} = \frac{|I_1|^{1/2} |J_1|^{1/2}}{|K_1|} \frac{|I_2|^{1/2} |J_2|^{1/2}}{|K_2|}.$$

Bi-Parameter Shifts vs Operator-Valued Shifts

We can write a bi-parameter shift

$$S_{\mathcal{D}}^{\mathbf{i},\mathbf{j}} f = \sum_{K \in \mathcal{D}} \sum_{\substack{I,J \in \mathcal{D} \\ I^{(\mathbf{i})} = J^{(\mathbf{j})} = K}} a_{KIJ} \langle f, h_I \rangle h_J,$$

in the form

$$\sum_{K_1 \in \mathcal{D}_1} \sum_{\substack{I_1, J_1 \in \mathcal{D}_1 \\ I_1^{(i_1)} = J_1^{(j_1)} = K_1}} S^{i_2,j_2}_{K_1 I_1 J_1} \langle f, h_{I_1} \rangle h_{J_1},$$

where $S^{i_2,j_2}_{K_1 I_1 J_1} = S^{i_2,j_2}_{\mathcal{D}_2, K_1 I_1 J_1}$ is a one-parameter dyadic shift in $\mathbb{R}$ defined by

$$S^{i_2,j_2}_{K_1 I_1 J_1} g = \sum_{K_2 \in \mathcal{D}_2} \sum_{\substack{I_2, J_2 \in \mathcal{D}_2 \\ I_2^{(i_2)} = J_2^{(j_2)} = K_2}} a_{KIJ} \langle g, h_{I_2} \rangle h_{J_2}.$$

Bi-Parameter Shifts vs Operator-Valued Shifts

In this sense the bi-parameter shift $S_{\mathcal{D}}^{\mathbf{i},\mathbf{j}}$ is a dyadic shift in $\mathbb{R}$ of complexity (i_1, j_1) but with **operator** coefficients $S^{i_2,j_2}_{K_1 I_1 J_1}$.

This leads us to study a general one-parameter operator-valued dyadic shift

$$S_{\mathcal{D}}^{i,j} f = \sum_{K \in \mathcal{D}} \sum_{\substack{I,J \in \mathcal{D} \\ I^{(i)} = J^{(j)} = K}} b_{KIJ} \langle f, h_I \rangle h_J,$$

where $\mathcal{D}$ is again just a dyadic grid in $\mathbb{R}$ (not the collection of dyadic rectangles in $\mathbb{R}^2$ like just above), $b_{KIJ} \in \mathcal{L}(X, Y)$ are bounded linear operators between two UMD spaces X and Y and $f : \mathbb{R} \to X$.

Under which conditions on the operators b_{KIJ} is the operator-valued shift $S_{\mathcal{D}}^{i,j}$ bounded $L^p(X) \to L^p(Y)$?

The proof of the UMD-valued boundedness of the usual scalar-valued shifts works also here. Following the proof we still reduce to bounding

$$\Big(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\Big|\sum_{K\in\mathcal{D}_{v,i}}\epsilon_K b_K(x,y_K)\Delta^i_K f(y_K)\Big|^p_Y \,\mathrm{d}\nu(y)\,\mathrm{d}x\Big)^{1/p},$$

where this time the kernels

$$b_K(x,y) := |K| \sum_{\substack{I,J\in\mathcal{D}\\ I^{(i)}=J^{(j)}=K}} b_{KIJ} h_I(y) h_J(x),$$

are not scalar-valued and bounded (so that we could use Kahane's contraction principle), but rather take values in $\mathcal{L}(X,Y)$.

Operator-Valued Shifts

Notice that with a fixed K and x, y, we have with some unique I and J (depending on x, y) that

$$b_K(x, y_K) = \pm \frac{|K|}{|I|^{1/2}|J|^{1/2}} b_{KIJ}.$$

To end the proof in exactly the same way as in the scalar-valued case, we simply need to assume that the family of normalised operators

$$\frac{|K|}{|I|^{1/2}|J|^{1/2}} b_{KIJ}$$

can be removed as if the Kahane's contraction principle would hold for them. This is called $\mathcal{R}$-boundedness – a notion which is more demanding than assuming that this normalised family of operators is uniformly bounded.

$\mathcal{R}$-boundedness

Definition

If X and Y are Banach spaces and $\mathcal{T} \subset \mathcal{L}(X, Y)$, we say that $\mathcal{T}$ is $\mathcal{R}$-bounded if there exists a constant C such that for all integers $K \geq 1$, all $T_k \in \mathcal{T}$ and for all $x_k \in X$, the inequality

$$\mathbb{E}\Big|\sum_{k=1}^{K} \varepsilon_k T_k x_k\Big|_Y \leq C\mathbb{E}\Big|\sum_{k=1}^{K} \varepsilon_k x_k\Big|_X$$

holds. The smallest constant C is denoted by $\mathcal{R}(\mathcal{T})$.

