FOURIER ANALYSIS I
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CONTENTS

1. INTRODUCTION: FOURIER COEFFICIENTS AND FOURIER SERIES

The lectures are for the most part based on the books by Duoandikoetxea [1] and Grafakos [2], and the lecture notes of Salo [3].

We study functions $f\colon \mathbb{R} \to \mathbb{C}$ that are 1-periodic:

$$f(x+1) = f(x), \qquad x \in \mathbb{R}.$$

It is equivalent to say that f is a function defined on the 1-torus

$$\mathbb{T} = \mathbb{T}^1 := \mathbb{R}/\mathbb{Z},$$

which consists of the equivalence classes determined by the equivalence relation $x \equiv y$ if and only $x - y \in \mathbb{Z}$. We will use this latter point of view $f\colon \mathbb{T} \to \mathbb{C}$ simply as a short way to say that $f\colon \mathbb{R} \to \mathbb{C}$ is 1-periodic. That is, we do not need to think about the torus more than that.

We note that many of the results could be stated and proved for functions defined in the n-Torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. This means studying functions $f\colon \mathbb{R}^n \to \mathbb{C}$ which satisfy $f(x+m) = f(x)$ for all x and $m \in \mathbb{Z}^n$. We will not pursue this but see Grafakos [2].

A trigonometric polynomial P has the form

$$P(x) = \sum_{m \in \mathbb{Z}} a_m e^{2\pi i m x},$$

where $a_m \neq 0$ for only finitely many m. Recall that

$$e^{ix} = \cos x + i \sin x$$

so that $x \mapsto e^{2\pi i m x}$ is 1-periodic. Notice that we can recover the coefficients a_m with the following calculation:

$$\int_0^1 P(x) e^{-2\pi i m x}\,\mathrm{d}x = \sum_{k\in\mathbb{Z}} a_k \int_0^1 e^{2\pi i(k-m)x}\,\mathrm{d}x = \sum_{k\in\mathbb{Z}} a_k \delta_{k,m} = a_m,$$

where $\delta_{k,m} = 1$ if $k = m$ and zero otherwise. Motivated by this we make the following definition.

1.1. **Definition.** Let $f \in L^1(\mathbb{T})$ – i.e., $f\colon \mathbb{R} \to \mathbb{C}$ is 1-periodic and $\|f\|_{L^1} := \int_0^1 |f| < \infty$. The mth Fourier coefficient $\widehat{f}(m)$ of f is defined by

$$\widehat{f}(m) := \int_0^1 f(x) e^{-2\pi i m x}\,\mathrm{d}x, \qquad m \in \mathbb{Z}.$$

So for a trigonometric polynomial P we have

$$P(x) = \sum_{m\in\mathbb{Z}} \widehat{P}(m) e^{2\pi i m x}.$$

1.2. **Definition.** The Nth partial sum of the Fourier series of $f \in L^1(\mathbb{T})$ is

$$S_N f(x) := \sum_{|m|\leq N} \widehat{f}(m) e^{2\pi i m x}.$$

We are interested in questions with the following flavour:

- Do the Fourier coefficients $\widehat{f}(m)$ determine f?
- Does $S_N f(x)$ converge in some sense (pointwise, in L^p) as $N \to \infty$? Under what assumptions? Does it converge to f?
- What kind of estimates do the coefficients $\widehat{f}(m)$ satisfy, and do they correlate with the regularity (smoothness) properties of f?

1.3. **Lemma** (Some basic properties of Fourier coefficients). *Let $f, g \in L^1(\mathbb{T})$. Then we have*

(1) $\widehat{f+g}(m) = \widehat{f}(m) + \widehat{g}(m)$;
(2) $\widehat{\lambda f}(m) = \lambda \widehat{f}(m)$, $\lambda \in \mathbb{C}$;
(3) $\widehat{\tau_y f}(m) = \widehat{f}(m) e^{-2\pi i m y}$, $\tau_y f := f(x-y)$, $y \in \mathbb{R}$;
(4) $\widehat{e^{2\pi i k \cdot} f}(m) = \widehat{f}(m-k)$.

Proof. We only check the property (3). Using the change of variables $u = x - y$ we get

$$\begin{aligned}\widehat{\tau_y f}(m) &= \int_0^1 f(x-y) e^{-2\pi i m x}\,\mathrm{d}x \\ &= \int_{-y}^{1-y} f(u) e^{-2\pi i m(u+y)}\,\mathrm{d}u = e^{-2\pi i m y} \int_{-y}^{1-y} f(u) e^{-2\pi i m u}\,\mathrm{d}u\end{aligned}$$

$$= e^{-2\pi i m y} \int_0^1 f(u) e^{-2\pi i m u} \, \mathrm{d}u = e^{-2\pi i m y} \widehat{f}(m).$$

The penultimate step used the 1-periodicity of the function $u \mapsto f(u)e^{-2\pi i m u}$. The fact that

$$\int_a^{a+1} g = \int_0^1 g$$

for all $a \in \mathbb{R}$ and $g \in L^1(T)$ is left as an exercise. □

It is clear that

$$\sup_{m \in \mathbb{Z}} |\widehat{f}(m)| \le \|f\|_{L^1},$$

where $\|f\|_{L^1} = \|f\|_{L^1(0,1)} = \int_0^1 |f|$. However, more is true even with the minimal regularity assumption $f \in L^1(\mathbb{T})$.

1.4. **Lemma** (Riemann-Lebesgue). *If $f \in L^1(\mathbb{T})$ then*

$$\lim_{|m| \to \infty} \widehat{f}(m) = 0.$$

Proof. Using the property (3) of Lemma 1.3 we get that

$$\int_0^1 f\Big(x - \frac{1}{2m}\Big) e^{-2\pi i m x} \, \mathrm{d}x = \widehat{\tau_{\frac{1}{2m}} f}(m) = \widehat{f}(m) e^{-\pi i} = -\widehat{f}(m).$$

Therefore, we get the identity

$$\widehat{f}(m) = \frac{1}{2} \int_0^1 \Big[f(x) - f\Big(x - \frac{1}{2m}\Big) \Big] e^{-2\pi i m x} \, \mathrm{d}x,$$

and thus the estimate

$$|\widehat{f}(m)| \lesssim \int_0^1 \Big| f(x) - f\Big(x - \frac{1}{2m}\Big) \Big| \, \mathrm{d}x.$$

Here we used that $|e^{ix}| = 1$. The result follows now directly from the next lemma. □

We need the following intuitive (but non-trivial) result from Real Analysis I. This is one of the few results we take from that course as given.

1.5. **Lemma.** *Suppose $F \in L^p(\mathbb{R})$ (i.e. $\int_{\mathbb{R}} |F|^p < \infty$) for $1 \le p < \infty$. Then*

$$\lim_{h \to 0} \int_{\mathbb{R}} |F(x+h) - F(x)|^p \, \mathrm{d}x = 0.$$

1.6. **Corollary.** *Suppose $f \in L^p(\mathbb{T})$ (i.e. $\int_0^1 |f|^p < \infty$ and f is 1-periodic), $1 \le p < \infty$. Then*

$$\lim_{h \to 0} \int_0^1 |f(x+h) - f(x)|^p \, \mathrm{d}x = 0.$$

Proof. Define $F := f 1_{(-1,2)} \in L^p(\mathbb{R})$ and notice that for all small enough h we have

$$\int_0^1 |f(x+h) - f(x)|^p \, \mathrm{d}x = \int_0^1 |F(x+h) - F(x)|^p \, \mathrm{d}x \le \int_{\mathbb{R}} |F(x+h) - F(x)|^p \, \mathrm{d}x,$$

and use Lemma 1.5. □

1.7. *Remark.* We introduce here the following highly convenient notation. We denote $A \lesssim B$ if $A \le CB$ for some unimportant constant C. This means that C cannot depend on anything relevant like some important parameter ϵ. That is, C can e.g. be some uniform constant, or some constant depending on some fixed integrability exponent p. We can write $A \lesssim_\epsilon B$ to mean that $A \le C(\epsilon)B$ for some constant $C(\epsilon)$ that is now allowed to depend on some given parameter ϵ. We will also write $A \sim B$ if $A \lesssim B \lesssim A$.

2. Convolution and approximate identities

Let $f, g \in L^1(\mathbb{T})$, and define for $x \in [0,1]$ the convolution

$$f * g(x) = \int_0^1 f(y)g(x-y)\,\mathrm{d}y.$$

This is well-defined, since $\int_0^1 |f(y)g(x-y)|\,\mathrm{d}y < \infty$ for a.e. $x \in [0,1]$. The latter follows from

$$\int_0^1 \int_0^1 |f(y)g(x-y)|\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 |f(y)| \int_0^1 |g(x-y)|\,\mathrm{d}x\,\mathrm{d}y = \Big(\int_0^1 |f|\Big)\Big(\int_0^1 |g|\Big).$$

Therefore, we have $f * g \in L^1(\mathbb{T})$ and

$$\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}.$$

The following properties of the convolution are left as an exercise (here $f, g, h \in L^1(\mathbb{T})$):

(1) $f * (g + h) = f * g + f * h$;
(2) $(\lambda f) * g = \lambda(f * g)$, $\lambda \in \mathbb{C}$;
(3) $f * g = g * f$;
(4) $f * (g * h) = (f * g) * h$;
(5) $f * g$ is continuous if f or g is.
(6) $\widehat{f * g}(m) = \widehat{f}(m)\widehat{g}(m)$.

2.1. **Definition.** A family $\varphi_r \in L^1(\mathbb{T})$, $r > 0$, is an approximate identity (as $r \to \infty$) if the following conditions hold.

(1) We have $\int_{-1/2}^{1/2} \varphi_r = 1$ for all $r > 0$.
(2) We have $\sup_r \|\varphi_r\|_{L^1(-1/2,1/2)} < \infty$. (Follows from (1) if always $\varphi_r \ge 0$.)
(3) For every $\delta > 0$ we have

$$\lim_{r\to\infty} \int_{\delta \le |x| \le 1/2} |\varphi_r(x)|\,\mathrm{d}x = 0.$$

Convolutions with approximate identities $f * \varphi_r$ are a very important way to approximate a given function f as $r \to \infty$. We will see that $f * \varphi_r \to f$ in many senses. Before we can prove the convergence results we need one of the absolutely most fundamental inequalities of analysis.

2.2. **Lemma** (Hölder's inequality). *Let $p \in (1,\infty)$ and $p' \in (1,\infty)$ be defined via the relation*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then

$$\|fg\|_{L^1} \le \|f\|_{L^p}\|g\|_{L^{p'}},$$

where

$$\|f\|_{L^p} = \Big(\int |f|^p \Big)^{1/p}.$$

Proof. The short proof is given in Real Analysis I, but we also give a quick proof here. Young's inequality says that

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}, \qquad a, b \ge 0.$$

This can be proved by defining

$$h(x) = \frac{x^p}{p} + \frac{1}{p'} - x, \qquad x \ge 0,$$

and noticing that, by elementary analysis (differentiation),

$$h(x) \le h(1) = 0, \text{ i.e., } x \le \frac{x^p}{p} + \frac{1}{p'}.$$

Apply this with $x = ab^{1/(1-p)}$ to get

$$ab^{1/(1-p)} \le \frac{a^p}{p} b^{-p'} + \frac{1}{p'}.$$

Here we used that $p' = p/(p-1)$. Multiply both sides of this inequality with

$$b^{1-\frac{1}{1-p}} = b^{\frac{-p}{1-p}} = b^{p'}$$

to establish Young's inequality.

Apply Young's inequality with

$$a = \frac{|f(x)|}{\|f\|_{L^p}} \qquad \text{and} \qquad b = \frac{|g(x)|}{\|g\|_{L^{p'}}},$$

and integrate the resulting pointwise inequality to get

$$\frac{1}{\|f\|_{L^p}\|g\|_{L^{p'}}} \|fg\|_{L^1} \le \frac{1}{p} + \frac{1}{p'} = 1.$$

This is Hölder's inequality and we are done. □

A small argument involving Hölder's inequality shows the triangle inequality of the L^p-norm:

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

See Real Analysis I.

2.3. *Remark.* We take this opportunity to remark that $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$ for all $1 \le p \le \infty$ (so that in particular $\widehat{f}(m)$ is defined also for $f \in L^p(\mathbb{T})$). To recap, here $f \in L^p(\mathbb{T})$ means that $f\colon \mathbb{R} \to \mathbb{C}$ is 1-periodic and $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} = \Big(\int_0^1 |f|^p \Big)^{1/p}$$

for $p < \infty$ and $\|f\|_{L^\infty}$ is the essential supremum, i.e.,

$$\|f\|_{L^\infty} = \inf\{C \ge 0 \colon |f(x)| \le C \text{ for a.e. } x \in [0,1]\}.$$

Indeed, from Hölder's inequality it follows that for $p < \infty$ we have

$$\|f\|_{L^1} = \int_0^1 |f| = \int_0^1 |f| \cdot 1 \le \|f\|_{L^p} \|1\|_{L^{p'}} = \|f\|_{L^p},$$

while the case $p = \infty$ is obvious.

2.4. **Proposition.** *Let $1 \le p < \infty$, $f \in L^p(\mathbb{T})$ and $(\varphi_r)_{r>0}$ be an approximate identity. Then we have*

$$\|f - f * \varphi_r\|_{L^p} \to 0, \qquad r \to \infty.$$

Proof. Using $\int_{-1/2}^{1/2} \varphi_r = 1$ and $f * \varphi_r = \varphi_r * f$ we write the pointwise identity

$$\begin{aligned} f(x) - f * \varphi_r(x) &= f(x) \int_{-1/2}^{1/2} \varphi_r(y)\,\mathrm{d}y - \int_{-1/2}^{1/2} f(x-y)\varphi_r(y)\,\mathrm{d}y \\ &= \int_{-1/2}^{1/2} [f(x) - f(x-y)]\varphi_r(y)\,\mathrm{d}y. \end{aligned}$$

For the moment let $p > 1$. We get using Hölder's inequality that

$$\begin{aligned} |f(x) - f * \varphi_r(x)| &\le \int_{-1/2}^{1/2} |f(x) - f(x-y)||\varphi_r(y)|^{1/p} |\varphi_r(y)^{1/p'}|\,\mathrm{d}y \\ &\le \Big(\int_{-1/2}^{1/2} |f(x) - f(x-y)|^p |\varphi_r(y)|\,\mathrm{d}y \Big)^{1/p} \Big(\int_{-1/2}^{1/2} |\varphi_r(y)|\,\mathrm{d}y \Big)^{1/p'} \\ &\lesssim \Big(\int_{-1/2}^{1/2} |f(x) - f(x-y)|^p |\varphi_r(y)|\,\mathrm{d}y \Big)^{1/p}, \end{aligned}$$

where the last step used that $\sup_r \|\varphi_r\|_{L^1} \lesssim 1$. Therefore, we have

$$|f(x) - f * \varphi_r(x)|^p \lesssim \int_{-1/2}^{1/2} |f(x) - f(x-y)|^p |\varphi_r(y)|\,\mathrm{d}y,$$

which also clearly holds with $p = 1$. We integrate this over $x \in [-1/2, 1/2]$, and use Fubini's theorem, to get that

$$\|f - f * \varphi_r\|_{L^p}^p \lesssim \int_{-1/2}^{1/2} |\varphi_r(y)| \int_{-1/2}^{1/2} |f(x) - f(x-y)|^p\,\mathrm{d}x\,\mathrm{d}y.$$

Let $\epsilon > 0$. Using Corollary (1.6) we find $\delta > 0$ so that

$$\int_{-1/2}^{1/2} |f(x) - f(x-y)|^p\,\mathrm{d}x < \epsilon$$

whenever $|y| < \delta$. Using property (3) of Definition 2.1 we find r_0 so that

$$\int_{\delta \le |y| \le 1/2} |\varphi_r(y)|\,\mathrm{d}y < \epsilon$$

for all $r \geq r_0$. For all $r \geq r_0$ we therefore have

$$\|f - f * \varphi_r\|_{L^p}^p \lesssim \epsilon \int_{-1/2}^{1/2} |\varphi_r(y)| \, \mathrm{d}y + \epsilon \|f\|_{L^p}^p.$$

The claim follows recalling $\sup_r \|\varphi_r\|_{L^1} \lesssim 1$. □

2.5. **Proposition.** *Let $f \colon \mathbb{T} \to \mathbb{C}$ be continuous and $(\varphi_r)_{r>0}$ be an approximate identity. Then we have*

$$\|f - f * \varphi_r\|_{L^\infty} \to 0, \qquad r \to \infty.$$

Proof. Given $\epsilon > 0$ we find, using the uniform continuity of f in the interval $[-1, 1]$, a $\delta \in (0, 1/2)$ so that

$$|f(x) - f(x - y)| < \epsilon$$

whenever $x, y \in [-1/2, 1/2]$ and $|y| < \delta$. We use this in combination with the estimate from the proof of the previous Proposition:

$$\begin{aligned} |f(x) - f * \varphi_r(x)| &\lesssim \int_{-1/2}^{1/2} |f(x) - f(x-y)||\varphi_r(y)| \, \mathrm{d}y \\ &\lesssim \epsilon \int_{|y| \leq \delta} |\varphi_r(y)| \, \mathrm{d}y + \|f\|_{L^\infty} \int_{\delta \leq |y| \leq 1/2} |\varphi_r(y)| \, \mathrm{d}y \\ &\lesssim \epsilon + \epsilon \|f\|_{L^\infty} \end{aligned}$$

for large enough r. □

2.6. *Remark.* If $f \in L^\infty(\mathbb{T})$ is continuous in some single point x, the previous proof shows that then

$$\lim_{r \to \infty} f * \varphi_r(x) = f(x).$$

3. The Dirichlet kernel

If $f \in L^1(\mathbb{T})$ and

$$P(x) = \sum_{|m| \leq N} a_m e^{2\pi i m x},$$

then we have the convolution identity

$$f * P(x) = \sum_{|m| \leq N} a_m \int_0^1 f(y) e^{2\pi i m (x-y)} \, \mathrm{d}y = \sum_{|m| \leq N} a_m \widehat{f}(m) e^{2\pi i m x}.$$

If we apply this with $P(x) = D_N(x)$, where D_N is the Dirichlet kernel

$$D_N(x) := \sum_{|m| \leq N} e^{2\pi i m x},$$

we see that we recover the Fourier series of f, i.e., we have

$$S_N f(x) = \sum_{|m| \leq N} \widehat{f}(m) e^{2\pi i m x} = f * D_N(x).$$

By summing up the geometric series we obtain that

$$D_N(x) = e^{-2\pi i N x}\frac{e^{2\pi i(2N+1)x}-1}{e^{2\pi i x}-1} = \frac{e^{2\pi i(N+1)x}-e^{-2\pi i N x}}{e^{\pi i x}[e^{\pi i x}-e^{-i\pi x}]}$$

Notice that from $e^{ix} = \cos x + i\sin x$ it follows that

$$\sin x = \frac{1}{2i}(e^{ix}-e^{-ix}),$$

and thus we get an alternative formula for the Dirichlet kernel:

$$D_N(x) = \frac{e^{\pi i(2N+1)x}-e^{-\pi i(2N+1)x}}{e^{\pi i x}-e^{-i\pi x}} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}. \tag{3.1}$$

It makes sense to ask whether $(D_N)_{N=0}^\infty$ is an approximate identity. We at least have that

$$\int_{-1/2}^{1/2} D_n(x)\,\mathrm{d}x = \sum_{|m|\le N}\int_0^1 e^{2\pi i m x}\,\mathrm{d}x = \sum_{|m|\le N}\delta_{0,m} = 1.$$

Unfortunately, we will next show that $\sup_N \|D_N\|_{L^1} = \infty$. As $|\sin(\pi x)| \le \pi|x|$ we have that

$$\int_{-1/2}^{1/2}|D_N(x)|\,\mathrm{d}x \gtrsim \int_{-1/2}^{1/2}\frac{|\sin((2N+1)\pi x)|}{|x|}\,\mathrm{d}x \sim \int_0^{1/2}|\sin((2N+1)\pi x)|\frac{\mathrm{d}x}{x}.$$

Performing the change of variables $u = (2N+1)\pi x$ we get

$$\int_{-1/2}^{1/2}|D_N(x)|\,\mathrm{d}x \gtrsim \int_0^{(N+1/2)\pi}|\sin u|\frac{\mathrm{d}u}{u} \gtrsim \sum_{k=1}^N \frac{1}{k}\int_{(k-1)\pi}^{k\pi}|\sin(u)|\,\mathrm{d}u \sim \sum_{k=1}^N\frac{1}{k},$$

since $\int_{(k-1)\pi}^{k\pi}|\sin(u)|\,\mathrm{d}u = 2$ for all k. The harmonic series diverges and so we get that $\sup_N\|D_N\|_{L^1} = \infty$. Thus, $(D_N)_{N=0}^\infty$ is not an approximate identity, and understanding the convergence properties of Fourier series becomes hard. Despite this, there are still many positive results of varying difficulty, with some of them very deep, concerning the convergence of Fourier series. We will discuss them later.

4. The Fejér kernel and applications

Despite the negative result that $(D_N)_{N=0}^\infty$ is not an approximate identity, we can still get various interesting results utilising the theory of approximate identities. The trick is to study the averaged kernels

$$F_N(x) = \frac{1}{N+1}(D_0(x)+D_1(x)+\cdots+D_N(x)) = \frac{1}{N+1}\sum_{k=0}^N D_k(x).$$

These are called the Fejér kernels, and they turn out to be an approximate identity – even positive functions. As $\int_{-1/2}^{1/2} D_k = 1$ for all k it is clear that also $\int_{-1/2}^{1/2} F_N = 1$. To see the other properties we need to perform some algebraic manipulations, which are left as an exercise. It follows that we can write

$$F_N(x) = \frac{1}{N+1}\Big(\frac{\sin((N+1)\pi x)}{\sin(\pi x)}\Big)^2 \ge 0. \tag{4.1}$$

Recall that in the case of positive functions the property (1) in Definition 2.1 implies the property (2). The property (3) of Definition 2.1 is left as an exercise. We conclude the following lemma.

4.2. **Lemma.** *The family of Fejér kernels $(F_N)_{N=0}^{\infty}$ is an approximate identity consisting of positive functions.*

This means that we can get various convergence results concerning

$$f * F_N(x) = \frac{1}{N+1} \sum_{k=0}^{N} f * D_k(x) = \frac{1}{N+1} \sum_{k=0}^{N} S_k f(x) =: \sigma_N f(x).$$

Notice that σ_N is the arithmetic mean of the Fourier partial sums. In general, Cesàro summability of a series $\sum_k a_k$ means that the average of the partial sums $s_n := \sum_{k=0}^{n} a_k$, i.e.,

$$\frac{s_0 + s_1 + \cdots + s_N}{N+1},$$

has a limit. It is an exercise to show that if $\sum_k a_k$ converges to A, then also

$$\lim_{N\to\infty} \frac{s_0 + s_1 + \cdots + s_N}{N+1} = A.$$

However, Cesàro summability is a weaker notion – the converse is not true. To reiterate, we can use Section 2 to at least get results concerning the Cesàro summability of the Fourier series $\sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi i m x}$. Indeed, as a corollary of Lemma 4.2, Proposition 2.4 and Proposition 2.5 we immediately get:

4.3. **Theorem.** *Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T})$, or $p = \infty$ and f be continuous. Then we have*

$$\|\sigma_N f - f\|_{L^p} \to 0, \qquad \text{when } N \to \infty.$$

A useful corollary of this result is the following.

4.4. **Corollary.**
(1) The trigonometric polynomials are dense in $L^p(\mathbb{T})$, $1 \le p < \infty$.
(2) Every continuous function on the torus is a uniform limit of trigonometric polynomials.
(3) If $f \in L^1(\mathbb{T})$ and $\widehat{f}(m) = 0$ for all $m \in \mathbb{Z}$, then $f = 0$ almost everywhere.

Proof. We notice that (1) and (2) are now obvious as clearly $\sigma_N f$ is a trigonometric polynomial. The part (3) follows by noticing that if $\widehat{f}(m) = 0$ for all $m \in \mathbb{Z}$, then $\sigma_N f = 0$ for all N. Hence $\|f\|_{L^1} = \lim_{N\to\infty} \|f - \sigma_N f\|_{L^1} = 0$. □

4.5. *Remark.* From (3) it follows that if $f, g \in L^1(\mathbb{T})$ and $\widehat{f}(m) = \widehat{g}(m)$ for all $m \in \mathbb{Z}$, then $f = g$ almost everywhere.

We may also give a new proof of the Riemann–Lebesgue lemma, Lemma 1.4.

New proof of Lemma 1.4. Let $f \in L^1(\mathbb{T})$ and $\epsilon > 0$. Choose a trigonometric polynomial P so that $\|f - P\|_{L^1} < \epsilon$. For large enough $|m|$ we have $\widehat{P}(m) = 0$, and so for these m we also have

$$|\widehat{f}(m)| = |\widehat{f}(m) - \widehat{P}(m)| \le \|f - P\|_{L^1} < \epsilon.$$

We have shown that $\lim_{|m|\to\infty} \widehat{f}(m) = 0$. □

Finally, we record the following consequence.

4.6. **Proposition.** *Suppose that* $f \in L^1(\mathbb{T})$ *is such that*

$$\sum_{m\in\mathbb{Z}} |\widehat{f}(m)| < \infty.$$

Then for almost every x *we have*

$$f(x) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imx}.$$

Proof. Under the assumptions that $\sum_m |\widehat{f}(m)| < \infty$ the function

$$g(x) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imx}$$

satisfies $g \in L^1(\mathbb{T})$ and $\widehat{f}(m) = \widehat{g}(m)$. Therefore $f = g$ a.e. □

4.7. *Remark.* Notice that the function g in the above proof is actually continuous under the assumptions of the Proposition. Indeed, by Weierstrass M-test the series converges uniformly (as $|f(m)e^{2\pi imx}| = |\widehat{f}(m)|$). A uniform limit of continuous functions is continuous. Thus, in this situation f agrees with a continuous function almost everywhere.

5. Pointwise convergence of the Fejér means $\sigma_N f$: Improved results

In view of Lemma 4.2 and Remark 2.6 we know that if $f \in L^\infty(\mathbb{T})$ is continuous in some point x, then

$$\lim_{N\to\infty} \sigma_N f(x) = f(x).$$

In fact, for this we only need that $f \in L^1(\mathbb{T})$ is continuous in some point x (and not that $f \in L^\infty(\mathbb{T})$). This is because the Fejér kernels actually satisfy the pointwise bound

$$\sup_{\delta\le|y|\le 1/2} F_N(y) \lesssim_\delta \frac{1}{N} \tag{5.1}$$

instead of just the property (3) in Definition 2.1. Mimicking the proof of the Proposition 2.5 it is then easy to see that $f \in L^1(\mathbb{T})$ is enough (see also the exercises). It follows that if $f \in L^1(\mathbb{T})$ is continuous in x and we know that the limit $\lim_{N\to\infty} S_N f(x)$ exists, then we must have

$$f(x) = \lim_{N\to\infty} \sigma_N f(x) = \lim_{N\to\infty} S_N f(x).$$

The latter equality follows from the discussion concerning Cesàro summability. This result can be further improved – and we do this below.

5.2. **Theorem.** *Suppose that a function* $f \in L^1(\mathbb{T})$ *has the left and right limits at a point* x_0*, denoted by* $f(x_0-)$ *and* $f(x_0+)$*, respectively. Then we have*

$$\lim_{N\to\infty} \sigma_N f(x_0) = \frac{f(x_0+) + f(x_0-)}{2}.$$

Proof. Let $\epsilon > 0$ and choose $\delta \in (0, 1/2)$ so that

$$\Big|\frac{f(x_0+y)+f(x_0-y)}{2} - \frac{f(x_0+)+f(x_0-)}{2}\Big| < \epsilon \tag{5.3}$$

whenever $0 < y < \delta$. Using (5.1) we find N_0 so that for all $N \ge N_0$ we have

$$\sup_{\delta \le |y| \le 1/2} F_N(y) < \epsilon. \tag{5.4}$$

By usual manipulations we get the identities

$$\begin{aligned}\sigma_N f(x_0) - f(x_0+) &= \int_{-1/2}^{1/2} F_N(-y)[f(x_0+y) - f(x_0+)]\,\mathrm{d}y \\ &= \int_{-1/2}^{1/2} F_N(y)[f(x_0+y) - f(x_0+)]\,\mathrm{d}y\end{aligned}$$

and

$$\sigma_N f(x_0) - f(x_0-) = \int_{-1/2}^{1/2} F_N(y)[f(x_0-y) - f(x_0-)]\,\mathrm{d}y.$$

This leads to the identity

$$\begin{aligned}&\sigma_N f(x_0) - \frac{f(x_0+)+f(x_0-)}{2} \\ &\quad= \int_{-1/2}^{1/2} F_N(y)\Big[\frac{f(x_0+y)+f(x_0-y)}{2} - \frac{f(x_0+)+f(x_0-)}{2}\Big]\,\mathrm{d}y \\ &\quad= 2\int_{0}^{1/2} F_N(y)\Big[\frac{f(x_0+y)+f(x_0-y)}{2} - \frac{f(x_0+)+f(x_0-)}{2}\Big]\,\mathrm{d}y,\end{aligned}$$

where we used the fact that the integrand is even. Split this into two pieces I and II, where in I we integrate over $y \in [0, \delta)$ and in II we integrate over $y \in [\delta, 1/2]$. By (5.3) we have that

$$|I| \lesssim \epsilon \int_{-1/2}^{1/2} F_N(y)\,\mathrm{d}y = \epsilon.$$

By (5.4) we have for all $N \ge N_0$ that

$$|II| \le \epsilon(\|f - f(x_0+)\|_{L^1} + \|f - f(x_0-)\|_{L^1}) = \epsilon C(f, x_0).$$

This proves the claim as $C(f, x_0)$ is just a finite constant depending on f and x_0. □

The corresponding elementary corollary about the behaviour of Fourier series is recorded below.

5.5. **Proposition.** *Suppose that a function $f \in L^1(\mathbb{T})$ has the left and right limits at a point x_0, denoted by $f(x_0-)$ and $f(x_0+)$, respectively. Suppose also that the limit $\lim_{N\to\infty} S_N f(x_0)$ exists. Then we must have*

$$\lim_{N\to\infty} S_N f(x_0) = \frac{f(x_0+)+f(x_0-)}{2}.$$

Proof. If $A = \lim_{N\to\infty} S_N f(x_0)$ then we know (see the discussion about Cesàro summability and the exercises) that also $\lim_{N\to\infty} \sigma_N f(x) = A$. But so using Theorem 5.2 we have

$$\frac{f(x_0+) + f(x_0-)}{2} = \lim_{N\to\infty} \sigma_N f(x) = A = \lim_{N\to\infty} S_N f(x_0).$$

□

While Proposition 5.5 is not so satisfactory as we need to assume that the limit $\lim_{N\to\infty} S_N f(x)$ exists, it is still quite useful. The following example showcases a situation, where such principles can be used. However, in the example we actually only need the easier version where f is continuous at a given point, and not this further refined version.

5.6. **Example.** In the exercises we show that if

$$f(x) = 1/2 - |x|, \qquad -1/2 \le |x| \le 1/2,$$

(and then continued periodically) we have

$$\widehat{f}(m) = \begin{cases} \frac{1}{4} & \text{if } m = 0; \\ 0 & \text{if } m \neq 0 \text{ is even;} \\ \frac{1}{\pi^2 m^2} & \text{if } m \text{ is odd.} \end{cases}$$

It is clear that f is continuous at $x = 0$ and also that the limit

$$\lim_{N\to\infty} S_N f(0) = \sum_{m\in\mathbb{Z}} \widehat{f}(m) = \frac{1}{4} + \frac{2}{\pi^2}(1 + 3^{-2} + 5^{-2} + \cdots) = \frac{1}{4} + \frac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

exists. Therefore, we know that

$$\frac{1}{4} + \frac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \lim_{N\to\infty} S_N f(0) = f(0) = \frac{1}{2},$$

and we can conclude that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

5.1. **Almost everywhere convergence of $\sigma_N f$.** We return to discussing the convergence properties of $\sigma_N f$ (as opposed to $S_N f$). Theorem 5.2 shows that if $f \in L^1(\mathbb{T})$ is relatively nice (it has left and right limit everywhere), then $\sigma_N f(x)$ converges pointwise everywhere (but not necessarily to $f(x)$). A lot can be said assuming only that $f \in L^1(\mathbb{T})$ – and this is what we will aim to do next. Our goal is to prove the following theorem.

5.7. **Theorem.** *Let $f \in L^1(\mathbb{T})$. Then we have that*

$$\sigma_N f(x) \to f(x)$$

for almost every x.

The proof contains many very important principles of real analysis. For the following few lemmas, we will be working with ordinary (non-periodic) functions defined in $\mathbb{R}$. As previously, we denote such functions usually by F instead of f. For $x \in \mathbb{R}$ define

$$m(x) = \frac{1}{1+|x|^2} \qquad \text{and} \qquad m_\epsilon(x) = \frac{1}{\epsilon} m\Big(\frac{x}{\epsilon}\Big).$$

5.8. **Lemma.** *The operator*

$$G(F)(x) := \sup_{\epsilon>0} \int_{\mathbb{R}} m_\epsilon(y)|F(x-y)| \, \mathrm{d}y$$

satisfies for all $x \in \mathbb{R}$ the pointwise bound

$$G(F)(x) \lesssim M(F)(x),$$

where M is the (centred) Hardy–Littlewood maximal function

$$M(F)(x) := \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |F(y)| \, \mathrm{d}y.$$

Proof. Fix an arbitrary $\epsilon > 0$. We have by change of variables that

$$\begin{aligned} \int_{\mathbb{R}} m_\epsilon(y)|F(x-y)| \, \mathrm{d}y &= \int_{\mathbb{R}} \frac{1}{\epsilon} m\Big(\frac{y}{\epsilon}\Big)|F(x-y)| \, \mathrm{d}y \\ &= \int_{\mathbb{R}} m(y)|F(x-\epsilon y)| \, \mathrm{d}y = \int_{\mathbb{R}} \frac{|F(x-\epsilon y)|}{1+|y|^2} \, \mathrm{d}y. \end{aligned}$$

We now estimate

$$\begin{aligned} \int_{\mathbb{R}} \frac{|F(x-\epsilon y)|}{1+|y|^2} \, \mathrm{d}y &\le \int_{-1}^{1} |F(x-\epsilon y)| \, \mathrm{d}y + \sum_{k=1}^{\infty} \int_{2^{k-1} \le |y| \le 2^k} \frac{|F(x-\epsilon y)|}{1+|y|^2} \, \mathrm{d}y \\ &\lesssim \sum_{k=0}^{\infty} 2^{-2k} \int_{-2^k}^{2^k} |F(x-\epsilon y)| \, \mathrm{d}y \\ &= \sum_{k=0}^{\infty} 2^{-2k} \epsilon^{-1} \int_{x-\epsilon 2^k}^{x+\epsilon 2^k} |F(y)| \, \mathrm{d}y \lesssim M(F)(x) \sum_{k=0}^{\infty} 2^{-k} = 2M(F)(x). \end{aligned}$$

□

Define the space $L^{1,\infty}(\mathbb{R})$ (the weak-L^1) via the requirement that $F \in L^{1,\infty}(\mathbb{R})$ if

$$\|F\|_{L^{1,\infty}(\mathbb{R})} := \sup_{\lambda>0} \lambda |\{x \in \mathbb{R} \colon |F(x)| > \lambda\}| < \infty.$$

Here $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$.