Recall that by Kahane–Khintchine inequality we can use whatever exponent in the random sums.

Operator-Valued Shifts

Theorem (Boundedness of OP-valued shifts)

Let X and Y be UMD spaces. The one-parameter operator-valued shift

$$S_{\mathcal{D}}^{i,j} f = \sum_{K \in \mathcal{D}} \sum_{\substack{I,J \in \mathcal{D} \\ I^{(i)} = J^{(j)} = K}} b_{KIJ} \langle f, h_I \rangle h_J$$

is bounded $L^p(X) \to L^p(Y)$, $p \in (1, \infty)$, if the family of operators

$$\mathcal{C}(S_{\mathcal{D}}^{i,j}) := \left\{ \frac{|K|}{|I|^{1/2} |J|^{1/2}} b_{KIJ} \in \mathcal{L}(X, Y) \colon K = I^{(i)} = J^{(j)} \right\}$$

is $\mathcal{R}$-bounded. In fact, we have

$$\|S_{\mathcal{D}}^{i,j} f\|_{L^p(Y)} \lesssim \mathcal{R}(\mathcal{C}(S_{\mathcal{D}}^{i,j}))(1 + \min(i,j)) \|f\|_{L^p(X)}.$$

Back to Bi-Parameter Shifts

We now return to our bi-parameter shift, and recall that for a certain one-parameter dyadic shift $S^{i_2,j_2}_{K_1 I_1 J_1}$ we could write

$$S^{\mathbf{i},\mathbf{j}}_{\mathcal{D}} f = \sum_{K_1 \in \mathcal{D}_1} \sum_{\substack{I_1, J_1 \in \mathcal{D}_1 \\ I_1^{(i_1)} = J_1^{(j_1)} = K_1}} S^{i_2,j_2}_{K_1 I_1 J_1} \langle f, h_{I_1} \rangle h_{J_1}.$$

We have been writing $L^p(X)$ for $L^p(\mathbb{R}; X)$ all the time – but now in the bi-parameter setting this gets confusing as we need to consider $L^p(\mathbb{R}^2; X)$ etc.

Moreover, if here $f\colon \mathbb{R}^2 \to X$ belongs to $L^p(\mathbb{R}^2; X)$ we should think (for the OP-valued theory) of this space in the form $L^p(\mathbb{R}; L^p(\mathbb{R}; X))$. For this reason, we might as well consider the more general mixed-norm spaces $L^{p_1}(\mathbb{R}; L^{p_2}(\mathbb{R}; X)) =: L^{p_1} L^{p_2}(X)$. We mean specifically $L^{p_1}_{x_1} L^{p_2}_{x_2}$ in what follows.

Bi-Parameter Shifts: Boundedness in $L^{p_1}L^{p_2}(X)$

Suppose X is UMD, $f\colon \mathbb{R}^2 \to X$ and fix $p_1, p_2 \in (1,\infty)$. By the OP-valued theory of shifts the bi-parameter shift satisfies

$$\|S_{\mathcal{D}}^{\mathbf{i},\mathbf{j}} f\|_{L^{p_1}L^{p_2}(X)} \lesssim C(1+\min(i_1,j_1))\|f\|_{L^{p_1}L^{p_2}(X)},$$

where

$$C := \mathcal{R}\left(\left\{ \frac{|K_1|}{|I_1|^{1/2}|J_1|^{1/2}} S^{i_2,j_2}_{K_1 I_1 J_1} \in \mathcal{L}(L^{p_2}(X))\colon K_1 = I_1^{(i_1)} = J_1^{(j_1)} \right\}\right).$$

For each fixed K_1, I_1, J_1, the scalar-coefficients, indexed by K_2, I_2, J_2, of the shift

$$\frac{|K_1|}{|I_1|^{1/2}|J_1|^{1/2}} S^{i_2,j_2}_{K_1 I_1 J_1}$$

satisfy precisely the usual normalisation of a one-parameter shift

$$\Big|\frac{|K_1|}{|I_1|^{1/2}|J_1|^{1/2}} a_{KIJ}\Big| \le \frac{|I_2|^{1/2}|J_2|^{1/2}}{|K_2|}.$$

$\mathcal{R}$-boundedness of one-parameter shifts

We have reduced to a question concerning a family of one-parameter shifts: if for each k we are given a one-parameter shift $S_k^{i,j}$, is the family $(S_k^{i,j})$ $\mathcal{R}$-bounded in $L^p(X)$ for every $p \in (1, \infty)$? This actually requires more than just X being UMD.

Definition

A Banach space X has Pisier's property (α) if for all N, all $\alpha_{i,j}$ in the complex unit disc and all $x_{i,j} \in X$, $1 \le i,j \le N$, there holds

$$\mathbb{E}\mathbb{E}'\Big|\sum_{1 \le i,j \le N} \epsilon_i \epsilon'_j \alpha_{i,j} x_{i,j}\Big|_X \lesssim \mathbb{E}\mathbb{E}'\Big|\sum_{1 \le i,j \le N} \epsilon_i \epsilon'_j x_{i,j}\Big|_X.$$

Here (ϵ_i) and (ϵ'_j) are sequences of independent random signs.

By Kahane–Khintchine we can use whatever exponent here. The scalars have Pisier's (α), and if X has Pisier's (α) so does $L^p(X)$.

$\mathcal{R}$-boundedness of one-parameter shifts

Pisier's property (α) arises naturally in multi-parameter X-valued analysis, together with the already familiar UMD condition. One reason is that the $\mathcal{R}$-boundedness of one-parameter shifts, or the boundedness of bi-parameter shifts, requires this.

Theorem

Let X be a UMD space satisfying Pisier's property (α). Suppose that we are given a family $\{S_k^{i,j} : k \in \mathcal{K}\}$ of dyadic one-parameter shifts of fixed complexity (i,j). Then for all $p \in (1,\infty)$ we have

$$\mathcal{R}(\{S_k^{i,j} \in \mathcal{L}(L^p(X)) \colon k \in \mathcal{K}\}) \lesssim 1 + \min(i,j).$$

Corollary

Let X be a UMD space satisfying Pisier's property (α) and $p_1, p_2 \in (1,\infty)$. Then a bi-parameter shift $S_{\mathcal{D}}^{\mathbf{i},\mathbf{j}}$ satisfies

$$\|S_{\mathcal{D}}^{\mathbf{i},\mathbf{j}} f\|_{L^{p_1}L^{p_2}(X)} \lesssim (1+\min(i_1,j_1))(1+\min(i_2,j_2))\|f\|_{L^{p_1}L^{p_2}(X)}.$$

Proof of the $\mathcal{R}$-boundedness of one-parameter shifts

We give the proof in the martingale transform – i.e., zero-complexity case. Suppose $f_k \in L^p(X)$, where X is UMD with Pisier's (α), and that $|\lambda_{I,k}| \le 1$. Then we have

$$\begin{aligned}
\mathbb{E}\Big\| \sum_k \varepsilon_k \sum_{I\in\mathcal{D}} \lambda_{I,k}\langle f_k, h_I\rangle h_I \Big\|_{L^p(X)}
&= \mathbb{E}\Big\| \sum_{I\in\mathcal{D}} \Delta_I\Big(\sum_k \varepsilon_k \lambda_{I,k} f_k\Big)\Big\|_{L^p(X)} \\
&\sim \mathbb{E}\mathbb{E}'\Big\| \sum_{I\in\mathcal{D}} \epsilon'_I \sum_k \varepsilon_k \lambda_{I,k}\langle f_k, h_I\rangle h_I\Big\|_{L^p(X)} \\
&\lesssim \mathbb{E}\mathbb{E}'\Big\| \sum_{I\in\mathcal{D}} \varepsilon'_I \sum_k \varepsilon_k \langle f_k, h_I\rangle h_I\Big\|_{L^p(X)} \\
&= \mathbb{E}\mathbb{E}'\Big\| \sum_{I\in\mathcal{D}} \varepsilon'_I \Delta_I\Big(\sum_k \varepsilon_k f_k\Big)\Big\|_{L^p(X)} \sim \mathbb{E}\Big\| \sum_k \varepsilon_k f_k\Big\|_{L^p(X)}.
\end{aligned}$$

The shift case is again more difficult and needs to go through the decoupling based proof – but this is how Pisier's (α) appears.

Bi-Parameter SIOs

We model a tensor-product of two SIOs $T_1 \otimes T_2$, which acts on a tensor-product function $f_1 \otimes f_2$ via the formula

$$(T_1 \otimes T_2)(f_1 \otimes f_2)(x) = T_1 f_1(x_1) T_2 f_2(x_2), \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

Notice that the relation

$$\langle (T_1 \otimes T_2)(f_1 \otimes f_2), g_1 \otimes g_2 \rangle = \langle T_1 f_1, g_1 \rangle \langle T_2 f_2, g_2 \rangle$$

implies

- The full kernel representation

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} K_1(x_1, y_1) K_2(x_2, y_2) (f_1 \otimes f_2)(y_1, y_2)(g_1 \otimes g_2)(x_1, x_2) \, \mathrm{d}y \, \mathrm{d}x$$

if $\operatorname{spt} f_1 \cap \operatorname{spt} g_1 = \emptyset$ and $\operatorname{spt} f_2 \cap \operatorname{spt} g_2 = \emptyset$;

Bi-Parameter SIOs

- The partial kernel representation

$$\iint_{\mathbb{R}\times\mathbb{R}} \langle T_2 f_2, g_2\rangle K_1(x_1, y_1) f_1(y_1) g_1(x_1)\, \mathrm{d}y_1\, \mathrm{d}x_1$$

  if spt $f_1 \cap$ spt $g_1 = \emptyset$;
- The partial kernel representation

$$\iint_{\mathbb{R}\times\mathbb{R}} \langle T_1 f_1, g_1\rangle K_2(x_2, y_2) f_2(y_2) g_2(x_2)\, \mathrm{d}y_2\, \mathrm{d}x_2$$

  if spt $f_2 \cap$ spt $g_2 = \emptyset$.