5.9. **Lemma.** *We have that $G \colon L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})$ boundedly – i.e.,*

$$\|G(F)\|_{L^{1,\infty}(\mathbb{R})} \lesssim \|F\|_{L^1(\mathbb{R})}.$$

Proof. Fix $\lambda > 0$. If $G(F)(x) > \lambda$ then by Lemma 5.8 we have that

$$M(F)(x) > c_0 \lambda$$

for some absolute constant c_0. This implies that

$$\sup_{\lambda>0} \lambda|\{x \in \mathbb{R}\colon G(F)(x) > \lambda\}| \lesssim \sup_{\lambda>0} \lambda|\{x \in \mathbb{R}\colon M(F)(x) > \lambda\}|.$$

It is a fundamental basic fact of real analysis – see Real Analysis I or the lemma below – that

$$\sup_{\lambda>0} \lambda|\{x \in \mathbb{R}\colon M(F)(x) > \lambda\}| \lesssim \int_{\mathbb{R}} |F|.$$

This ends the proof. □

5.10. **Lemma.** *We have that $M\colon L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})$ boundedly – i.e.,*

$$\|M(F)\|_{L^{1,\infty}(\mathbb{R})} \lesssim \|F\|_{L^1(\mathbb{R})}.$$

Proof. Fix $\lambda > 0$ and define

$$\Omega_\lambda := \{x \in \mathbb{R}\colon M(F)(x) > \lambda\}.$$

Let $K \subset \Omega_\lambda$ be an arbitrary compact set, and for every $x \in K$ choose an interval $I_x = (x - r_x, x + r_x)$ so that

$$\frac{1}{|I_x|}\int_{I_x} |F| > \lambda.$$

By compactness choose a finite subfamily $I_{x_1}, \ldots, I_{x_m}$ so that

$$K \subset \bigcup_{j=1}^{m} I_{x_j}.$$

By reordering we may assume that $r_{x_j} \ge r_{x_{j+1}}$ for $j = 1, \ldots, m-1$. Let $I_1 = I_{x_1}$, and then let I_2 be the biggest interval I_{x_j} so that $I_{x_j} \not\subset 3I_1$ (if it exists). Let then I_3 be the biggest interval I_{x_j} so that $I_{x_j} \not\subset 3I_1 \cup 3I_2$ (if it exists). We continue this selection process as long as possible – the process finishes after a finite, say M, number of steps. It is clear that

$$K \subset \bigcup_{j=1}^{m} I_{x_j} \subset \bigcup_{i=1}^{M} 3I_i.$$

What is of real importance is that the intervals I_i, $i = 1, \ldots, M$, are disjoint. To see this suppose that $I_{i_1} \cap I_{i_2} \ne \emptyset$ for some $1 \le i_1 < i_2 \le M$. We have that $|I_{i_1}| \ge |I_{i_2}|$ and so $I_{i_2} \subset 3I_{i_1}$ – a contradiction with the selection process.

We now get

$$|K| \le \sum_{i=1}^{M} |3I_i| \lesssim \sum_{i=1}^{M} |I_i| \le \frac{1}{\lambda}\sum_{i=1}^{M}\int_{I_i} |F| \le \frac{1}{\lambda}\int_{\mathbb{R}} |F|.$$

As $K \subset \Omega_\lambda$ was an arbitrary compact subset, the claim follows. □

5.11. *Remark.* It follows that $\|M(F)\|_{L^p(\mathbb{R})} \lesssim \|F\|_{L^p(\mathbb{R})}$ for every $1 < p < \infty$. Indeed this follows from the previous lemma and the trivial estimate $\|M(F)\|_{L^\infty(\mathbb{R})} \lesssim \|F\|_{L^\infty(\mathbb{R})}$ by standard interpolation – see the very straightforward Marcinkiewicz interpolation theorem, Theorem A.1 in the Appendix.

We return to the periodic setting. Using Lemma 5.9 we will show that the maximal function

$$\mathscr{H} f(x) := \sup_{N\in\mathbb{N}} |\sigma_N f(x)|, \qquad f \in L^1(\mathbb{T}),$$

satisfies $\|\mathscr{H} f(x)\|_{L^{1,\infty}(\mathbb{T})} \lesssim \|f\|_{L^1(\mathbb{T})}$. (As in the previous remark this implies the $L^p(\mathbb{T}) \to L^p(\mathbb{T})$, $1 < p < \infty$, boundedness of $\mathscr{H}$ immediately.)

5.12. **Theorem.** *We have for every $f \in L^1(\mathbb{T})$ that*

$$\|\mathscr{H} f(x)\|_{L^{1,\infty}(\mathbb{T})} \lesssim \|f\|_{L^1(\mathbb{T})}.$$

Proof. In combination with the identity (4.1) for the Fejér kernels we use the elementary facts that $|\sin y| \le |y|$ for all $y \in \mathbb{R}$ and $|\sin y| \ge \frac{2}{\pi}|y|$ if $|y| \le \frac{\pi}{2}$. This gives that for all $|y| \le 1/2$ we have

$$\begin{aligned} F_N(y) = \frac{1}{N+1}\Big|\frac{\sin((N+1)\pi y)}{\sin(\pi y)}\Big|^2 &\lesssim \frac{1}{N+1}\Big|\frac{\sin((N+1)\pi y)}{y}\Big|^2 \\ &= (N+1)\Big|\frac{\sin((N+1)\pi y)}{(N+1)y}\Big|^2 \\ &\lesssim (N+1)\min\Big(1, \frac{1}{(N+1)^2|y|^2}\Big) \\ &\lesssim \frac{N+1}{1+(N+1)^2|y|^2} = m_\epsilon(y), \end{aligned}$$

where $\epsilon = \epsilon(N) = (N+1)^{-1}$. In the last estimate we used that $\min(1, 1/t) \lesssim 1/(1+t)$ for $t \ge 0$.

Define $F = f 1_{[-1,1]} \in L^1(\mathbb{R})$, and notice that we now have for $|x| \le 1/2$ that

$$\begin{aligned} \mathscr{H} f(x) &= \sup_{N\in\mathbb{N}} \Big| \int_{-1/2}^{1/2} F_N(y) f(x-y)\,\mathrm{d}y\Big| \\ &\lesssim \sup_{\epsilon>0} \int_{\mathbb{R}} m_\epsilon(y)|F(x-y)|\,\mathrm{d}y = G(F)(x). \end{aligned}$$

Using Lemma 5.9 we now get

$$\begin{aligned} \|\mathscr{H} f(x)\|_{L^{1,\infty}(\mathbb{T})} &= \sup_{\lambda>0} \lambda|\{x \in [-1/2, 1/2] \colon \mathscr{H} f(x) > \lambda\}| \\ &\lesssim \sup_{\lambda>0} \lambda|\{x \in \mathbb{R} \colon G(F)(x) > \lambda\}| \\ &\lesssim \int_{\mathbb{R}} |F(x)|\,\mathrm{d}x = \int_{-1}^{1} |f(x)|\,\mathrm{d}x = 2\int_{-1/2}^{1/2} |f(x)|\,\mathrm{d}x, \end{aligned}$$

and so we are done. □

We are now ready to prove Theorem 5.7. The proof involves again an important standard argument of real analysis.

Proof Theorem 5.7. The standard way to show almost everywhere convergence for integrable functions is via the following two steps: 1) show convergence in a dense subset and then 2) prove the boundedness of a relevant maximal operator.

If you have taken Real Analysis I compare to the proof of Lebesgue differentiation theorem via Lemma 5.10.

Let now $f \in L^1(\mathbb{T})$. It is enough to show that

$$|\{|x| \le 1/2\colon \limsup_{N\to\infty} |\sigma_N f(x) - f(x)| > 0\}| = 0.$$

We fix an arbitrary $\lambda > 0$ and show that

$$|\{|x| \le 1/2\colon \limsup_{N\to\infty} |\sigma_N f(x) - f(x)| > \lambda\}| = 0,$$

which is enough. Let $\epsilon > 0$. Choose a continuous $g \in L^1(\mathbb{T})$ (e.g. a trigonometric polynomial using Corollary (4.4)) so that

$$\int_{-1/2}^{1/2} |f(x) - g(x)|\,\mathrm{d}x < \epsilon.$$

We know that because g is continuous we have

$$\lim_{N\to\infty} \sigma_N g(x) = g(x)$$

for every $x \in [-1/2, 1/2]$. Estimating

$$|\sigma_N f(x) - f(x)| \le |\sigma_N(f-g)(x)| + |\sigma_N g(x) - g(x)| + |g(x) - f(x)|$$

we see that

$$\begin{aligned}\limsup_{N\to\infty} |\sigma_N f(x) - f(x)| &\le \sup_{N\in\mathbb{N}} |\sigma_N(f-g)(x)| + 0 + |f(x) - g(x)| \\ &= \mathscr{H}(f-g)(x) + |f(x) - g(x)|.\end{aligned}$$

Therefore, we have by Theorem 5.12 and the trivial inequality (Chebyshev's inequality)

$$|\{x \in A\colon |h(x)| > \lambda\}| = \int_A 1_{|h|>\lambda} = \lambda^{-1}\int_A \lambda 1_{|h|>\lambda} \le \lambda^{-1}\int_A |h|$$

that

$$\begin{aligned}&\Big|\Big\{|x| \le \frac{1}{2}\colon \limsup_{N\to\infty} |\sigma_N f(x) - f(x)| > \lambda\Big\}\Big| \\ &\le \Big|\Big\{|x| \le \frac{1}{2}\colon \mathscr{H}(f-g)(x) > \frac{\lambda}{2}\Big\}\Big| + \Big|\Big\{|x| \le \frac{1}{2}\colon |f(x) - g(x)| > \frac{\lambda}{2}\Big\}\Big| \\ &\lesssim \frac{1}{\lambda}\int_{-1/2}^{1/2} |f(x) - g(x)|\,\mathrm{d}x < \frac{\epsilon}{\lambda}.\end{aligned}$$

This ends the proof. □

6. Criteria for the pointwise convergence of Fourier series

Thus far we have a very satisfactory theory of the convergence properties of $\sigma_N f$, including the following key results:

(1) If $f \in L^p(\mathbb{T})$, $1 \le p < \infty$, then

$$\lim_{N\to\infty} \|\sigma_N f - f\|_{L^p} = 0$$

by Theorem 4.3.

(2) If $f \in L^1(\mathbb{T})$ has the left and right limits at a point x_0, denoted by $f(x_0-)$ and $f(x_0+)$, respectively, then we have

$$\lim_{N\to\infty} \sigma_N f(x_0) = \frac{f(x_0+)+f(x_0-)}{2}$$

by Theorem 5.2.

(3) For all $f \in L^1(\mathbb{T})$ we have that

$$\lim_{N\to\infty} \sigma_N f(x) = f(x)$$

for almost every x by Theorem 5.7.

The only result we have so far about $S_N f$ concerns the pointwise behaviour recorded in Proposition 5.5. While useful, it requires the a priori knowledge of the convergence of the series $\lim_{N\to\infty} S_N f(x_0)$. In this section we look at some pointwise results for $S_N f(x)$ – but these will require more than continuity. Later we will discuss the convergence in L^p norm. We warn the reader that some results that hold for $\sigma_N f$ fail spectacularly for $S_N f$. The following proposition is the first warning sign.

6.1. **Proposition** (duBois Reymond). *There exists a continuous function $f\colon \mathbb{T} \to \mathbb{C}$ so that for some x_0 we have*

$$\limsup_{N\to\infty} |S_N f(x_0)| = \infty.$$

Proof. Such a function can be constructed explicitly, but it is less tedious to show its existence abstractly. Recall the Banach–Steinhaus theorem (also known as the uniform boundedness principle) from functional analysis: a family of bounded linear operators $T_s\colon X \to Y$ between Banach spaces X and Y is either uniformly bounded ($\sup_s \|T_s\|_{X\to Y} < \infty$) or there is some single vector $x \in X$ such that $\sup_s \|T_s x\|_Y = \infty$. For $N \in \mathbb{N}$ we define the functionals $T_N\colon C(\mathbb{T}) \to \mathbb{C}$ by

$$T_N(f) := S_N f(0) = \int_{|y|\le 1/2} D_N(y) f(y)\,\mathrm{d}y.$$

Notice that if we could choose $f = f_N = \operatorname{sgn} D_N$ then we would get by Section 3.

$$T_N(f_N) = \int_{|y|\le 1/2} |D_N(y)|\,\mathrm{d}y \gtrsim \sum_{k=1}^{N} \frac{1}{k}.$$

This is not a continuous function as D_N has zeroes – the idea is to simply modify this choice of a f a little bit.

Let $N \ge 100$ be an integer and let $f_N \in C(\mathbb{T})$ be an even function bounded by 1, which is equal to $\operatorname{sgn} D_N$ except at small intervals of length $(2N)^{-2}$ around the $2N$ zeos of D_N. Let I_N denote the union of these intervals. Then we have

$$\|T_N\|_{C(\mathbb{T})\to\mathbb{C}} = \sup_{\substack{g\in C(\mathbb{T})\\ \|g\|_{L^\infty}\le 1}} |T_N(g)| \ge |T_N(f_N)| = \Big|\int_{|y|\le 1/2} D_N(y) f_N(y)\,\mathrm{d}y\Big|$$

so that by triangle inequality we have

$$\|T_N\|_{C(\mathbb{T})\to\mathbb{C}} \ge \int_{[-\frac12,\frac12]\setminus I_N} |D_N(y)|\,\mathrm{d}y - \Big|\int_{I_N} D_N(y) f_N(y)\,\mathrm{d}y\Big|$$

$$
\begin{aligned}
&= \int_{|y|\le \frac12} |D_N(y)|\,\mathrm{d}y - \int_{I_N} |D_N(y)|\,\mathrm{d}y - \Big|\int_{I_N} D_N(y) f_N(y)\,\mathrm{d}y\Big| \\
&\ge \int_{|y|\le \frac12} |D_N(y)|\,\mathrm{d}y - 2(2N+1)|I_N| \\
&\ge \int_{|y|\le \frac12} |D_N(y)|\,\mathrm{d}y - 6N(2N)^{-1} \ge c_0 \sum_{k=1}^N \frac{1}{k} - 3.
\end{aligned}
$$

It follows that $\sup_N \|T_N\|_{C(\mathbb{T})\to\mathbb{C}} = \infty$ and so by the uniform boundedness principle there has to be an $f \in C(\mathbb{T})$ for which we have

$$
\infty = \sup_N |T_N(f)| = \sup_N |S_N f(0)|.
$$

It follows that $\sup_{N\ge N_0} |S_N f(0)| = \infty$ for all N_0, and the claim follows. □

If f is somewhat more than just continuous, we have positive results. The results are very local in nature.

6.2. **Theorem** (Dini). *Let $f \in L^1(\mathbb{T})$. Suppose that for some x and $\delta > 0$ we have*

$$
\int_{|y|<\delta} \frac{|f(x+y)-f(x)|}{|y|}\,\mathrm{d}y < \infty.
$$

Then we have

$$
\lim_{N\to\infty} S_N f(x) = f(x).
$$

Proof. It follows that

$$
\int_{|y|\le \frac12} \frac{|f(x-y)-f(x)|}{|y|}\,\mathrm{d}y < \infty.
$$

Using $\int_{-1/2}^{1/2} D_N(y)\,\mathrm{d}y = 1$ write

$$
\begin{aligned}
S_N f(x) - f(x) &= \int_{|y|\le \frac12} [f(x-y)-f(x)] D_N(y)\,\mathrm{d}y \\
&= \int_{|y|\le \frac12} [f(x-y)-f(x)] \frac{\sin((2N+1)\pi y)}{\sin(\pi y)}\,\mathrm{d}y.
\end{aligned}
$$

For $y \in [-1/2, 1/2)$ define

$$
g(y) = \frac{1}{2i}\frac{f(x-y)-f(x)}{\sin(\pi y)}.
$$

Recalling that $|\sin y| \ge \frac{2}{\pi}|y|$ if $|y| \le \frac{\pi}{2}$ we have

$$
|g(y)| \le \frac{|f(x-y)-f(x)|}{|y|},
$$

and so $g \in L^1(\mathbb{T})$. Writing

$$
\sin((2N+1)\pi y) = \frac{1}{2i}[e^{2\pi i N y} e^{i\pi y} - e^{-2\pi i N y} e^{-i\pi y}]
$$

we see that

$$S_N f(x) - f(x) = \widehat{ge^{i\pi \cdot}}(-N) - \widehat{ge^{-i\pi \cdot}}(N).$$

The claim now follows from the Riemann–Lebesgue lemma, Lemma 1.4. □

We readily get the following corollary.

6.3. **Corollary.** *Let $f \in L^1(\mathbb{T})$.*

(1) Suppose that for some $\alpha \in (0,1]$, $C < \infty$, and for some x we have $|f(x) - f(y)| \le C|x-y|^\alpha$ for all y in some interval around x. Then $\lim_{N\to\infty} S_N f(x) = f(x)$.

(2) Suppose f is differentiable at x. Then $\lim_{N\to\infty} S_N f(x) = f(x)$.

Another direct corollary of Dini's criterion is the following.

6.4. **Corollary** (Riemann's localisation principle)**.** *If $f \in L^1(\mathbb{T})$ vanishes in some interval around x, then*

$$\lim_{N\to\infty} S_N f(x) = 0.$$

7. L^2 CONVERGENCE OF FOURIER SERIES

We recall the following fundamental proposition from functional analysis.

7.1. **Proposition.** *Let H be a Hilbert space with the inner product $\langle x \,|\, y\rangle$, $x, y \in H$. Let $(e_m)_{m\in\mathbb{Z}}$ be an orthonormal sequence in H – i.e., $\langle e_{m_1} \,|\, e_{m_2}\rangle = \delta_{m_1,m_2}$. Then the following are equivalent:*

(1) The sequence $(e_m)_{m\in\mathbb{Z}}$ is a complete orthonormal system – i.e.,

$$\overline{\operatorname{span}}(\{e_m \colon m \in \mathbb{Z}\}) = H.$$

(2) We have $\langle x \,|\, e_m\rangle = 0$ for all m if and only if $x = 0$.

(3) For all $x \in H$ we have

$$\lim_{N\to\infty} \Big\| x - \sum_{|m|\le N} \langle x \,|\, e_m\rangle e_m \Big\|_H = 0.$$

(4) For all $x \in H$ we have

$$\|x\|_H^2 = \sum_{m\in\mathbb{Z}} |\langle x \,|\, e_m\rangle|^2.$$

(5) For all $x, y \in H$ we have

$$\langle x \,|\, y\rangle = \sum_{m\in\mathbb{Z}} \langle x \,|\, e_m\rangle \overline{\langle y \,|\, e_m\rangle}.$$

Notice that $fg \in L^1(\mathbb{T})$ for $f, g \in L^2(\mathbb{T})$ by Hölder's inequality. We consider the Hilbert space $L^2(\mathbb{T})$ equipped with the inner product

$$\langle f \,|\, g\rangle = \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x.$$

The completeness of L^2 (and L^p spaces) is proved at least in Real Analysis I, so L^2 is a Hilbert space. This is why L^2 is very special compared to L^p, $p \ne 2$. Consequently, the L^2 convergence, together with some additional important properties, of Fourier series turns out to be simple.