Bi-Parameter SIOs

The full kernel $K(x,y) = K_1(x_1,y_1)K_2(x_2,y_2)$ satisfies many natural estimates, like

$$|K(x,y)| \lesssim \frac{1}{|x_1 - y_1|}\frac{1}{|x_2 - y_2|}$$

and

$$\begin{aligned}|K(x,y) - K(x,(y_1,y_2')) - K(x,(y_1',y_2)) + K(x,y')| \\ \lesssim \frac{|y_1 - y_1'|^{\alpha}}{|x_1 - y_1|^{1+\alpha}}\frac{|y_2 - y_2'|^{\alpha}}{|x_2 - y_2|^{1+\alpha}}\end{aligned}$$

whenever $|y_1 - y_1'| \le |x_1 - y_1|/2$ and $|y_2 - y_2'| \le |x_2 - y_2|/2$.

Bi-Parameter SIOs

The partial kernel

$$K_{f_2,g_2}(x_1,y_1) := \langle T_2 f_2, g_2\rangle K_1(x_1,y_1)$$

satisfies the usual one-parameter kernel estimates with the constant $C(f_2,g_2) = \|K_1\|_{CZ_\alpha} |\langle T_2 f_2, g_2\rangle|$.

If T_2 is a CZO, then we have

$$C(1_I, g_I) + C(g_I, 1_I) \lesssim \int_I |T_2 1_I| + \int_I |T_2^* 1_I| \lesssim |I|$$

for all intervals I and functions g_I supported on I satisfying $\|g_I\|_{L^\infty} \le 1$

Partial kernels $K_{f_1,g_1}(x_2,y_2)$ behave analogously if T_1 is a CZO.

Bi-Parameter SIOs

However, tensor-products $T_1 \otimes T_2$ are not particularly interesting. Indeed, we can write

$$(T_1 \otimes T_2)f = T_1^1 T_2^2 f,$$

where e.g. $T_1^1 f(x) := T_1(f(\cdot, x_2))(x_1)$, and then Fubini shows

$$\|(T_1 \otimes T_2)f\|_{L^p} \le \|T_1\|_{L^p \to L^p} \|T_2\|_{L^p \to L^p} \|f\|_{L^p}.$$

Definition

A linear operator acting on suitable functions defined in $\mathbb{R}^2$ is a bi-parameter SIO, if it has full and partial kernel representations that satisfy the estimates a tensor product model does.

This means that we require that the pairing

$$\langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle$$

has, under the natural support conditions, the full kernel representation

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} K(x,y)(f_1 \otimes f_2)(y_1, y_2)(g_1 \otimes g_2)(x_1, x_2)\,\mathrm{d}y\,\mathrm{d}x,$$

for some K satisfying the various product estimates from above, and partial kernel representations like

$$\iint_{\mathbb{R} \times \mathbb{R}} K_{f_2,g_2}(x_1, y_1) f_1(y_1) g_1(x_1)\,\mathrm{d}y_1\,\mathrm{d}x_1,$$

where the kernel satisfies the 1-parameter kernel bounds with a constant $C(f_2, g_2)$ satisfying

$$C(1_I, g_I) + C(g_I, 1_I) \lesssim |I|$$

for all intervals I and functions g_I supported on I satisfying $\|g_I\|_{L^\infty} \leq 1$.

Bi-Parameter CZOs

So what are the conditions we should impose on 2-parameter SIOs to make them bounded (in the 1-parameter setting we demanded WBP and $T1, T^*1 \in \mathrm{BMO}$.)

Recall that for 1-parameter T the BMO condition we really used was that for all $p \in (1, \infty)$ we have

$$\sup_{I_0 \in \mathcal{D}} \frac{1}{|I_0|^{1/p}} \Big\| \Big(\sum_{I \subset I_0} |a_I|^2 \frac{1_I}{|I|} \Big)^{1/2} \Big\|_{L^p} < \infty, \qquad a_I = \langle T1, h_I \rangle.$$

For a 2-parameter SIO the analog of '$T1 \in \mathrm{BMO}$' will be that uniformly for all dyadic grids $\mathcal{D}^1, \mathcal{D}^2$ we have

$$\sup_{\Omega} \frac{1}{|\Omega|^{1/p}} \Big\| \Big(\sum_{\substack{I \in \mathcal{D}^1 \times \mathcal{D}^2 \\ I \subset \Omega}} |a_I|^2 \frac{1_I}{|I|} \Big)^{1/2} \Big\|_{L^p} < \infty, \qquad a_I = \langle T1, h_I \rangle,$$

where the supremum is over all **open** sets Ω of finite measure. The exponent p does not matter here – the conditions are the same for all $p \in (0, \infty)$ (by a John–Nirenberg style argument).

We will abbreviate this condition with '$T1 \in \mathrm{BMO}_{\mathrm{prod}}$'. We will have to assume this not only for T and T^* but also for the **partial adjoint** T_1 and its dual:

$$\langle T_1(f_1 \otimes f_2), g_1 \otimes g_2 \rangle := \langle T(g_1 \otimes f_2), f_1 \otimes g_2 \rangle.$$

Thus, '$T1$, T^*1, $T_1(1), (T_1)^*(1) \in \mathrm{BMO}_{\mathrm{prod}}$' will be among our assumptions. What about WBP? We will assume

$$|\langle T(1_{I_1} \otimes 1_{I_2}), 1_{I_1} \otimes 1_{I_2} \rangle| \lesssim |I_1||I_2|$$

for all intervals $I_1, I_2 \subset \mathbb{R}$. However, this is not enough. We will, in fact, need to incorporate some 'BMO' here as well and assume

$$|\langle T(a_{I_1} \otimes 1_{I_2}), 1_{I_1} \otimes 1_{I_2} \rangle| \lesssim |I_1||I_2|$$

whenever $\operatorname{spt} a_{I_1} \subset I_1$ and $\|a_{I_1}\|_{L^\infty} \leq 1$, and the three symmetric conditions.

Bi-Parameter CZOs

Definition

A bi-parameter SIO T satisfying

$$T1,\, T^*1,\, T_1(1), (T_1)^*(1) \in \mathrm{BMO}_{\mathrm{prod}},$$

$$|\langle T(a_{I_1} \otimes 1_{I_2}), 1_{I_1} \otimes 1_{I_2}\rangle| \lesssim |I_1||I_2|$$

whenever $\operatorname{spt} a_{I_1} \subset I_1$ and $\|a_{I_1}\|_{L^\infty} \le 1$, and the three other symmetric conditions, is a bi-parameter CZO.

Bi-Parameter Representation Theorem

Theorem (M., 2011)

Suppose T is a bi-parameter CZO. Then T is an average (over all dyadic grids $\mathcal{D}^1$ and $\mathcal{D}^2$) of a rapidly converging sum of bi-parameter dyadic model operators: bi-parameter shifts, bi-parameter partial paraproducts (hybrids of shifts and paraproducts) and full paraproducts.

Corollary

Let T be a bi-parameter CZO and $p_1, p_2 \in (1, \infty)$. Then we have

$$\|Tf\|_{L^{p_1}L^{p_2}} \lesssim \|f\|_{L^{p_1}L^{p_2}}.$$

We have only considered bi-parameter shifts at this point, so the rest of the dyadic model operators need to still be defined and bounded to get this result.

We did the shift result already even in the vector-valued situation. Thus, we have already proved the following result:

Corollary

Let T be a bi-parameter CZO, X be a UMD space with Pisier's (α) and $p_1, p_2 \in (1, \infty)$. If T is free of paraproducts in the sense that it has a representation with shifts only, then

$$\|Tf\|_{L^{p_1}L^{p_2}(X)} \lesssim \|f\|_{L^{p_1}L^{p_2}(X)}.$$

The 'paraproduct free' can be phrased concretely in terms of $T1 = 0$ type conditions – however, it is more than $T1 = T^*1 = T_1(1) = (T_1)^*(1) = 0$ as also the so-called 'partial paraproducts' are assumed to vanish here.

Remarks

- In the 1-parameter situation it was easy that if T is an L^p bounded SIO, then T is a CZO. The non-trivial part being that a CZO is L^p bounded.
- In the bi-parameter situation this converse is also hard. Using a so-called Journé's covering lemma, it can be proved that if a bi-parameter SIO T is L^p bounded, then $T1 \in \mathrm{BMO}_{\mathrm{prod}}$.
- However, we also need that $T_1(1)\,\mathrm{BMO}_{\mathrm{prod}}$. It is not true, though, that if T is e.g. L^2 bounded, then so is T_1! Thus, we can only prove that if T is a 2-parameter SIO so that T and T_1 are bounded, then T is a CZO. This detail means that the CZO theory is not characterising just the L^p boundedness of T but the simultaneous L^p boundedness of T and T_1.