Let $e_m(x) := e^{2\pi i m x}$, $m \in \mathbb{Z}$. As we have

$$\int_0^1 e_{m_1}(x)\overline{e_{m_2}(x)}\,\mathrm{d}x = \int_0^1 e^{2\pi i(m_1-m_2)x}\,\mathrm{d}x = \delta_{m_1,m_2},$$

$(e_m)_{m\in\mathbb{Z}}$ is an orthonormal sequence in H. Moreover, if

$$0 = \langle f\,|e_m\rangle = \int_0^1 f(x)e^{-2\pi i m x}\,\mathrm{d}x = \widehat{f}(m)$$

for all $m \in \mathbb{Z}$, then $f = 0$ by Corollary 4.4. Therefore, by Proposition 7.1 we have that $(e_m)_{m\in\mathbb{Z}}$ is a complete orthonormal system in $L^2(\mathbb{T})$. We are ready to state the L^2 theory of Fourier series.

7.2. **Theorem.** *For $f, g \in L^2(\mathbb{T})$ the following holds.*

(1) We have

$$\lim_{N\to\infty} \|S_N f - f\|_{L^2} = 0.$$

(2) We have the Plancherel's identity

$$\|f\|_{L^2}^2 = \sum_{m\in\mathbb{Z}} |\widehat{f}(m)|^2.$$

(3) We have the Parseval's relation

$$\int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x = \sum_{m\in\mathbb{Z}} \widehat{f}(m)\overline{\widehat{g}(m)}.$$

(4) The map $f \mapsto (\widehat{f}(m))_{m\in\mathbb{Z}}$ is an isometry from $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$.

(5) For all $k \in \mathbb{Z}$ we have

$$\widehat{fg}(k) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)\widehat{g}(k-m) = \sum_{m\in\mathbb{Z}} \widehat{f}(k-m)\widehat{g}(m).$$

Proof. Notice that (1), (2) and (3) follow directly from Proposition 7.1. For (4) it only remains to show that the mapping is onto (i.e., surjective). We leave this and (5) as an exercise. □

8. Decay of Fourier coefficients and Sobolev spaces

We know by the Riemann-Lebesgue lemma that if $f \in L^1(\mathbb{T})$ then $|\widehat{f}(m)| \to 0$ when $|m| \to \infty$. It can be shown that given a sequence of positive real numbers d_m with $d_m \to 0$ as $|m| \to \infty$, there exists $f \in L^1(\mathbb{T})$ so that $|\widehat{f}(m)| \geq d_m$ for all $m \in \mathbb{Z}$. That is, the convergence in the Riemann-Lebesgue lemma can be arbitrarily slow. See Grafakos [2].

For more regular f we get better decay. Let $C(\mathbb{T}) = C^0(\mathbb{T})$ consist of continuous 1-periodic functions $f\colon \mathbb{R} \to \mathbb{R}$, $C^k(\mathbb{T})$ consist of k-times differentiable 1-periodic functions $f\colon \mathbb{R} \to \mathbb{R}$, and let $C^\infty(\mathbb{T}) = \bigcap_{k\geq 0} C^k(\mathbb{T})$. Let us denote ordinary pointwise derivatives as $f'(x) = f^{(1)}(x)$, $f''(x) = f^{(2)}(x)$ or $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)$, and so on.

8.1. **Definition.** Let $f \in L^1(\mathbb{T})$ and $k = 1, 2, \ldots$. If there exists $g \in L^1(\mathbb{T})$ so that for all $\varphi \in C^\infty(\mathbb{T})$ we have

$$\int_0^1 f\varphi^{(k)} = (-1)^k \int_0^1 g\varphi,$$

then g is called the kth weak derivative of f, and we denote $g = D^k f$.

If g_1 and g_2 are both kth weak derivatives of f, then $g_1 = g_2$ almost everywhere. Indeed, if for $g \in L^1(\mathbb{T})$ we have

$$0 = \int_0^1 g(y)\varphi(y)\,\mathrm{d}y$$

for all $\varphi \in C^\infty(\mathbb{T})$, then $g = 0$ almost everywhere. To see this, fix x, N and let $\varphi(y) = F_N(x-y)$. Then we have

$$0 = F_N * g(x) = \sigma_N g(x)$$

but $\sigma_N g \to g$ in $L^1(\mathbb{T})$.

8.2. **Definition.** Let $1 \le p < \infty$ and $k = 1, 2, \ldots$. We say that $f \in W^{k,p}(\mathbb{T})$ if $f \in L^p(\mathbb{T})$ has weak-derivatives $D^1 f, \ldots, D^k f \in L^p(\mathbb{T})$. We norm this space with the norm

$$\|f\|_{W^{k,p}(\mathbb{T})} := \Big(\sum_{i=0}^k \|D^i f\|_{L^p}^p \Big)^{1/p},$$

where we agree $D^0 f = f$. These are called Sobolev spaces.

8.3. **Example.** Suppose $f(x) = x$ for $0 \le x < 1$ and $f(1) = 0$, and extend f periodically. Then $f \in L^1(\mathbb{T}) \setminus W^{1,1}(\mathbb{T})$. To see this, let $\varphi \in C^\infty(\mathbb{T})$ and notice that by integration by parts we have

$$\int_0^1 f(x)\varphi'(x)\,\mathrm{d}x = \int_0^1 x\varphi'(x)\,\mathrm{d}x = (1\cdot\varphi(1) - 0\cdot\varphi(0)) - \int_0^1 \varphi(x)\,\mathrm{d}x. \tag{8.4}$$

Therefore, we have

$$\varphi(1) = \int_0^1 f(x)\varphi'(x)\,\mathrm{d}x + \int_0^1 \varphi(x)\,\mathrm{d}x$$

Aiming for a contradiction suppose that $g := D^1 f \in L^1$ exists. Then we have

$$\varphi(1) = -\int_0^1 g(x)\varphi(x)\,\mathrm{d}x + \int_0^1 \varphi(x)\,\mathrm{d}x = \int_0^1 (1-g(x))\varphi(x)\,\mathrm{d}x.$$

Choose a sequence $\varphi_j \in C^\infty(\mathbb{T})$ with $0 \le \varphi_j \le 1$, $\varphi_j(m) = 1$ for all $m \in \mathbb{Z}$ and $\lim_{j\to\infty} \varphi_j(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$. Then by dominated convergence we have

$$1 = \lim_{j\to\infty} \varphi_j(1) = \lim_{j\to\infty} \int_0^1 (1-g(x))\varphi_j(x)\,\mathrm{d}x = \int_0^1 (1-g(x)) \lim_{j\to\infty} \varphi_j(x)\,\mathrm{d}x = 0.$$

If $f \in W^{k,1}(\mathbb{T})$ there is an easy but useful formula connecting $\widehat{f}(m)$ and $\widehat{D^k f}(m)$.

8.5. **Lemma.** *Suppose $f \in W^{k,1}(\mathbb{T})$ for some $k \geq 1$. Then for $m \in \mathbb{Z} \setminus \{0\}$ we have*

$$\widehat{f}(m) = \frac{1}{(2\pi i m)^k} \widehat{D^k f}(m)$$

and $\widehat{D^k f}(0) = 0$.

Proof. Using $e^{-2\pi i m x} = \frac{(-1)^k}{(2\pi i m)^k} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-2\pi i m x}$ and that $x \mapsto e^{-2\pi i m x}$ is in $C^\infty(\mathbb{T})$ we get that

$$\widehat{f}(m) = \int_0^1 f(x) e^{-2\pi i m x}\,\mathrm{d}x = \frac{1}{(2\pi i m)^k} \int_0^1 D^k f(x) e^{-2\pi i m x}\,\mathrm{d}x = \frac{1}{(2\pi i m)^k} \widehat{D^k f}(m).$$

We also have

$$\widehat{D^k f}(0) = \int_0^1 D^k f \cdot 1 = (-1)^k \int_0^1 f \frac{\mathrm{d}^k}{\mathrm{d}x^k} 1 = 0.$$

$\square$

8.6. **Corollary.** *Suppose $f \in W^{k,1}(\mathbb{T})$ for some $k \geq 1$. Then we have*

$$\lim_{|m| \to \infty} |m|^k |\widehat{f}(m)| = 0$$

and for all $m \in \mathbb{Z}$ we have

$$|\widehat{f}(m)| \lesssim_k \frac{\max(\|f\|_{L^1}, \|D^k f\|_{L^1})}{(1 + |m|)^k}.$$

Proof. For $m \neq 0$ we have

$$|m|^k |\widehat{f}(m)| = \frac{1}{(2\pi)^k} |\widehat{D^k f}(m)| \to 0$$

by applying the Riemann-Lebesgue lemma with $D^k f \in L^1$. The desired estimate follows by using that $|\widehat{D^k}(m)| \leq \|D^k f\|_{L^1}$ and $|\widehat{f}(m)| \leq \|f\|_{L^1}$ (the latter is needed for $m = 0$). $\square$

8.7. **Example.** We continue with the Example 8.3. So let again $f(x) = x$ for $0 \leq x < 1$ and $f(1) = 0$, and extend f periodically. Applying Equation (8.4) with $\varphi(x) = \varphi_m(x) = e^{-2\pi i m x}$ (i.e., integration by parts) we get for $m \neq 0$ that

$$(-2\pi i m)\widehat{f}(m) = \int_0^1 x\varphi'(x)\,\mathrm{d}x = 1 - \int_0^1 \varphi(x)\,\mathrm{d}x = 1.$$

Thus, we have $\lim_{|m| \to \infty} |\widehat{f}(m)| = 0$ as we should have by the Riemann-Lebesgue lemma. However, we have

$$|m||\widehat{f}(m)| = \frac{1}{2\pi},$$

and so in particular $\lim_{|m| \to \infty} |m||\widehat{f}(m)| \neq 0$. We can again conclude that $f \notin W^{1,1}(\mathbb{T})$ by Corollary 8.6.

8.1. **Absolutely continuous functions.** We now introduce/recall the notion of absolute continuity from Real Analysis I as it is very closely related.

8.8. **Definition.** A function $F\colon [a,b] \to \mathbb{R}$ is absolutely continuous on the interval $[a,b]$ if for all $\epsilon > 0$ there is a $\delta > 0$ so that for any finite collection of disjoint intervals $(a_1, b_1), \ldots, (a_m, b_m) \subset [a,b]$ with

$$\sum_{i=1}^{m} (b_i - a_i) < \delta$$

we have

$$\sum_{i=1}^{m} |f(b_i) - f(a_i)| < \epsilon.$$

We do not need the definition too much – rather, the following fundamental basic facts that are proved in Real Analysis I are useful:

(1) If $G \in L^1([a,b])$ then the function

$$H(x) := \int_a^x G, \qquad x \in [a,b],$$

is absolutely continuous and $H'(x) = G(x)$ for almost every $x \in [a,b]$.

(2) F is absolutely continuous if and only if $F'(x)$ exists for almost every $x \in [a,b]$, $F' \in L^1([a,b])$ and

$$F(x) - F(a) = \int_a^x F'$$

for every $x \in [a,b]$.

Suppose now $F, G\colon [a,b] \to \mathbb{R}$ are absolutely continuous. The product FG is absolutely continuous and for almost every $y \in [a,b]$ we have

$$(FG)'(y) = F'(y)G(y) + F(y)G'(y).$$

Integrating this over $y \in [a,x]$, where $x \in [a,b]$, we get

$$F(x)G(x) - F(a)G(a) = \int_a^x (FG)' = \int_a^x F'G + \int_a^x FG'.$$

Written in a different order we arrive at the integration by parts formula

$$\int_a^x FG' = [F(x)G(x) - F(a)G(a)] - \int_a^x F'G, \qquad x \in [a,b].$$

8.9. **Proposition.** *A function $f \in L^1(\mathbb{T})$ satisfies $f \in W^{1,1}(\mathbb{T})$ if and only if $f = \tilde{f}$ almost everywhere, where $\tilde{f} \in L^1(\mathbb{T})$ is absolutely continuous on $[0,1]$.*

Proof. Suppose $f \in L^1(\mathbb{T})$ is such that $f = \tilde{f}$ almost everywhere, where $\tilde{f} \in L^1(\mathbb{T})$ is absolutely continuous on $[0,1]$. Then for every $\varphi \in C^\infty(\mathbb{T})$ we have

$$\int_0^1 f\varphi' = [\tilde{f}(1)\varphi(1) - \tilde{f}(0)\varphi(0)] - \int_0^1 \tilde{f}'\varphi = -\int_0^1 \tilde{f}'\varphi,$$

where we used that $\tilde{f}(1) = \tilde{f}(0)$ and $\varphi(1) = \varphi(0)$. Therefore, we have that the weak derivative $D^1 f$ exists and equals $\tilde{f}' \in L^1$, and so $f \in W^{1,1}(\mathbb{T})$.

Conversely, suppose that $f \in W^{1,1}(\mathbb{T})$. Define

$$\tilde{f}(x) = \int_0^x D^1 f(y)\,\mathrm{d}y, \qquad 0 \le x \le 1.$$

Then $\tilde{f}$ is absolutely continuous on $[0,1]$, and $D^1\tilde{f}(x) = \tilde{f}'(x) = D^1 f(x)$ for almost every $x \in [0,1]$. It follows (see the exercises) that $f(x) = f(0) + \tilde{f}(x)$ for almost every $x \in [0,1]$. Notice also that

$$\tilde{f}(1) = \int_0^1 D^1 f(y)\,\mathrm{d}y = -\int_0^1 f(y)\frac{\mathrm{d}y}{y}(1)\,\mathrm{d}y = 0 = \tilde{f}(0).$$

We conclude that f agrees almost everywhere with the function $f(0) + \tilde{f} \in L^1(\mathbb{T})$, which is absolutely continuous on $[0,1]$. $\square$

8.10. *Remark.* We have to be careful not to get confused. Suppose again $f(x) = x$ for $0 \le x < 1$ and $f(1) = 0$, and extend f periodically. Define $\tilde{f}(x) = x$, $0 \le x \le 1$. Of course $f(x) = \tilde{f}(x)$ for almost every $x \in [0,1]$ and $\tilde{f}$ is absolutely continuous on $[0,1]$. However, as $\tilde{f}(0) \ne \tilde{f}(1)$ we cannot extend $\tilde{f}$ into a 1-periodic function.

9. Sobolev spaces $H^s(\mathbb{T})$ and Sobolev embeddings

The exponent $p = 2$ is again in a special role even in the Sobolev range $W^{k,p}(\mathbb{T})$. We now study

$$H^k(\mathbb{T}) := W^{k,2}(\mathbb{T}), \qquad k = 1, 2, \ldots.$$

For $m \in \mathbb{Z}$ denote

$$\langle m \rangle := (1 + |m|^2)^{1/2}.$$

This is sometimes called the Japanese bracket of m.

9.1. **Proposition.** *For $k = 1, 2, \ldots$, we have that $f \in L^1(\mathbb{T})$ satisfies $f \in H^k(\mathbb{T})$ if and only if*

$$\Big(\sum_{m\in\mathbb{Z}} |\langle m \rangle^k \widehat{f}(m)|^2\Big)^{1/2} < \infty.$$

Moreover, for $f \in H^k(\mathbb{T})$ we have

$$\|f\|_{H^k(\mathbb{T})} \sim \Big(\sum_{m\in\mathbb{Z}} |\langle m \rangle^k \widehat{f}(m)|^2\Big)^{1/2}.$$

Proof. For convenience we assume that $k = 1$. Suppose that $f \in H^1(\mathbb{T}) \subset W^{1,1}(\mathbb{T})$. Then we have by Lemma 8.5 and Theorem 7.2 that

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |\langle m \rangle \widehat{f}(m)|^2 &= \sum_{m\in\mathbb{Z}} |\widehat{f}(m)|^2 + \sum_{m\in\mathbb{Z}} ||m|\widehat{f}(m)|^2 \\ &\sim \sum_{m\in\mathbb{Z}} |\widehat{f}(m)|^2 + \sum_{m\in\mathbb{Z}} |\widehat{D^1 f}(m)|^2 \\ &= \|f\|_{L^2}^2 + \|D^1 f\|_{L^2}^2 = \|f\|_{H^1(\mathbb{T})}^2.\end{aligned}$$

We now suppose that $f \in L^1(\mathbb{T})$ is such that

$$\Big(\sum_{m\in\mathbb{Z}} |\langle m\rangle \widehat{f}(m)|^2\Big)^{1/2} < \infty.$$

It follows that $((2\pi i m)\widehat{f}(m))_{m\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. By (4) of Theorem 7.2 it follows that $f \in L^2(\mathbb{T})$ and we find $g \in L^2(\mathbb{T})$ so that $\widehat{g}(m) = (2\pi i m)\widehat{f}(m)$ for all $m \in \mathbb{Z}$. We will show that $g = D^1 f$. Let $\varphi \in C^\infty(\mathbb{T})$. Notice that for all N we have

$$\begin{aligned}\Big|\int_0^1 f\varphi' + \int_0^1 g\varphi\Big| \\ \le \Big|\int_0^1 (f - S_N f)\varphi'\Big| + \Big|\int_0^1 S_N f\varphi' + \int_0^1 S_N g\varphi\Big| + \Big|\int_0^1 (g - S_N g)\varphi\Big|.\end{aligned}$$

As $S_N f \to f$ and $S_N g \to g$ in $L^2(\mathbb{T})$, we have by Hölder's inequality that the first and second term vanish at the limit $N \to \infty$. Thus, we only need to note that

$$\int_0^1 S_N f\varphi' = -\int_0^1 (S_N f)'\varphi$$

and that

$$(S_N f)'(x) = \sum_{|m|\le N} \widehat{f}(m)(2\pi i m)e^{2\pi i m x} = \sum_{|m|\le N} \widehat{g}(m)e^{2\pi i m x} = S_N g,$$

so that the term in the middle vanishes for all N. It follows that $f \in L^2(\mathbb{T})$ and $D^1 f = g \in L^2(\mathbb{T})$. Thus $f \in H^1(\mathbb{T})$, and we are done. □

By redefining (switching to an equivalent norm)

$$\|f\|_{H^k(\mathbb{T})} := \Big(\sum_{m\in\mathbb{Z}} |\langle m\rangle^k \widehat{f}(m)|^2\Big)^{1/2}$$

we have

$$H^k(\mathbb{T}) = \{f \in L^1(\mathbb{T}) \colon \|f\|_{H^k(\mathbb{T})} < \infty\}.$$

This is convenient: we do not need the a priori existence of the weak derivatives – everything is determined simply by the finiteness of this constant which makes sense even with the minimal assumption $f \in L^1(\mathbb{T})$. We can even make sense of non-integer values of k by defining things in the Fourier side like this.