Remarks

- In the 1-parameter theory we can upgrade L^2 boundedness to L^p boundedness by proving $L^1 \to L^{1,\infty}$, interpolating and using duality.
- However, the end point $L^1 \to L^{1,\infty}$ is not true in the bi-parameter world.
- For this reason it is quite powerful that the representation point of view gives that a CZO T is L^p bounded for every $p \in (1, \infty)$ (even mixed-norm bounded).
- Indeed, interpolating the only known end point $L^\infty \to \mathrm{BMO}_{\mathrm{prod}}$ is very difficult, although still doable. This interpolation requires an extensive theory of product Hardy spaces.

The Rest of the Bi-Parameter Model Operators

As we are modeling $T_1 \otimes T_2$, the model operators need to include suitable generalisations of all $U_1 \otimes U_2$, where $U_1 \in \{S_1, \pi_1, \pi_1^*\}$ and similarly for U_2. Bi-parameter shifts generalise $S_1 \otimes S_2$.

For example, if $\pi_{\mathcal{D}^1} f_1 = \sum_{I_1 \in \mathcal{D}^1} a_{I_1} \langle f_1 \rangle_{I_1} h_{I_1}$ and $\pi_{\mathcal{D}^2} f_2 = \sum_{I_2 \in \mathcal{D}^2} b_{I_2} \langle f_2 \rangle_{I_2} h_{I_2}$, then $(\pi_{\mathcal{D}^1} \otimes \pi_{\mathcal{D}^2}) f$ looks like

$$\sum_{I_1, I_2} a_{I_1} b_{I_2} \langle f \rangle_{I_1 \times I_2} h_{I_1 \times I_2}.$$

It is not so hard to see that $(a_{I_1} b_{I_2})_{I_1, I_2}$ satisfies the product BMO condition. The correct generalisation then is

$$\sum_{I_1, I_2} a_{I_1, I_2} \langle f \rangle_{I_1 \times I_2} h_{I_1 \times I_2},$$

where

$$\sup_{\Omega} \frac{1}{|\Omega|^{1/2}} \Big(\sum_{I_1 \times I_2 \subset \Omega} |a_{I_1, I_2}|^2 \Big)^{1/2} \leq 1.$$

Remarks on Bi-Parameter X-Valued theory

We knew how to bound bi-parameter shifts in $L^{p_1}L^{p_2}(X)$, where X is UMD with Pisier's (α). We do not know how to do this for quite all these spaces X for the other model operators – the above introduced full paraproducts or the partial paraproducts (that generalise $S_1 \otimes \pi_2$).

In practice, all known UMD spaces satisfying **Pisier's** (α) are function lattices – this means that $x \in X$ is actually a function $x\colon \Omega \to \mathbb{R}$ with suitable assumptions. We could develop the vector-valued theory of bi-parameter full and partial paraproducts in UMD function lattices. This then would imply that all bi-parameter CZOs are bounded in $L^{p_1}L^{p_2}(X)$ whenever X is a UMD function lattice.

For simplicity, however, in these lectures we show the boundedness of these other bi-parameter model operators only in the scalar-valued case.

H^1-BMO duality

Let $\mathcal{D} = \mathcal{D}^1 \times \mathcal{D}^2$. Below $R \in \mathcal{D}$. We will prove that

$$\sum_R |a_R||b_R| \lesssim \Big[\sup_\Omega \frac{1}{|\Omega|^{1/2}} \Big(\sum_{R \subset \Omega} |a_R|^2 \Big)^{1/2} \Big] \cdot \Big\| \Big(\sum_R |b_R|^2 \frac{1_R}{|R|} \Big)^{1/2} \Big\|_{L^1}.$$

Given $k \in \mathbb{Z}$ we define

$$U_k = \Big\{ x \colon \Big(\sum_R |b_R|^2 \frac{1_R(x)}{|R|} \Big)^{1/2} > 2^{-k} \Big\},$$

and

$$\widehat{\mathcal{R}}_k = \{ R \in \mathcal{D} \colon |R \cap U_k| > |R|/2 \}.$$

If $R \in \widehat{\mathcal{R}}_k$, then

$$R \subset \tilde{U}_k := \{ x \colon M_{\mathcal{D}} 1_{U_k} > 1/2 \},$$

where $M_{\mathcal{D}} f = \sup_R 1_R \langle |f| \rangle_R$. As $M_{\mathcal{D}} \colon L^2 \to L^2$, we have $|\tilde{U}_k| \lesssim |U_k|$.

H^1-BMO duality

For all $R_0 \in \mathcal{D}$ and for all $x \in R_0$ we have

$$\Big(\sum_{R \in \mathcal{D}} |b_R|^2 \frac{1_R(x)}{|R|} \Big)^{1/2} \geq \frac{|b_{R_0}|}{|R_0|^{1/2}}.$$

If $b_{R_0} \neq 0$, then this implies that $R_0 \subset U_k$ and so $R_0 \in \widehat{\mathcal{R}}_k$ for all large enough k.