9.2. **Definition.** Let $s \in [0,\infty)$. Define

$$H^s(\mathbb{T}) = \{f \in L^1(\mathbb{T}) \colon \|f\|_{H^s(\mathbb{T})} < \infty\},$$

where

$$\|f\|_{H^s(\mathbb{T})} := \Big(\sum_{m\in\mathbb{Z}} |\langle m\rangle^s \widehat{f}(m)|^2\Big)^{1/2}.$$

Notice that for $s \ge 0$ we always have $1 \le \langle m\rangle^s$, and so by Theorem 7.2 we have $H^s(\mathbb{T}) \subset L^2(\mathbb{T})$ (with $H^0(\mathbb{T}) = L^2(\mathbb{T})$).

9.3. **Theorem.** *Let $s \in (0,\infty)$. The space $H^s(\mathbb{T})$ is a Hilbert space with the inner product*

$$\langle f \mid g\rangle_{H^s(\mathbb{T})} = \sum_{m\in\mathbb{Z}} \langle m\rangle^{2s} \widehat{f}(m)\overline{\widehat{g}(m)}$$

Proof. Exercise. □

Let us also denote

$$\|f\|_{\dot{H}^s(\mathbb{T})} := \Big(\sum_{m\neq 0} |m|^{2s}|\widehat{f}(m)|^2\Big)^{1/2}.$$

9.1. **Sobolev embeddings.**

9.4. **Definition.** For $0 < \alpha < 1$ define

$$\|f\|_{\dot{\Lambda}_\alpha(\mathbb{T})} := \sup_{x,y\in[0,1]} \frac{|f(x+y)-f(x)|}{|y|^\alpha}$$

and

$$\|f\|_{\Lambda_\alpha(\mathbb{T})} = \|f\|_{L^\infty} + \|f\|_{\dot{\Lambda}_\alpha(\mathbb{T})}.$$

We prove the following Sobolev embedding stating that functions in $H^s(\mathbb{T})$, $s > 1/2$, are continuous and in fact belong to some space Λ_α.

9.5. **Theorem.** *Suppose $s = \frac{1}{2} + \alpha$, $\alpha \in (0,1)$. Then we have the Sobolev embedding*

$$\|f\|_{\Lambda_\alpha(\mathbb{T})} \lesssim \|f\|_{H^s(\mathbb{T})}.$$

9.6. *Remark.* Strictly speaking $f = g$ a.e. for some $g \in H^s(\mathbb{T}) \cap \Lambda_\alpha(\mathbb{T})$.

Proof of Theorem 9.5. Notice that as $2s > 1$ we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |\widehat{f}(m)| = \sum_{m\in\mathbb{Z}} \langle m\rangle^{-s}|\langle m\rangle^s\widehat{f}(m)| &\le \Big(\sum_{m\in\mathbb{Z}}\langle m\rangle^{-2s}\Big)^{1/2}\Big(\sum_{m\in\mathbb{Z}}|\langle m\rangle^s\widehat{f}(m)|^2\Big)^{1/2}\\ &\lesssim \Big(\sum_{m\in\mathbb{Z}}|\langle m\rangle^s\widehat{f}(m)|^2\Big)^{1/2} = \|f\|_{H^s(\mathbb{T})}.\end{aligned}$$

By Proposition 4.6 we have that

$$f(x) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imx}$$

almost everywhere. We replace f by this continuous representative. Notice first that

$$\|f\|_{L^\infty} \le \sum_{m\in\mathbb{Z}} |\widehat{f}(m)| \lesssim \|f\|_{H^s(\mathbb{T})}.$$

We now prove that $\|f\|_{\dot{\Lambda}_\alpha(\mathbb{T})} \lesssim \|f\|_{\dot{H}^s(\mathbb{T})}$, $s = 1/2+\alpha$. Fix $x, y \in [-1/2, 1/2]$. We have

$$\begin{aligned}|f(x+y)-f(x)| &= \Big|\sum_{m\neq 0}\widehat{f}(m)[e^{2\pi imy}-1]e^{2\pi imx}\Big|\\ &= \Big|\sum_{m\neq 0}|m|^s\widehat{f}(m)\cdot[e^{2\pi imy}-1]e^{2\pi imx}|m|^{-s}\Big|\\ &\le \Big(\sum_{m\neq 0}|m|^{2s}|\widehat{f}(m)|^2\Big)^{1/2}\Big(\sum_{m\neq 0}|e^{2\pi imy}-1|^2|m|^{-2s}\Big)^{1/2}\end{aligned}$$

$$= \|f\|_{\dot H^s(\mathbb{T})}\Big(\sum_{m\neq 0}|e^{2\pi imy}-1|^2|m|^{-2s}\Big)^{1/2}.$$

We will prove that for $|y|\le 1/2$ we have

$$\sum_{m\neq 0}|e^{2\pi imy}-1|^2|m|^{-2s}\lesssim |y|^{2\alpha},$$

which ends the proof. We estimate using the facts that $|e^{it}|=1$ and $|e^{it}-1|\le |t|$, $t\in\mathbb{R}$, and get that

$$|e^{2\pi imy}-1|^2|m|^{-2s}\lesssim \min(|m|^{-2s},|y|^2|m|^{2-2s}).$$

We may then estimate only the part $m>0$. Here m^{-2s} is the better estimate of the two precisely when $m\ge |y|^{-1}$. Accordingly, given y satisfying $0<|y|\le 1/2$ we fix $A\in\{2,3,\ldots\}$ so that $A\le |y|^{-1}<A+1$, and then write

$$\sum_{m=1}^{\infty}\min(m^{-2s},|y|^2m^{2-2s}) = |y|^2\sum_{m=1}^{A}m^{2-2s}+\sum_{m=A+1}^{\infty}m^{-2s} = I+II.$$

We have

$$II=\sum_{m=A+1}^{\infty}\int_{m-1}^{m}m^{-2s}\,\mathrm{d}t\le\sum_{m=A+1}^{\infty}\int_{m-1}^{m}t^{-2s}\,\mathrm{d}t=\int_A^\infty t^{-2s}\,\mathrm{d}t$$

and

$$\int_A^\infty t^{-2s}\,\mathrm{d}t\sim A^{-2s+1}\sim |y|^{2s-1}=|y|^{2\alpha}.$$

It remains to estimate I. We prove $\sum_{m=1}^{A}m^{2-2s}\lesssim |y|^{2\alpha-2}$. For $\theta>-1$ we have by the integral test (similarly as above) that

$$\sum_{m=1}^{A}m^{\theta}\lesssim A^{\theta+1}+1,$$

and so

$$\sum_{m=1}^{A}m^{2-2s}\lesssim A^{3-2s}+1\sim |y|^{2\alpha-2}+1\lesssim |y|^{2\alpha-2}.$$

We are done. □

9.7. *Remark.* The case $s=1$ can also be proved as follows. Notice $H^1(\mathbb{T})=W^{1,2}(\mathbb{T})\subset W^{1,1}(\mathbb{T})$ so that f is absolutely continuous with $f'\in L^2$. Writing $f(t)=f(0)+\int_0^t f'$, $0\le t\le 2$, we have for $x,y\in[0,1]$ that

$$|f(x+y)-f(x)|=\Big|\int_0^{x+y}f'-\int_0^x f'\Big|\le\int_x^{x+y}|f'|\le \|f'\|_{L^2}|y|^{1/2}.$$

In the last step we applied Hölder's inequality with $p=p'=2$. This proves that $\|f\|_{\dot\Lambda_{1/2}(\mathbb{T})}\lesssim \|f\|_{\dot H^1(\mathbb{T})}$. The estimate $\|f\|_{L^\infty}\lesssim\|f\|_{H^1(\mathbb{T})}$ is proved as in the beginning of the above proof.

What happens when $s\le\frac12$? We aim to answer this next. We need the following estimate first.

9.8. **Proposition** (Hausdorff-Young). *Let $0 \le M < N < \infty$ and let $a = (a_m)_{M \le |m| \le N}$ be a sequence of complex numbers. Define*

$$S_{M,N}a(x) := \sum_{M \le |m| \le N} a_m e^{2\pi i m x}.$$

Let $1 \le p \le 2$. Then we have

$$\|S_{M,N}a\|_{L^{p'}(\mathbb{T})} \le \|a\|_{\ell^p}.$$

Proof. Let $p = 2$. Then $p = p' = 2$, and we have by Plancherel's identity that

$$\|S_{M,N}a\|_{L^2(\mathbb{T})} = \Big(\sum_{m \in \mathbb{Z}} |\widehat{S_{M,N}a}|^2 \Big)^{1/2} = \Big(\sum_{M \le |m| \le N} |a_m|^2 \Big)^{1/2} = \|a\|_{\ell^2}.$$

Let $p = 1$ so that $p' = \infty$. Then we have

$$\|S_{M,N}a\|_{L^\infty(\mathbb{T})} \le \sum_{M \le |m| \le N} |a_m| = \|a\|_{\ell^1}.$$

We now use the Riesz–Thorin interpolation theorem, Theorem A.3, with $p_0 = 1$, $q_0 = \infty$ and $p_1 = q_1 = 2$. It follows that for all $\theta \in (0,1)$ we have

$$\|S_{M,N}a\|_{L^{q_\theta}(\mathbb{T})} \le \|a\|_{\ell^{p_\theta}},$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2}$$

and

$$\frac{1}{q_\theta} = \frac{1-\theta}{\infty} + \frac{\theta}{2} = \frac{\theta}{2}.$$

It follows that

$$\frac{1}{p_\theta} + \frac{1}{q_\theta} = 1,$$

and so $q_\theta = p_\theta'$, and p_θ is an arbitrary number in the interval $(1,2)$. □

9.9. **Corollary.** *Let $1 \le p \le 2$. For $a = (a_m)_{m \in \mathbb{Z}} \in \ell^p$ define*

$$S_N a(x) = \sum_{|m| \le N} a_m e^{2\pi i m x}.$$

Then $(S_N a)_{N \in \mathbb{N}}$ is a Cauchy sequence in $L^{p'}(\mathbb{T})$ and converges to

$$Sa =: \sum_{m \in \mathbb{Z}} a_m e^{2\pi i m x} \in L^{p'}(\mathbb{T})$$

satisfying

$$\|Sa\|_{L^{p'}(\mathbb{T})} \le \|a\|_{\ell^p(\mathbb{Z})}.$$

9.10. **Corollary.** *Let $1 \le p \le 2$ and suppose $f \in L^1(\mathbb{T})$ is such that $\mathcal{F}f := (\widehat{f}(m))_{m \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$ – i.e., we have*

$$\|(\widehat{f}(m))_{m \in \mathbb{Z}}\|_{\ell^p(\mathbb{Z})} = \Big(\sum_{m \in \mathbb{Z}} |\widehat{f}(m)|^p \Big)^{1/p} < \infty.$$

Then we have $f \in L^{p'}(\mathbb{T})$ with

$$\|f\|_{L^{p'}(\mathbb{T})} \le \|(\widehat{f}(m))_{m\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})}$$

and

$$\lim_{N\to\infty} \|S_N f - f\|_{L^{p'}(\mathbb{T})} = 0.$$

Proof. Let $a = (\widehat{f}(m))_{m\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$. Notice that $S_N f(x) = S_N a(x)$ (with the obvious abuse of notation that S_N can hit functions or sequences), and so by the previous corollary $(S_N f)_{N\in\mathbb{N}}$ converges in $L^{p'}(\mathbb{T})$ to some function $g \in L^{p'}(\mathbb{T})$ with $\|g\|_{L^{p'}(\mathbb{T})} \le \|a\|_{\ell^p(\mathbb{Z})}$. It remains to prove that $f = g$ almost everywhere. Notice that we have for every $k \in \mathbb{Z}$ that

$$|\widehat{S_N f}(k) - \widehat{g}(k)| \le \|S_N f - g\|_{L^1(\mathbb{T})} \le \|S_N f - g\|_{L^{p'}(\mathbb{T})} \to 0$$

as $N \to \infty$. But $\widehat{S_N f}(k) = \widehat{f}(k)$ if $N \ge |k|$, and so $\widehat{f}(k) = \widehat{g}(k)$ for every $k \in \mathbb{Z}$. It follows that $f = g$ almost everywhere, and we are done. □

We are ready to prove Sobolev's embedding theorem in the range $s \le 1/2$. Given $s \le 1/2$ define

$$p(s) = \frac{2}{1-2s}.$$

Notice that $p(0) = 2$ and $p(1/2) = \infty$. Recall that if $f \in H^s(\mathbb{T})$ then $f \in L^2(\mathbb{T})$. The following shows that in fact f is always p-integrable with $p > 2$.

9.11. **Theorem.** *Let $f \in H^s(\mathbb{T})$, where $0 < s \le 1/2$. Then for all $2 \le q < p(s)$ we have*

$$\|f\|_{L^q(\mathbb{T})} \lesssim \|f\|_{H^s(\mathbb{T})}.$$

Proof. Fix $q \in (2, p(s))$ and let $p := q' \in (1,2)$. By Corollary 9.10 it is enough to prove that

$$\|(\widehat{f}(m))_{m\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \lesssim \|f\|_{H^s(\mathbb{T})}.$$

We estimate using Hölder's inequality with $t = 2/p > 1$ and $t' = 2/(2-p)$ as follows:

$$\begin{aligned}
&\Big(\sum_{m\in\mathbb{Z}} |\widehat{f}(m)|^p\Big)^{1/p} \\
&= \Big(\sum_{m\in\mathbb{Z}} \langle m\rangle^{sp} |\widehat{f}(m)|^p \langle m\rangle^{-sp}\Big)^{1/p} \\
&\le \Big(\sum_{m\in\mathbb{Z}} \langle m\rangle^{spt} |\widehat{f}(m)|^{pt}\Big)^{1/(pt)} \Big(\sum_{m\in\mathbb{Z}} \langle m\rangle^{-spt'}\Big)^{1/(pt')} \\
&= \Big(\sum_{m\in\mathbb{Z}} \langle m\rangle^{2s} |\widehat{f}(m)|^2\Big)^{1/2} \Big(\sum_{m\in\mathbb{Z}} \langle m\rangle^{-sp\frac{2}{2-p}}\Big)^{\frac{2-p}{2p}} = \|f\|_{H^s(\mathbb{T})} \Big(\sum_{m\in\mathbb{Z}} \langle m\rangle^{-sp\frac{2}{2-p}}\Big)^{\frac{2-p}{2p}}.
\end{aligned}$$

It is enough to prove that

$$sp\frac{2}{2-p} > 1.$$

But this is seen to be equivalent to

$$\frac{1}{p} < \frac{2s+1}{2}.$$

Therefore, as $1/p + 1/q = 1$, we get that this is further equivalent to

$$q < \frac{2}{1-2s} = p(s).$$

But this is our assumption, and so we are done. □

9.12. *Remark.* If $s < 1/2$ the theorem is true even with $q = p(s)$, but this requires a different proof.

9.2. **Compact embeddings.** A general philosophy is that sequences bounded in a high regularity space (and constrained to lie in a compact domain such as the torus) usually have convergent subsequences in low regularity spaces.

9.13. **Theorem** (Rellich–Kondrachov theorem)**.** *Let $s > 0$ and $0 \le s_0 < s$. Suppose f_n, $n = 1, 2, 3 \ldots$, is a sequence of functions in $H^s(\mathbb{T})$ such that*

$$\sup_n \|f_n\|_{H^s(\mathbb{T})} < \infty.$$

Then there is a function $f \in H^s(\mathbb{T})$ and a subsequence f_{n_k}, $k \in \mathbb{N}$, of f_n, $n \in \mathbb{N}$, such that

$$\lim_{k\to\infty} \|f_{n_k} - f\|_{H^{s_0}(\mathbb{T})} = 0.$$

Proof. Let $M = \sup_n \|f_n\|_{H^s(\mathbb{T})} < \infty$. For every $l \in \mathbb{Z}$ we have

$$\sup_n |\widehat{f_n}(l)|^2 \le \sum_{m\in\mathbb{Z}} |\langle m\rangle^s \widehat{f_n}(m)|^2 = \|f_n\|^2_{H^s(\mathbb{T})} \le M^2.$$

Bounded sequences of scalars have convergent subsequences. First, choose a subsequence $(n_{0,k})_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} \widehat{f_{n_{0,k}}}(0) = a_0.$$

Then choose a subsequence $(n_{1,k})_{k\in\mathbb{N}}$ of $(n_{0,k})_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} \widehat{f_{n_{1,k}}}(1) = a_1.$$

Next, choose a subsequence $(n_{-1,k})_{k\in\mathbb{N}}$ of $(n_{1,k})_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} \widehat{f_{n_{-1,k}}}(-1) = a_{-1},$$

and then a subsequence $(n_{2,k})_{k\in\mathbb{N}}$ of $(n_{-1,k})_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} \widehat{f_{n_{2,k}}}(2) = a_2.$$

Continue like this and let $n_k = n_{k,k}$, $k \in \mathbb{N}$. Then we have for all $m \in \mathbb{Z}$ that

$$\lim_{k\to\infty} \widehat{f_{n_k}}(m) = a_m.$$

Notice that

$$\sum_{m\in\mathbb{Z}} |\langle m\rangle^s a_m|^2 = \lim_{k\to\infty} \sum_{m\in\mathbb{Z}} |\langle m\rangle^s \widehat{f_{n_k}}(m)|^2 \le M^2.$$

Define $f \in H^s(\mathbb{T})$ by setting

$$f = \sum_{m\in\mathbb{Z}} a_m e^{2\pi i m\cdot}.$$

For an arbitrary $k, N \in \mathbb{N}$ we estimate $\|f_{n_k} - f\|_{H^{s_0}(\mathbb{T})}$ up by

$$\Big(\sum_{|m|\le N} \langle m\rangle^{2s_0}|\widehat{f_{n_k}}(m) - a_m|^2\Big)^{1/2} + \Big(\sum_{|m|>N} \langle m\rangle^{2s_0}|\widehat{f_{n_k}}(m) - a_m|^2\Big)^{1/2} =: I_{k,N} + II_{k,N}.$$

We further estimate

$$\begin{aligned} II_{k,N} &\le \Big(\sum_{|m|>N} \langle m\rangle^{2(s_0-s)}\langle m\rangle^{2s}|\widehat{f_{n_k}}(m)|^2\Big)^{1/2} + \Big(\sum_{|m|>N} \langle m\rangle^{2(s_0-s)}\langle m\rangle^{2s}|a_m|^2\Big)^{1/2} \\ &\le N^{s_0-s}\Big[\Big(\sum_{m\in\mathbb{Z}}\langle m\rangle^{2s}|\widehat{f_{n_k}}(m)|^2\Big)^{1/2} + \Big(\sum_{m\in\mathbb{Z}}\langle m\rangle^{2s}|a_m|^2\Big)^{1/2}\Big] \le \frac{2M}{N^{s-s_0}}. \end{aligned}$$

Let $\epsilon > 0$. First choose N so that $II_{k,N} < \epsilon$ for all $k \in \mathbb{Z}$. Then choose $k_0 \in \mathbb{N}$ so that for all $k \ge k_0$ we have for all $|m| \le N$ that

$$|\widehat{f_{n_k}}(m) - a_m| < \frac{\epsilon}{2^{s_0/2}N^{s_0}(3N)^{1/2}}.$$

Then for all $k \ge k_0$ we have by trivial upper bounds that $I_{k,N} < \epsilon$. We conclude that for $k \ge k_0$ we have

$$\|f_{n_k} - f\|_{H^{s_0}(\mathbb{T})} \le I_{k,N} + II_{k,N} < 2\epsilon.$$

□

9.14. *Remark.* Because of this result we say that $H^s(\mathbb{T})$ is compactly embedded to $H^{s-\epsilon}(\mathbb{T})$. A particular case is that $H^s(\mathbb{T})$, $s > 0$, is compactly embedded to $L^2(\mathbb{T})$.

10. Periodic distributions

Thus far we have only been able to define $\widehat{f}(m)$ and the associated Fourier series $S_N f$ if $f \in L^1(\mathbb{T})$. We now develop a more general viewpoint, where a Fourier series can be formed of objects which are not even functions.

10.1. **Test functions.** We call the space $C^\infty(\mathbb{T})$ (consisting of smooth periodic functions) the space of test functions. We equip this space with the metric

$$d_{C^\infty(\mathbb{T})}(f, g) := \sum_{N=0}^{\infty} 2^{-N}\frac{\|f - g\|_{C^N(\mathbb{T})}}{1 + \|f - g\|_{C^N(\mathbb{T})}},$$

where

$$\|f\|_{C^N(\mathbb{T})} = \sum_{|m|\le N} \|f^{(m)}\|_{L^\infty}.$$

10.1. **Proposition.** *The following holds.*

1. *The mapping $(f, g) \mapsto d_{C^\infty(\mathbb{T})}(f, g)$ is a metric on $C^\infty(\mathbb{T})$.*
2. *We have that $f_j \to f$ in the metric space $(C^\infty(\mathbb{T}), d_{C^\infty(\mathbb{T})})$ if and only if for all $m = 0, 1, 2, \ldots$ we have*

$$\lim_{j\to\infty} \|f_j^{(m)} - f^{(m)}\|_{L^\infty} = 0.$$

(3) The space $C^\infty(\mathbb{T})$ is a complete metric space.

Proof. Exercise. □

10.2. **Proposition.** *The following operations are continuous maps from $C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$.*

(1) $f \mapsto \tilde{f}$, where $\tilde{f}(x) = f(-x)$. (reflection)
(2) $f \mapsto \bar{f}$ (conjugation)
(3) $f \mapsto \tau_y f$, where $\tau_y f := f(x-y)$, $y \in \mathbb{R}$. (translation)
(4) $f \mapsto f^{(m)}$ (derivative)
(5) $f \mapsto fg$, where $g \in C^\infty(\mathbb{T})$. (multiplication)
(6) $f \mapsto f * g$, where $g \in C^\infty(\mathbb{T})$. (convolution) Moreover, we have $(f*g)^{(m)} = f^{(m)} * g = f * g^{(m)}$.

Proof. Exercise. □

10.3. **Definition.** A sequence $a = (a_m)_{m\in\mathbb{Z}}$ is said to be rapidly decreasing if for any $N = 0, 1, 2, \dots$ we have

$$(\langle m\rangle^N a_m)_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}).$$

This sequence space is denoted by $\mathscr{S}(\mathbb{Z})$ and is equipped with the metric

$$d_{\mathscr{S}(\mathbb{Z})}(a,b) := \sum_{N=0}^{\infty} 2^{-N} \frac{\|a-b\|_{\mathscr{S}_N(\mathbb{Z})}}{1+\|a-b\|_{\mathscr{S}_N(\mathbb{Z})}},$$

where

$$\|a\|_{\mathscr{S}_N(\mathbb{Z})} := \|(\langle m\rangle^N a_m)_{m\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}.$$

10.4. *Remark.* If $a^k, a \in \mathscr{S}(\mathbb{Z})$ then again $a^k \to a$ in $\mathscr{S}(\mathbb{Z})$ if and only if

$$\lim_{k\to\infty} \|a^k - a\|_{\mathscr{S}_N(\mathbb{Z})} = 0$$

for all N.

We denote

$$\mathcal{F}\colon C^\infty(\mathbb{T}) \to \mathscr{S}(\mathbb{Z}), \qquad \mathcal{F}f = (\widehat{f}(m))_{m\in\mathbb{Z}}.$$

10.5. **Proposition.** *$\mathcal{F}$ is an isomorphism (a linear bijective continuous map with continuous inverse) from $C^\infty(\mathbb{T})$ onto $\mathscr{S}(\mathbb{Z})$. Moreover, we have*

$$S_N f \to f$$

in $C^\infty(\mathbb{T})$.

Proof. We essentially already know all the claims made here. For example, that the target space is indeed $\mathscr{S}(\mathbb{Z})$ is a very special case of Corollary 8.6. Injectivity follows from Corollary 4.4. For surjectivity remember arguments like in Proposition 4.6 and the remark after that (involving Weierstrass M-test). You should think through the details.

We only comment in detail on the new aspect involving the metric topologies introduced above. We show the continuity of $\mathcal{F}\colon C^\infty(\mathbb{T}) \to \mathscr{S}(\mathbb{Z})$. Suppose $f_k \to 0$ in $C^\infty(\mathbb{T})$. We need to show that $\mathcal{F}f_k \to 0$ in $\mathscr{S}(\mathbb{Z})$. For this we need to fix N and show that

$$\lim_{k\to\infty} \|\mathcal{F}f_k\|_{\mathscr{S}_N(\mathbb{Z})} = 0.$$

But

$$\|\mathcal{F}f_k\|_{\mathscr{S}_N(\mathbb{Z})} = \|(\langle m\rangle^N \widehat{f}_k(m))_{m\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} = \sup_{m\in\mathbb{Z}} \langle m\rangle^N |\widehat{f}_k(m)|$$

and for every $m \in \mathbb{Z}$ we have

$$\langle m\rangle^N |\widehat{f}_k(m)| \lesssim |\widehat{f}_k(m)| + |m|^N |\widehat{f}_k(m)| \lesssim |\widehat{f}_k(m)| + |\widehat{f_k^{(N)}}(m)| \le \|f_k\|_{L^\infty} + \|f_k^{(N)}\|_{L^\infty}.$$

The claim follows. Think how to prove the continuity of the inverse. □

10.2. **Distributions.** For convenience we denote the test functions also by $\mathcal{D} = \mathcal{D}(\mathbb{T}) = C^\infty(\mathbb{T})$ (equipped with the notion of convergence as above). Then we define that $\mathcal{D}'$ consists of all the continuous linear functionals on $\mathcal{D}$, that is,

$$\mathcal{D}' := \{T : \mathcal{D} \to \mathbb{C} \colon T \text{ is linear and } T\varphi_j \to 0 \text{ whenever } \varphi_j \to 0 \text{ in } \mathcal{D}\}.$$

Elements of $\mathcal{D}'$ are called periodic distributions. If $T \in \mathcal{D}'$ we also denote

$$\langle T, \varphi\rangle := T\varphi, \qquad \varphi \in \mathcal{D}.$$

10.6. **Example.** Suppose $f \in L^1(\mathbb{T})$ and define the functional $T_f : \mathcal{D} \to \mathbb{C}$ by setting

$$\langle T_f, \varphi\rangle = \int_0^1 f(x)\varphi(x)\,\mathrm{d}x.$$

To check that $T_f \in \mathcal{D}'$ let $\varphi_j \to 0$ in $\mathcal{D}$. Then we have

$$|\langle T_f, \varphi\rangle| \le \|f\|_{L^1}\|\varphi_j\|_{L^\infty} \to 0$$

as $j \to \infty$.

The argument after the definition of weak derivatives, Definition 8.1, shows that if $T_f = T_g$, then $f = g$ almost everywhere. This means that any $f \in L^1(\mathbb{T})$ determines a unique element of $\mathcal{D}'$.

Therefore, $\mathcal{D}'$ is a set that contains all reasonable periodic functions (if we identify $f \in L^1(\mathbb{T})$ with the associated distribution T_f). However, it contains much more.

10.7. **Example.** For $\varphi \in \mathcal{D}$ set

$$\langle \delta_0, \varphi\rangle = \varphi(0).$$

It is clear that $\delta_0 \in \mathcal{D}'$. An argument analogous to the one in Example 8.3 shows that δ_0 is not given by an $L^1(\mathbb{T})$ function: $\delta_0 \ne T_f$ for all $f \in L^1(\mathbb{T})$.

Most operations that are defined on test functions can also be defined for distributions by duality. This is the whole point of this theory. For example, consider the simple operation of reflection $\tilde{\varphi}(x) = \varphi(-x)$, $\varphi \in \mathcal{D}$. We want to define the reflection $\tilde{T}$ of $T \in \mathcal{D}'$ so that if it happens that $T = T_f$ for some $f \in \mathcal{D}$, then $\tilde{T} = T_{\tilde{f}}$. For this to be true we need that for all $f, \varphi \in \mathcal{D}$ we have

$$\langle \tilde{T}_f, \varphi\rangle = \langle T_{\tilde{f}}, \varphi\rangle = \int_0^1 f(-x)\varphi(x)\,\mathrm{d}x = \int_0^1 f(x)\varphi(-x)\,\mathrm{d}x = \langle T_f, \tilde{\varphi}\rangle.$$

Motivated by this we **define** the reflection of $T \in \mathcal{D}'$ as the distribution $\tilde{T} \in \mathcal{D}'$ given by

$$\langle \tilde{T}, \varphi\rangle := \langle T, \tilde{\varphi}\rangle, \qquad \varphi \in \mathcal{D}.$$

Similar computations motivate other natural definitions like

$$\langle \tau_y T, \varphi \rangle := \langle T, \tau_{-y}\varphi \rangle.$$

There is even a natural notion of a derivative of a distribution. For $f \in \mathcal{D}$ we again want $D^k T_f = T_{f^{(k)}}$ which leads by integration by parts to the requirement

$$\langle D^k T_f, \varphi \rangle = \int_0^1 f^{(k)}\varphi = (-1)^k \int_0^1 f\varphi^{(k)} = (-1)^k \langle T_f, \varphi^{(k)} \rangle.$$

10.8. **Definition.** For $T \in \mathcal{D}'$ and $k = 1, 2, \ldots$ we define $D^k T \in \mathcal{D}'$ by setting

$$\langle D^k T, \varphi \rangle := (-1)^k \langle T, \varphi^{(k)} \rangle.$$

$D^k T$ is clearly a continuous linear functional on $\mathcal{D}$ as differentiation is a continuous operation in $\mathcal{D}$. The distribution $D^k T$ is called the kth distributional derivative or weak derivative of T.

10.9. **Example.** Let $f \in L^1(\mathbb{T})$. We can always differentiate f in the distributional sense by forming $D^1 T_f$. This is an extremely weak notion of a derivative as even very irregular functions have a derivative in this sense. For f to have a weak derivative in the sense of Definition 8.1 we require that $D^1 T_f$ is a function – i.e., there is $g \in L^1(\mathbb{T})$ so that $D^1 T_f = T_g$. In this situation we denoted $g = D^1 f$. So if f has a weak derivative $D^1 f \in L^1$, then $D^1 T_f = T_{D^1 f}$, and conversely if $D^1 T_f = T_g$ for some $g \in L^1(\mathbb{T})$, then $D^1 f$ exists and is equal to g.

In Example 8.3 we studied the $L^1(\mathbb{T})$ function $f(x) = x$ for $0 \le x < 1$ and $f(1) = 0$. From those calculations it follows that the distributional derivative $D^1 T_f = 1 - \delta_0$. Notice further that

$$\langle D^1 \delta_0, \varphi \rangle = -\langle \delta_0, \varphi' \rangle = -\varphi'(0).$$

We continue to define operations for distributions. The pointwise multiplication of $T \in \mathcal{D}'$ by a fixed function $f \in \mathcal{D}$ is defined by

$$\langle fT, \varphi \rangle := \langle T, f\varphi \rangle.$$

To define a convolution we again want $T_f * g = T_{f*g}$, and thus calculate for $f, g, \varphi \in \mathcal{D}$ that

$$\begin{aligned}
\langle T_f * g, \varphi \rangle &= \langle T_{f*g}, \varphi \rangle \\
&= \int_0^1 \int_0^1 f(y)g(x-y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 f(y) \int_0^1 g(x-y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 f(y)\tilde{g} * \varphi(y)\,\mathrm{d}x = \langle T_f, \tilde{g} * \varphi \rangle.
\end{aligned}$$

Here we used Fubini's theorem. For a general $T \in \mathcal{D}'$ and $g \in \mathcal{D}$ we thus define $T * g \in \mathcal{D}'$ by

$$\langle T * g, \varphi \rangle := \langle T, \tilde{g} * \varphi \rangle.$$

This defines a continuous linear functional as $\varphi \mapsto g * \varphi$ is continuous operation $\mathcal{D} \to \mathcal{D}$ by Proposition 10.2. It turns out that $T * g$ is in fact a function in $\mathcal{D}$, but we do not prove this now – we prove it soon using Fourier analysis. This is in line with the usual philosophy that convolution is a nice object if one of the objects involved in the convolution is nice.

The following is an important result that finally utilises the assumed continuity of distributions.

10.10. **Theorem** (All periodic distribution have finite order)**.** *For all $T \in \mathcal{D}'$ there exists $N > 0$ and $C < \infty$ such that*

$$|\langle T, \varphi \rangle| \le C \sum_{m=0}^{N} \|\varphi^{(m)}\|_{L^\infty}, \qquad \varphi \in \mathcal{D}.$$

Proof. Fix $T \in \mathcal{D}'$. To reach a contradiction assume that for all $N > 0$ there is $\varphi_N \in \mathcal{D}$ such that

$$|\langle T, \varphi_N \rangle| \ge N \sum_{m=0}^{N} \|\varphi_N^{(m)}\|_{L^\infty}.$$

Define

$$\psi_N := \frac{1}{N}\Big(\sum_{m=0}^{N} \|\varphi_N^{(m)}\|_{L^\infty} \Big)^{-1} \varphi_N.$$

For any fixed m we have for all $N \ge m$ that

$$\|\psi_N^{(m)}\|_{L^\infty} \le \frac{1}{N}.$$

It follows that $\psi_N \to 0$ in $\mathcal{D}$. By the continuity of the linear functional T we have $\langle T, \psi_N \rangle \to 0$ as $N \to \infty$. But by construction we also have for all N that $|\langle T, \psi_N \rangle| \ge 1$ – a contradiction. □

10.3. **Fourier series of periodic distributions.** For a distribution of a form T_f, $f \in \mathcal{D}$, we have

$$\widehat{f}(m) = \int_0^1 f(x) e^{-2\pi i m x}\, \mathrm{d}x = \langle T_f, e^{-2\pi i m \cdot} \rangle.$$

This motivates the following definition.

10.11. **Definition.** If $T \in \mathcal{D}'$ we define its Fourier coefficients via the formula

$$\widehat{T}(m) = \langle T, e^{-2\pi i m \cdot} \rangle, \qquad m \in \mathbb{Z}.$$

Now $\mathcal{F}$ can act on $T \in \mathcal{D}'$ and $\mathcal{F}(T) := (\widehat{T}(m))_{m \in \mathbb{Z}}$. A sequence $a = (a_m)_{m \in \mathbb{Z}}$ is said to have polynomial growth if for some $N > 0$ we have

$$\|(\langle m \rangle^{-N} a_m)_{m \in \mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} < \infty.$$

We denote the set of such sequences by $\mathscr{S}'(\mathbb{Z})$. Do not confuse this with the notation $\mathscr{S}(\mathbb{Z})$ – the sequences of rapid decay.

The following is the main result concerning the Fourier series of periodic distributions.

10.12. **Theorem.** *The mapping $\mathcal{F}$ is a bijective map from the set of periodic distributions $\mathcal{D}'$ to the set of sequences of polynomial growth $\mathscr{S}'(\mathbb{Z})$. If $T \in \mathcal{D}'$ we have*

$$\lim_{N \to \infty} \int_0^1 \Big(\sum_{|m| \le N} \widehat{T}(m) e^{2\pi i m x} \Big) \varphi(x)\, \mathrm{d}x = \langle T, \varphi \rangle, \qquad \varphi \in \mathcal{D}.$$

This means that the sequence of $L^1(\mathbb{T})$ functions

$$S_N(T)(x) := \sum_{|m|\le N} \widehat{T}(m)e^{2\pi i m x}$$

converge to T in the sense of distributions.

Proof. Let $T \in \mathcal{D}'$. Theorem 10.10 gives an N such that

$$|\langle T, \varphi\rangle| \lesssim \sum_{k=0}^{N} \|\varphi^{(k)}\|_{L^\infty}, \qquad \varphi \in \mathcal{D}.$$

In particular, we have

$$|\widehat{T}(m)| = |\langle T, e^{-2\pi i m \cdot}\rangle| \lesssim \sum_{k=0}^{N} |m|^k \lesssim \langle m\rangle^N,$$

and so $\mathcal{F}(T) \in \mathscr{S}'(\mathbb{Z})$.

For the converse direction, suppose that $(a_m)_{m\in\mathbb{Z}} \in \mathscr{S}'(\mathbb{Z})$. Let $N > 1$ be such that $\langle m\rangle^{-N}|a_m| \lesssim 1$, $m \in \mathbb{Z}$. Let $b_m = \langle m\rangle^{-2N}a_m$. As $|b_m| \lesssim \langle m\rangle^{-N}$ and $N > 1$ we have that $b_m \in \ell^1(\mathbb{Z})$. Using Proposition 4.6 we can define the continuous function

$$f(x) = \sum_{m\in\mathbb{Z}} b_m e^{2\pi i m x}$$

with $\widehat{f}(m) = b_m$. Define the differential operator $L = 1 - \frac{1}{4\pi^2}D^2$, and then define the periodic distribution

$$T := L^N T_f \in \mathcal{D}'$$

with

$$\widehat{T}(m) = \langle m\rangle^{2N}\widehat{f}(m) = \langle m\rangle^{2N} b_m = a_m$$

as desired.