We may obviously assume

$$\Big\| \Big(\sum_{R} |b_R|^2 \frac{1_R}{|R|} \Big)^{1/2} \Big\|_{L^1} < \infty.$$

Then we have $|U_k| \to 0$ when $k \to -\infty$, and so we also have that $R_0 \notin \widehat{\mathcal{R}}_k$ for all small enough k.

Let $\mathcal{R}_k = \widehat{\mathcal{R}}_k \setminus \widehat{\mathcal{R}}_{k-1}$, $k \in \mathbb{Z}$, and notice that we have deduced that all relevant R_0 (i.e. those for which $b_{R_0} \neq 0$) belong to one and exactly one $\mathcal{R}_k$.

H^1-BMO duality

$$
\begin{aligned}
\sum_R |a_R||b_R| &= \sum_{k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_k} |a_R||b_R| \\
&\leq 2\int \sum_{k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_k} |a_R||b_R| \frac{1_R}{|R|} 1_{\tilde{U}_k} 1_{U^c_{k-1}} \\
&\lesssim \sum_{k\in\mathbb{Z}} \Big\| \Big(\sum_{R\subset\tilde{U}_k} |a_R|^2 \frac{1_R}{|R|}\Big)^{1/2} \Big\|_{L^2} \Big\| \Big(\sum_R |b_R|^2 \frac{1_R}{|R|}\Big)^{1/2} 1_{\tilde{U}_k} 1_{U^c_{k-1}} \Big\|_{L^2} \\
&\lesssim \Big[\sup_\Omega \frac{1}{|\Omega|^{1/2}} \Big(\sum_{R\subset\Omega} |a_R|^2 \Big)^{1/2} \Big] \sum_{k\in\mathbb{Z}} 2^{-k} |U_k| \\
&\sim \Big[\sup_\Omega \frac{1}{|\Omega|^{1/2}} \Big(\sum_{R\subset\Omega} |a_R|^2 \Big)^{1/2} \Big\| \Big(\sum_R |b_R|^2 \frac{1_R}{|R|}\Big)^{1/2} \Big\|_{L^1}.
\end{aligned}
$$

Boundedness of Full Paraproducts

It is enough to bound

$$\sum_R |a_R| \langle |f| \rangle_R |\langle g, h_R \rangle|.$$

The H^1-BMO duality bounds this with

$$\int \Big(\sum_R \langle |f| \rangle_R^2 |\langle g, h_R \rangle|^2 \frac{1_R}{|R|} \Big)^{1/2} \le \int M_{\mathcal{D}} f \Big(\sum_R |\langle g, h_R \rangle|^2 \frac{1_R}{|R|} \Big)^{1/2}.$$

It remains to use Hölder's inequality and the boundedness of the maximal function $M_{\mathcal{D}}$ and the square function involving rectangles. How to see that the latter is bounded? (The mixed-norm boundedness of the rectangular maximal function takes some thinking as well, but we omit it for now.)

Boundedness of the Bi-Parameter Square Function

Let $\Delta_R f = \langle f, h_R\rangle h_R = \Delta^1_{I_1}\Delta^2_{I_2} f$, if $R = I_1 \times I_2 \in \mathcal{D}$ and e.g. $\Delta^1_{I_1} f(x) = \Delta_{I_1}(f(\cdot, x_2))(x_1)$. Then

$$\begin{aligned}
&\Big\| \Big(\sum_R |\Delta_R f|^2 \Big)^{1/2} \Big\|_{L^{p_1} L^{p_2}} \\
&\sim \mathbb{E} \Big\| \sum_R \epsilon_R \Delta_R f \Big\|_{L^{p_1} L^{p_2}} \\
&\sim \mathbb{E}\mathbb{E}' \Big\| \sum_{I_1} \epsilon_{I_1} \Delta^1_{I_1} \Big(\sum_{I_2} \epsilon'_{I_2} \Delta^2_{I_2} f \Big) \Big\|_{L^{p_1} L^{p_2}} \\
&\sim \mathbb{E}' \Big\| \sum_{I_2} \epsilon'_{I_2} \Delta^2_{I_2} f \Big\|_{L^{p_1} L^{p_2}} \sim \|f\|_{L^{p_1} L^{p_2}}.
\end{aligned}$$

We used Kahane–Khintchine multiple times and the known UMD space estimates.

Boundedness of Full Paraproducts

There is a genuinely different full paraproduct as well (the partial adjoint of the previous one – or the one modeling $\pi_1 \otimes \pi_2^*$):

$$\sum_{R=I_1\times I_2} a_R \Big\langle f, \frac{1_{I_1}}{|I_1|} \otimes h_{I_2} \Big\rangle h_{I_1} \otimes \frac{1_{I_2}}{|I_2|}.$$

It is bounded by H^1-BMO duality as well, but the end is different reducing e.g. to bounding

$$\Big\| \Big(\sum_{I_2} |M_{\mathcal{D}^1} \langle f, h_{I_2} \rangle_2|^2 \otimes \frac{1_{I_2}}{|I_2|} \Big)^{1/2} \Big\|_{L^{p_1} L^{p_2}}.$$

First, remove the maximal function by using that for all $p_1, p_2, r \in (1, \infty)$ we have

$$\Big\| \Big(\sum_j |M^1_{\mathcal{D}^1} f_j|^r \Big)^{1/r} \Big\|_{L^{p_1} L^{p_2}} \lesssim \Big\| \Big(\sum_j |f_j|^r \Big)^{1/r} \Big\|_{L^{p_1} L^{p_2}}.$$

Then we are left with

$$\Big\| \Big(\sum_{I_2} |\langle f, h_{I_2} \rangle_2|^2 \otimes \frac{1_{I_2}}{|I_2|} \Big)^{1/2} \Big\|_{L^{p_1} L^{p_2}} = \Big\| \Big(\sum_{I_2} |\Delta^2_{I_2} f|^2 \Big)^{1/2} \Big\|_{L^{p_1} L^{p_2}},$$

after which this is just a square function estimate in L^{p_2}.

We have dealt with all the full paraproducts. It remains to deal with the last remaining family of model operators – the partial paraproducts.

A proper generalisation of $S_1 \otimes \pi_2$ is

$$Pf = \sum_{K = K_1 \times K_2 \in \mathcal{D}} \sum_{\substack{I_1, J_1 \in \mathcal{D}_1 \\ I_1^{(i_1)} = J_1^{(j_1)} = K_1}} a_{K I_1 J_1} \Big\langle f, h_{I_1} \otimes \frac{1_{K_2}}{|K_2|} \Big\rangle h_{J_1} \otimes h_{K_2},$$

where for each fixed K_1, I_1, J_1 we have the one-parameter BMO estimate

$$\sup_{I_2 \in \mathcal{D}^2} \frac{1}{|I_2|^{1/2}} \Big(\sum_{\substack{K_2 \in \mathcal{D}^2 \\ K_2 \subset I_2}} |a_{K I_1 J_1}|^2 \Big)^{1/2} \le \frac{|I_1|^{1/2} |J_1|^{1/2}}{|K_1|}.$$

We have a choice to try to bound this directly, or to use the operator-valued theory of shifts again. We could see a bi-parameter shift as a shift-valued shift, and we can view partial paraproducts as a paraproduct-valued shift. We use the op-valued approach.

Regarding the mixed-norm bounds, the op-valued approach has a detail: we can directly do $L^{p_1}_{x_1} L^{p_2}_{x_2}$ as now the shift structure is in the x_1 variable. Unlike in the bi-parameter shift case where we can, by symmetry, also do $L^{p_2}_{x_2} L^{p_1}_{x_1}$, here there is no symmetry. This would lead us to study shift-valued paraproducts, but there is no equally good theory for operator-valued paraproducts as there is for operator-valued shifts.

For this reason, we really only explictly tackle the $L^{p_1}_{x_1} L^{p_2}_{x_2}$ case now, while the other one is also true by modified arguments.

So we write

$$Pf = \sum_{K_1 \in \mathcal{D}^1} \sum_{\substack{I_1, J_1 \in \mathcal{D}_1 \\ I_1^{(i_1)} = J_1^{(j_1)} = K_1}} \pi_{K_1 I_1 J_1} \langle f, h_{I_1} \rangle h_{J_1},$$

where

$$\pi_{K_1 I_1 J_1} g = \sum_{K_2 \in \mathcal{D}^2} a_{K I_1 J_1} \langle g \rangle_{K_2} h_{K_2}.$$

Partiall Paraproducts

By the OP-valued theory of shifts the partial paraproduct satisfies

$$\|Pf\|_{L^{p_1}L^{p_2}} \lesssim C(1+\min(i_1,j_1))\|f\|_{L^{p_1}L^{p_2}},$$

where

$$C := \mathcal{R}\left(\left\{ \frac{|K_1|}{|I_1|^{1/2}|J_1|^{1/2}}\pi_{K_1 I_1 J_1} \in \mathcal{L}(L^{p_2})\colon\, K_1 = I_1^{(i_1)} = J_1^{(j_1)} \right\}\right).$$

With fixed K_1, I_1, J_1 the coefficients $b_{K_2} = \frac{|K_1|}{|I_1|^{1/2}|J_1|^{1/2}} a_{KI_1J_1}$ satisfy the natural normalisation

$$\sup_{I_2\in\mathcal{D}^2} \frac{1}{|I_2|^{1/2}}\Big(\sum_{\substack{K_2\in\mathcal{D}^2\\ K_2\subset I_2}} |b_{K_2}|^2\Big)^{1/2} \le 1.$$

So the question has reduced to the $\mathcal{R}$-boundedness of a family of normalised paraproducts – this can be done directly by using H^1-BMO duality, for example. We omit the details.

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