It remains to show that given $T \in \mathcal{D}'$ we have

$$\lim_{N\to\infty} \int_0^1 \Big(\sum_{|m|\le N} \widehat{T}(m)e^{2\pi i m x}\Big)\varphi(x)\,\mathrm{d}x = \langle T, \varphi\rangle, \qquad \varphi \in \mathcal{D}.$$

Notice simply that

$$\begin{aligned}\int_0^1 \Big(\sum_{|m|\le N} \widehat{T}(m)e^{2\pi i m x}\Big)\varphi(x)\,\mathrm{d}x &= \sum_{|m|\le N} \widehat{T}(m)\widehat{\varphi}(-m) \\ &= \sum_{|m|\le N} \langle T, e^{-2\pi i m \cdot}\rangle \widehat{\varphi}(-m) \\ &= \Big\langle T, \sum_{|m|\le N} \widehat{\varphi}(-m)e^{-2\pi i m \cdot}\Big\rangle = \langle T, S_N\varphi\rangle.\end{aligned}$$

As $S_N\varphi \to \varphi$ in $\mathcal{D}$ by Proposition 10.5 and T is continuous, the claim follows. □

10.13. **Corollary.** *If $T \in \mathcal{D}'$ and $\widehat{T}(m) = 0$ for all $m \in \mathbb{Z}$, then $T = 0$.*

10.14. **Proposition.** *If $T \in \mathcal{D}'$ has Fourier series*

$$T = \sum_{m \in \mathbb{Z}} \widehat{T}(m) e^{2\pi i m \cdot}$$

then for all $k = 1, 2, \dots$ we have

$$D^k T = \sum_{m \in \mathbb{Z}} \widehat{T}(m)(2\pi i m)^k e^{2\pi i m x}$$

with convergence in the sense of distributions.

Proof. Exercise. □

Recall that we can perform various natural operations on distributions – we can e.g. take $T \in \mathcal{D}'$ and $g \in \mathcal{D}$ and form $T * g \in \mathcal{D}'$. It then makes sense to calculate $\widehat{T * g}(m)$. All the natural formulas extend to this setting, and e.g.

$$\begin{aligned}
\widehat{T * g}(m) &= \langle T * g, e^{-2\pi i m \cdot} \rangle \\
&= \langle T, \tilde{g} * e^{-2\pi i m \cdot} \rangle \\
&= \Big\langle T, \int_0^1 g(-y) e^{-2\pi i m (\cdot - y)} \, \mathrm{d}y \Big\rangle = \langle T, e^{-2\pi i m \cdot} \rangle \widehat{g}(m) = \widehat{T}(m) \widehat{g}(m).
\end{aligned}$$

From here we may also conclude that as $\mathcal{F}(T) \in \mathscr{S}'(\mathbb{Z})$ and $\mathcal{F}g \in \mathscr{S}(\mathbb{Z})$ we have that $\mathcal{F}(T * g) = (\widehat{T}(m)\widehat{g}(m))_{m \in \mathbb{Z}} \in \mathscr{S}(\mathbb{Z})$. But this means that $T * g \in \mathcal{D}$ as by Proposition 10.5 there is some $f \in \mathcal{D}$ such that $\widehat{T_f}(m) = \widehat{f}(m) = \widehat{T * g}(m)$, which means by Corollary 10.13 that $T_f = T * g$. In fact, we have $f(x) = \langle T, \tau_x \tilde{g} \rangle$. We can now even convolve two periodic distributions.

10.15. **Definition.** Let $T, S \in \mathcal{D}'$. Then we define

$$\langle T * S, \varphi \rangle := \langle T, \tilde{S} * \varphi \rangle, \qquad \varphi \in \mathcal{D}.$$

This makes sense as $\tilde{S} * \varphi \in \mathcal{D}$.

11. Sobolev spaces $H^{-s}(\mathbb{T})$ and elliptic regularity

With the periodic distributions we can now even make sense of $H^{-s}(\mathbb{T})$ for $s > 0$.

11.1. **Definition.** Let $s > 0$. We say that $T \in H^{-s}(\mathbb{T})$ if $T \in \mathcal{D}'$ satisfies

$$\|T\|_{H^{-s}(\mathbb{T})} := \Big(\sum_{m \in \mathbb{Z}} |\langle m \rangle^{-s} \widehat{T}(m)|^2 \Big)^{1/2} < \infty.$$

11.2. *Remark.* Therefore, for all $s \in \mathbb{R}$ we have that $T \in H^s(\mathbb{T})$ if $T \in \mathcal{D}'$ satisfies

$$\|T\|_{H^{s}(\mathbb{T})} := \Big(\sum_{m \in \mathbb{Z}} |\langle m \rangle^{s} \widehat{T}(m)|^2 \Big)^{1/2} < \infty.$$

11.3. **Example.** Notice that

$$\widehat{\delta_0}(m) = \langle \delta_0, e^{-2\pi i m \cdot} \rangle = e^{-2\pi i m \cdot 0} = 1$$

for all $m \in \mathbb{Z}$. Thus $\delta_0 \in H^{-s}(\mathbb{T})$ for $s > 1/2$.

$H^s(\mathbb{T})$ is a Hilbert space for all $s \in \mathbb{R}$ with the inner product

$$\langle T|S\rangle_{H^s(\mathbb{T})} = \sum_{m\in\mathbb{Z}} \langle m\rangle^{2s} \widehat{T}(m)\overline{\widehat{S}(m)}.$$

Moreover, we have

$$\bigcup_{s\in\mathbb{R}} H^s(\mathbb{T}) = \mathcal{D}'.$$

These are exercises.

11.4. **Proposition.** *The trigonometric polynomials are dense in $H^s(\mathbb{T})$ for every $s \in \mathbb{R}$. In particular, the space $C^\infty(\mathbb{T})$ is dense in $H^s(\mathbb{T})$.*

Proof. Exercise. □

11.1. **Differential equations, elliptic regularity.** Define the polynomial

$$P(t) = \sum_{k=0}^{N} (2\pi i)^k \alpha_k t^k,$$

where $\alpha_0, \dots, \alpha_N$ are constants and $\alpha_N \neq 0$. The associated constant coefficient differential operator is

$$P(D)T := \sum_{k=0}^{N} \alpha_k D^k T, \qquad T \in \mathcal{D}',$$

in the sense that for all $T \in \mathcal{D}'$ and $m \in \mathbb{Z}$ we have

$$\widehat{P(D)T}(m) = \sum_{k=0}^{N} \alpha_k \widehat{D^k T}(m) = \sum_{k=0}^{N} \alpha_k (2\pi i m)^k \widehat{T}(m) = P(m)\widehat{T}(m).$$

Suppose now that $S \in H^s(\mathbb{T})$ for some $s \in \mathbb{R}$ and $T \in \mathcal{D}'$ solves (in the distributional sense) the differential equation

$$P(D)T = S.$$

It follows that for all $m \in \mathbb{Z}$ we have

$$P(m)\widehat{T}(m) = \widehat{S}(m).$$

To estimate $|\widehat{T}(m)|$ from up we estimate $|P(m)|$ from below. Notice that for $c_0 = |(2\pi i)^N \alpha_N| > 0$ and some $C_0 < \infty$ we have

$$\begin{aligned}
|P(m)| &= \Big|(2\pi i)^N \alpha_N m^N + \sum_{k=0}^{N-1} (2\pi i)^k \alpha_k m^k\Big| \\
&\geq c_0 |m|^N - \sum_{k=0}^{N-1} |(2\pi i)^k \alpha_k| |m|^k \\
&\geq c_0 |m|^N - C_0 |m|^{N-1}.
\end{aligned}$$

Thus, there is some M so that for all $|m| \geq M$ we have

$$|P(m)| \geq \frac{c_0}{2} |m|^N.$$

For $|m| \ge M$ we thus have

$$|\widehat{T}(m)| = \frac{|\widehat{S}(m)|}{|P(m)|} \lesssim |m|^{-N}|\widehat{S}(m)|.$$

Finally, we conclude that

$$\sum_{|m|\ge M} |\langle m\rangle^{s+N}\widehat{T}(m)|^2 \lesssim \sum_{m\in\mathbb{Z}} |\langle m\rangle^{s}\widehat{S}(m)|^2 < \infty,$$

where we used the assumption $S \in H^s(\mathbb{T})$. We conclude that $T \in H^{s+N}(\mathbb{T})$. This is an example of Elliptic regularity theory – here the solution T has N degrees more regularity than the given S.

Suppose for example that $S \in H^{-1/2}(\mathbb{T})$ and $N > 2$. Then $T \in H^{s_0}(\mathbb{T})$ for $s_0 := -1/2 + N > 3/2$. As $D^1T \in H^{s_0-1}(\mathbb{T})$, where $s_0 - 1 > 1/2$, it follows by the Sobolev embedding, Theorem 9.5, that $T \in C^1(\mathbb{T})$.

Similarly, if e.g. $S \in C^\infty(\mathbb{T})$ then always $T \in C^\infty(\mathbb{T})$.

11.2. **Duality.** Recall that the dual of a normed space X is denoted by X^* and that it consists of all the continuous linear functionals $\Lambda\colon X \to \mathbb{C}$ and is equipped with the operator norm

$$\|\Lambda\|_{X^*} := \|\Lambda\|_{X\to\mathbb{C}} := \sup_{\|x\|_X\le 1} |\Lambda x|.$$

11.5. **Theorem.** *Let $s \in \mathbb{R}$. For every $S \in H^{-s}(\mathbb{T})$ define the linear functional*

$$\Lambda_S\colon H^s(\mathbb{T}) \to \mathbb{C}, \qquad \Lambda_S T := \sum_{m\in\mathbb{Z}} \widehat{T}(m)\widehat{S}(m).$$

Then $\Lambda_S \in (H^s(\mathbb{T}))^*$ (the dual of $H^s(\mathbb{T})$) with

$$\|\Lambda_S\|_{(H^s(\mathbb{T}))^*} = \|\Lambda_S\|_{H^s(\mathbb{T})\to\mathbb{C}} = \sup_{\|T\|_{H^s(\mathbb{T})}\le 1} |\Lambda_S T| \le \|S\|_{H^{-s}(\mathbb{T})}.$$

Conversely, let $\Lambda \in (H^s(\mathbb{T}))^*$ be arbitrary. Then there exists $S \in H^{-s}(\mathbb{T})$ such that $\|S\|_{H^{-s}(\mathbb{T})} = \|\Lambda\|_{(H^s(\mathbb{T}))^*}$ and $\Lambda = \Lambda_S$.

It follows that for $T \in \mathcal{D}'$ we have

$$\|T\|_{H^s(\mathbb{T})} = \sup\Big\{\Big|\sum_{m\in\mathbb{Z}} \widehat{T}(m)\widehat{f}(m)\Big| \colon f \in C^\infty(\mathbb{T}),\ \|f\|_{H^{-s}(\mathbb{T})} = 1\Big\}.$$

Proof. Exercise (with hints). □

12. L^p convergence of Fourier series

We aim to prove here that $\|S_N f - f\|_{L^p} \to 0$ for $f \in L^p(\mathbb{T})$ and $1 < p < \infty$. We already know the $p = 2$ result, which is much easier. We will also prove that this result is not true for $p = 1$ in general.

For a bounded sequence $a = (a_m)_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ define

$$S_a\varphi(x) = \sum_{m\in\mathbb{Z}} a_m\widehat{\varphi}(m)e^{2\pi i m x}, \qquad \varphi \in \mathcal{P}.$$

Here $\mathcal{P} = \mathcal{P}(\mathbb{T})$ denotes the set of trigonometric polynomials. $S_a\varphi$ is clearly well-defined as the sum is finite for $\varphi \in \mathcal{P}$ (as $\widehat{\varphi}(m) = 0$ for all but finitely many m). This could also be defined for $\varphi \in \mathcal{D}$ as then $(\widehat{\varphi}(m))_m \in \mathscr{S}(\mathbb{Z})$. For each N define

$$S_{a,N}f(x) = \sum_{|m|\le N} a_m \widehat{f}(m)e^{2\pi i m x}, \qquad f \in L^1(\mathbb{T}).$$

This is well-defined for all N and all $f \in L^1(\mathbb{T})$ simply because it is a finite sum and $\widehat{f}(m)$ is well-defined.

12.1. **Lemma.** *Let $1 \le p < \infty$ and $a \in \ell^\infty(\mathbb{Z})$. We have that the sequence $(S_{a,N}f)_N$ is a converging sequence in $L^p(\mathbb{T})$ for all $f \in L^p(\mathbb{T})$ if and only if*

$$\sup_N \|S_{a,N}\|_{L^p\to L^p} < \infty.$$

Moreover, if this holds then S_a extends to a bounded operator from $L^p(\mathbb{T}) \to L^p(\mathbb{T})$ with

$$\|S_a\|_{L^p\to L^p} \le \sup_N \|S_{a,N}\|_{L^p\to L^p},$$

and for all $f \in L^p(\mathbb{T})$ we have

$$\lim_{N\to\infty} \|S_{a,N}f - S_a f\|_{L^p} = 0.$$

Proof. Suppose that the sequence $(S_{a,N}f)_N$ is a converging sequence in $L^p(\mathbb{T})$ for all $f \in L^p(\mathbb{T})$. Then in particular $\sup_N \|S_{a,N}f\|_{L^p} < \infty$ for all $f \in L^p(\mathbb{T})$. It follows from the Banach–Steinhaus theorem that $\sup_N \|S_{a,N}\|_{L^p\to L^p} < \infty$.

Suppose now that $\|S_{a,N}f\|_{L^p} \le C\|f\|_{L^p}$ for all N and $f \in L^p(\mathbb{T})$. Notice that we have for all $\varphi \in \mathcal{P}$ that for $N = \deg\varphi$ there holds that

$$\|S_a\varphi\|_{L^p}^p = \|S_{a,N}\varphi\|_{L^p}^p \le C^p\|\varphi\|_{L^p}^p.$$

By the density of $\mathcal{P}$ in $L^p(\mathbb{T})$ we know that S_a extends to a bounded operator from $L^p(\mathbb{T}) \to L^p(\mathbb{T})$, and then we have $\|S_a\|_{L^p\to L^p} \le C$.

Fix $f \in L^p(\mathbb{T})$. We will show that $\lim_{N\to\infty} \|S_{a,N}f - S_a f\|_{L^p} = 0$. Let $\epsilon > 0$ and choose a trigonometric polynomial φ so that $\|f - \varphi\|_{L^p} < \epsilon$. We estimate

$$\begin{aligned}\|S_{a,N}f - S_a f\|_{L^p} &\le \|S_{a,N}(f-\varphi)\|_{L^p} + \|S_{a,N}\varphi - S_a\varphi\|_{L^p} + \|S_a(\varphi - f)\|_{L^p} \\ &\le 2C\epsilon + \|S_{a,N}\varphi - S_a\varphi\|_{L^p}.\end{aligned}$$

Notice that $S_{a,N}\varphi = S_a\varphi$ for all $N \ge \deg\varphi$ – we are done. □

12.2. *Remark.* A particular case is that $a_m = 1$ for all m. Then we have that $S_N f \to f$ in $L^p(\mathbb{T})$ for all $f \in L^p(\mathbb{T})$ if and only if we have $\sup_N \|S_N\|_{L^p\to L^p} < \infty$.

We can already show the failure of the L^1 result.

12.3. **Proposition.** *There exists $f \in L^1(\mathbb{T})$ so that $\|f - S_N f\|_{L^1} \not\to 0$ when $N \to \infty$.*

Proof. By Lemma 12.1 (i.e. essentially because of Banach–Steinhaus theorem) it is enough to show that

$$\sup_N \|S_N\|_{L^1\to L^1} = \infty.$$

Fix N. Recall that $\|F_M\|_{L^1} = 1$ for all M. Therefore, we have

$$\|S_N\|_{L^1\to L^1} = \sup_{\|f\|_{L^1}=1} \|S_N f\|_{L^1} \ge \|S_N(F_M)\|_{L^1} = \|D_N * F_M\|_{L^1} = \|\sigma_M D_N\|_{L^1}.$$

Letting $M \to \infty$ we get by Theorem 4.3 that

$$\|S_N\|_{L^1 \to L^1} \geq \lim_{M\to\infty} \|\sigma_M D_N\|_{L^1} = \|D_N\|_{L^1}.$$

From Section 3 we know that $\sup_N \|D_N\|_{L^1} = \infty$, and so it also follows that $\sup_N \|S_N\|_{L^1\to L^1} = \infty$. We are done. □

12.4. **Definition.** For $\varphi \in \mathcal{P}$ (or $\varphi \in \mathcal{D}$) define the conjugate function $\tilde{\varphi}$ by

$$\tilde{\varphi}(x) = -i \sum_{m\in\mathbb{Z}} \operatorname{sgn}(m)\widehat{\varphi}(m)e^{2\pi imx},$$

where $\operatorname{sgn}(m) = 1$ for $m > 0$, -1 for $m < 0$, and 0 for $m = 0$. Define also the Riesz projections

$$P_+\varphi(x) = \sum_{m=1}^{\infty} \widehat{\varphi}(m)e^{2\pi imx},$$

$$P_-\varphi(x) = \sum_{m=-\infty}^{-1} \widehat{\varphi}(m)e^{2\pi imx}.$$

Notice that $\varphi(x) = P_+\varphi(x) + P_-\varphi(x) + \widehat{\varphi}(0)$, while $\tilde{\varphi}(x) = -iP_+\varphi(x) + iP_-\varphi(x)$.

12.5. *Remark.* The conjugate function $\tilde{\varphi}$, $\varphi \in \mathcal{D}$, also has the form

$$\varphi(x) = \lim_{r\to 1-} Q_r * f(x), \qquad Q_r(x) = \frac{2r\sin(2\pi x)}{1 - 2r\cos(2\pi x) + r^2},\ 0 < r < 1.$$

This is in a sense an analog of the so-called Hilbert transform, which is the simplest singular integral operator in $\mathbb{R}$. We will not use this viewpoint in the lecture notes. However, see the exercises for some further context.

12.6. **Proposition.** *Let $1 \leq p < \infty$. Then $S_N f \to f$ in $L^p(\mathbb{T})$ for all $f \in L^p(\mathbb{T})$ if and only if for all $\varphi \in \mathcal{P}$ we have*

$$\|\tilde{\varphi}\|_{L^p(\mathbb{T})} \lesssim \|\varphi\|_{L^p(\mathbb{T})}.$$

Proof. Notice that

$$P_+\varphi(x) = \frac{1}{2}(\varphi + i\tilde{\varphi} - \widehat{\varphi}(0)).$$

Thus $\|\tilde{\varphi}\|_{L^p(\mathbb{T})} \lesssim \|\varphi\|_{L^p(\mathbb{T})}$ if and only if $\|P_+\varphi\|_{L^p(\mathbb{T})} \lesssim \|\varphi\|_{L^p(\mathbb{T})}$.

Then we notice that as $\widehat{fe^{2\pi iN\cdot}}(m) = \widehat{f}(m-N)$ we have

$$S_N f(x) = e^{-2\pi iNx} \sum_{m=0}^{2N} \widehat{fe^{2\pi iN\cdot}}(m)e^{2\pi imx}.$$

Since multiplying by exponentials does not affect L^p norms we have that $\|S_N\|_{L^p\to L^p}$ is equal to $\|S'_N\|_{L^p\to L^p}$, where

$$S'_N f(x) := \sum_{m=0}^{2N} \widehat{f}(m)e^{2\pi imx}.$$

This means that $\sup_N \|S_N\|_{L^p\to L^p} = \sup_N \|S'_N\|_{L^p\to L^p}$.

Therefore, we have that $S_N f \to f$ in $L^p(\mathbb{T})$ for all $f \in L^p(\mathbb{T})$ if and only if $\sup_N \|S_N\|_{L^p\to L^p} < \infty$ if and only if $\sup_N \|S'_N\|_{L^p\to L^p} < \infty$. We only need to show that $\sup_N \|S'_N\|_{L^p\to L^p} < \infty$ if and only if $\|P_+\varphi\|_{L^p(\mathbb{T})} \lesssim \|\varphi\|_{L^p(\mathbb{T})}$ for all $\varphi \in \mathcal{P}$.

Suppose now that $\sup_N \|S'_N\|_{L^p\to L^p} < \infty$. Define the sequence $a_m = 1$ if $m \ge 0$ and $a_m = 0$ otherwise. Then we have $S'_N = S_{a,2N}$. By Lemma 12.1 we have that

$$S_a\varphi = P_+(\varphi) + \widehat{\varphi}(0), \qquad \varphi \in \mathcal{P},$$

extends to a bounded operator in $L^p(\mathbb{T})$. Hence also P_+ is bounded in $L^p(\mathbb{T})$.

Assume then that $\|P_+\varphi\|_{L^p(\mathbb{T})} \lesssim \|\varphi\|_{L^p(\mathbb{T})}$ for all $\varphi \in \mathcal{P}$. For $\varphi \in \mathcal{P}$ and an arbitrary N (which might also be significantly smaller than $\deg \varphi$) write

$$\begin{aligned} S'_N\varphi(x) &= P_+\varphi(x) + \widehat{\varphi}(0) - \sum_{m=2N+1}^{\infty} \widehat{\varphi}(m)e^{2\pi imx} \\ &= P_+\varphi(x) + \widehat{\varphi}(0) - e^{2\pi i(2N)x}\sum_{m=1}^{\infty} \widehat{\varphi}(m+2N)e^{2\pi imx} \\ &= P_+\varphi(x) + \widehat{\varphi}(0) - e^{2\pi i(2N)x}\sum_{m=1}^{\infty} \mathcal{F}(\varphi e^{-2\pi i(2N)\cdot})(m)e^{2\pi imx} \\ &= P_+\varphi(x) - e^{2\pi i(2N)x}P_+(\varphi e^{-2\pi i(2N)\cdot})(x) + \widehat{\varphi}(0). \end{aligned}$$

It follows that

$$\sup_N \|S'_N\varphi\|_{L^p(\mathbb{T})} \lesssim \|\varphi\|_{L^p(\mathbb{T})}, \qquad \varphi \in \mathcal{P}.$$

By density $\sup_N \|S'_N\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} < \infty$, and we are done. □

As the convergence $S_N f \to f$ fails in general in $L^1(\mathbb{T})$, we know that the conjugate operator $\varphi \to \tilde{\varphi}$ is not bounded in $L^1(\mathbb{T})$. However, we will show that it is bounded in $L^p(\mathbb{T})$ for $1 < p < \infty$, and thus $\lim_{N\to\infty} \|S_N f - f\|_{L^p(\mathbb{T})} = 0$ for all $f \in L^p(\mathbb{T})$, $1 < p < \infty$.

12.7. **Theorem** (L^p, $1 < p < \infty$, convergence of the Fourier series)**.** *Let* $1 < p < \infty$. *Then we have for all* $\varphi \in \mathcal{P}$ *that*

$$\|\tilde{\varphi}\|_{L^p(\mathbb{T})} \lesssim \|\varphi\|_{L^p(\mathbb{T})}.$$

Consequently, we have $\lim_{N\to\infty} \|S_N f - f\|_{L^p(\mathbb{T})} = 0$ *for all* $f \in L^p(\mathbb{T})$.

Proof. We assume that φ is a trigonometric polynomial of degree N_0. We first also assume that φ is real-valued and $\int_0^1 \varphi = 0$ – i.e., $\widehat{\varphi}(0) = 0$. We have $\widehat{\varphi}(-m) = \overline{\widehat{\varphi}(m)}$ as φ is real-valued. Thus, (by using $z + \bar{z} = 2\operatorname{Re} z$) we have

$$\begin{aligned} \tilde{\varphi}(x) &= -iP_+\varphi(x) + iP_-\varphi(x) \\ &= -i\sum_{m>0} \widehat{\varphi}(m)e^{2\pi imx} + i\sum_{m<0} \widehat{\varphi}(m)e^{2\pi imx} \\ &= -i\sum_{m>0} \widehat{\varphi}(m)e^{2\pi imx} + i\sum_{m>0} \overline{\widehat{\varphi}(m)}e^{-2\pi imx} \\ &= -i\sum_{m>0} \widehat{\varphi}(m)e^{2\pi imx} + \overline{(-i)\sum_{m>0} \widehat{\varphi}(m)e^{2\pi imx}} \end{aligned}$$

$$= 2\,\mathrm{Re}\Big[-i\sum_{m>0}\widehat{\varphi}(m)e^{2\pi imx}\Big].$$

This implies that also $\tilde{\varphi}$ is real-valued, which will be useful somewhat later.

Recalling $\widehat{\varphi}(0) = 0$ we can write

$$\begin{aligned}\varphi(x) + i\tilde{\varphi}(x) &= P_+\varphi(x) + P_-\varphi(x) + P_+\varphi(x) - P_-\varphi(x)\\ &= 2P_+\varphi(x) = 2\sum_{m>0}\widehat{\varphi}(m)e^{2\pi imx} = 2\sum_{m=1}^{N_0}\widehat{\varphi}(m)e^{2\pi imx}.\end{aligned}$$

This implies that for all $u \in \{1,2,\ldots\}$ we have

$$\begin{aligned}\int_0^1(\varphi(x) + i\tilde{\varphi}(x))^u\,\mathrm{d}x &= 2^u\int_0^1\Big(\sum_{m=1}^{N_0}\widehat{\varphi}(m)e^{2\pi imx}\Big)^u\,\mathrm{d}x\\ &= 2^u\sum_{m_1,\ldots,m_u=1}^{N_0}\prod_{j=1}^{u}\widehat{\varphi}(m_j)\int_0^1 e^{2\pi i(\sum_{j=1}^u m_j)x}\,\mathrm{d}x = 0\end{aligned}$$

simply because

$$\sum_{j=1}^{u} m_j \neq 0.$$

This calculation was based only on the fact that only strictly positive frequencies appear here.

We now expand this when $u = 2k$, $k \in \{1,2,\ldots\}$, is a positive even integer. Using the binomial formula we have

$$\begin{aligned}&(\varphi(x) + i\tilde{\varphi}(x))^{2k}\\ &\quad= \sum_{j=0}^{2k}\binom{2k}{j}\varphi(x)^j(i\tilde{\varphi}(x))^{2k-j}\\ &\quad= \sum_{j=0}^{k}\binom{2k}{2j}\varphi(x)^{2j}(i\tilde{\varphi}(x))^{2k-2j} + \sum_{j=0}^{k-1}\binom{2k}{2j+1}\varphi(x)^{2j+1}(i\tilde{\varphi}(x))^{2k-2j-1}\\ &\quad= \sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{2j}\varphi(x)^{2j}\tilde{\varphi}(x)^{2k-2j}\\ &\quad- i\sum_{j=0}^{k-1}(-1)^{k-j}\binom{2k}{2j+1}\varphi(x)^{2j+1}\tilde{\varphi}(x)^{2k-2j-1}.\end{aligned}$$

Since both φ and $\tilde{\varphi}$ are real-valued we have that the real-part is

$$\mathrm{Re}\,(\varphi(x) + i\tilde{\varphi}(x))^{2k} = \sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{2j}\varphi(x)^{2j}\tilde{\varphi}(x)^{2k-2j}.$$

Therefore, we also have

$$0 = \operatorname{Re} \int_0^1 (\varphi(x) + i\tilde{\varphi}(x))^{2k} \,\mathrm{d}x = \sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{2j} \int_0^1 \varphi(x)^{2j} \tilde{\varphi}(x)^{2k-2j} \,\mathrm{d}x.$$

Writing

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{2j} \int_0^1 \varphi(x)^{2j} \tilde{\varphi}(x)^{2k-2j} \,\mathrm{d}x$$
$$= (-1)^k \int_0^1 \tilde{\varphi}(x)^{2k} \,\mathrm{d}x + \sum_{j=1}^{k} (-1)^{k-j} \binom{2k}{2j} \int_0^1 \varphi(x)^{2j} \tilde{\varphi}(x)^{2k-2j} \,\mathrm{d}x$$

we get that

$$\|\tilde{\varphi}\|_{L^{2k}}^{2k} \le \sum_{j=1}^{k} \binom{2k}{2j} \int_0^1 \tilde{\varphi}(x)^{2k-2j} \varphi(x)^{2j} \,\mathrm{d}x.$$

We now apply Hölder's inequality with the exponent $2k/(2k-2j) > 1$ and the dual exponent $2k/2j$ to get

$$\|\tilde{\varphi}\|_{L^{2k}}^{2k} \le \sum_{j=1}^{k} \binom{2k}{2j} \|\tilde{\varphi}\|_{L^{2k}}^{2k-2j} \|\varphi\|_{L^{2k}}^{2j}.$$

Therefore, we have

$$1 \le \sum_{j=1}^{k} \binom{2k}{2j} \|\tilde{\varphi}\|_{L^{2k}}^{-2j} \|\varphi\|_{L^{2k}}^{2j} = \sum_{j=1}^{k} \binom{2k}{2j} R(\varphi)^{-2j},$$

where we set

$$R(\varphi) = \frac{\|\tilde{\varphi}\|_{L^{2k}}}{\|\varphi\|_{L^{2k}}}.$$

If there would exist trigonometric polynomials $(\varphi_u)_{u=1}^{\infty}$ so that $R(\varphi_u) \to \infty$ as $u \to \infty$ we can e.g. establish the contradiction $1 \le 1/2$. Therefore, we must have

$$\|\tilde{\varphi}\|_{L^p} \le C_p \|\varphi\|_{L^p}$$

whenever $p = 2k$ for some $k \in \{1, 2 \ldots\}$ and $\varphi \in \mathcal{P}$ is real-valued and satisfies $\widehat{\varphi}(m) = 0$.

We first dispose of the assumption $\widehat{\varphi}(m) = 0$. So suppose $\varphi \in \mathcal{P}$ is real-valued. Apply what we proved to the real-valued trigonometric polynomial $\varphi_0 := \varphi - \widehat{\varphi}(0) = \varphi - \int_0^1 \varphi$ satisfying $\widehat{\varphi}_0(0) = 0$. Observe that $\tilde{\varphi}_0 = \tilde{\varphi}$ (as $\widehat{C}(m) = C\delta_{0,m}$). We get

$$\|\tilde{\varphi}\|_{L^p} = \|\tilde{\varphi}_0\|_{L^p} \le C_p \|\varphi_0\|_{L^p}.$$

As $|\widehat{\varphi}(0)| \le \|\varphi\|_{L^1} \le \|\varphi\|_{L^p}$ we have $\|\varphi_0\|_{L^p} \le 2\|\varphi\|_{L^p}$. We conclude that

$$\|\tilde{\varphi}\|_{L^p} \le 2C_p \|\varphi\|_{L^p}$$

whenever $p = 2k$ for some $k \in \{1, 2 \ldots\}$ and $\varphi \in \mathcal{P}$ is real-valued.

Next, take a general $\varphi \in \mathcal{P}$ of some degree N_0 and let $p = 2k$ for some k. We write

$$\varphi(x) = \sum_{|m| \le N_0} a_m e^{2\pi i m x} = \varphi_1(x) + i\varphi_2(x),$$

where

$$\varphi_1(x) = \sum_{|m| \le N_0} \frac{a_m + \overline{a_{-m}}}{2} e^{2\pi i m x}$$

and

$$\varphi_2(x) = \sum_{|m| \le N_0} \frac{a_m - \overline{a_{-m}}}{2i} e^{2\pi i m x}.$$

Notice that

$$\overline{\varphi_2(x)} = \sum_{|m| \le N_0} \frac{\overline{a_m} - a_{-m}}{-2i} e^{-2\pi i m x} = \sum_{|m| \le N_0} \frac{\overline{a_{-m}} - a_m}{-2i} e^{2\pi i m x} = \varphi_2(x)$$

and similarly $\overline{\varphi_1(x)} = \varphi_1(x)$. Thus, $\varphi_1, \varphi_2 \in \mathcal{P}$ are real-valued. We have

$$\|\tilde{\varphi}\|_{L^p} = \|\tilde{\varphi}_1 + i\tilde{\varphi}_2\|_{L^p} \le \sum_{j=1}^{2} \|\tilde{\varphi}_j\|_{L^p} \le 2C_p \sum_{j=1}^{2} \|\varphi_j\|_{L^p} \le 4C_p\|\varphi\|_{L^p}.$$

The last inequality is simply the fact that e.g. $\varphi_1 = \operatorname{Re}\varphi$ and $|\operatorname{Re} z| \le |z|$.

We have now proved that

$$\|\tilde{\varphi}\|_{L^p} \le 4C_p\|\varphi\|_{L^p}$$

whenever $p = 2k$ for some $k \in \{1, 2 \ldots\}$ and $\varphi \in \mathcal{P}$. This can also be extended to all $\varphi \in L^p(\mathbb{T})$, $p = 2k$, by density. We can then get any $p \ge 2$ by choosing $k = \{1, 2, \ldots\}$ so that $p \in [2k, 2k+2]$ and interpolating the estimates valid for $2k$ and $2(k+1)$.

Finally, let $1 < p < 2$. We will prove this range by a standard duality argument. To this end, let $\varphi_1, \varphi_2 \in \mathcal{P}$. We have

$$\begin{aligned}\Big|\int_0^1 \tilde{\varphi}_1(x)\varphi_2(x)\,\mathrm{d}x\Big| &= \Big|\int_0^1 \varphi_1(x)(-\tilde{\varphi}_2(x))\,\mathrm{d}x\Big| \\ &\le \int_0^1 |\varphi_1(x)||\tilde{\varphi}_2(x)|\,\mathrm{d}x \le \|\varphi_1\|_{L^p}\|\tilde{\varphi}_2\|_{L^{p'}} \lesssim \|\varphi_1\|_{L^p}\|\varphi_2\|_{L^{p'}}.\end{aligned}$$

Here the first equality is an easy direct calculation, while the last estimate uses that $p' > 2$ and the bound we just proved above in this regime. In the middle we used Hölder's inequality. It is an easy direct calculation (and also follows from the duality of L^p and $L^{p'}$) that

$$\|\tilde{\varphi}_1\|_{L^p} = \sup_{\substack{\varphi_2 \in \mathcal{P} \\ \|\varphi_2\|_{L^{p'}} = 1}} \Big|\int_0^1 \tilde{\varphi}_1(x)\varphi_2(x)\,\mathrm{d}x\Big| \lesssim \|\varphi_1\|_{L^p}.$$

We are done. □

It is also true that we have

$$S_N f(x) \to f(x)$$

pointwise almost everywhere whenever $f \in L^p(\mathbb{T})$, $1 < p < \infty$. This is a very difficult result of Carleson ($p = 2$) and Hunt ($p \in (1, \infty)$). Before Carleson's result the answer was not known even for continuous functions. The almost everywhere result is not true for $p = 1$, and in fact Kolmogorov gave an example of an $L^1(\mathbb{T})$ function whose Fourier series diverges at almost every point. Compare this with Theorem 5.7.

APPENDIX A. INTERPOLATION

Let (X, μ) be a measure space. For $0 < p < \infty$ and a measurable $f\colon X \to \mathbb{C}$ define

$$\begin{aligned} \|f\|_{L^p(X)} &= \Big(\int_X |f|^p \,\mathrm{d}\mu \Big)^{1/p}, \\ \|f\|_{L^{p,\infty}(X)} &= \sup_{\lambda > 0} \lambda \mu(\{x \in X \colon |f(x)| > \lambda\})^{1/p}, \\ \|f\|_{L^\infty(X)} &= \inf\{C \ge 0 \colon |f(x)| \le C \text{ for } \mu\text{-a.e. } x \in X\}, \\ \|f\|_{L^{\infty,\infty}(X)} &= \|f\|_{L^\infty(X)}. \end{aligned}$$

A.1. **Theorem** (Marcinkiewicz interpolation theorem)**.** *Let (X, μ) and (Y, ν) be measure spaces and let $0 < p_0 < p_1 \le \infty$. Let T be a* ***sublinear*** *operator defined on the space $L^{p_0}(X) + L^{p_1}(X)$ and taking values in the space of measurable functions on Y. Assume that there exists two constants A_0 and A_1 such that*

$$\begin{aligned} \|Tf\|_{L^{p_0,\infty}(Y)} &\le A_0 \|f\|_{L^{p_0}(X)}, \qquad f \in L^{p_0}(X), \\ \|Tf\|_{L^{p_1,\infty}(Y)} &\le A_1 \|f\|_{L^{p_1}(X)}, \qquad f \in L^{p_1}(X). \end{aligned}$$

Let $p \in (p_0, p_1)$ and write

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \theta \in (0,1).$$

Then we have

$$\|Tf\|_{L^p(Y)} \le 2\Big(\frac{p}{p - p_0} + \frac{p}{p_1 - p} \Big)^{1/p} A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p(X)}.$$

A.2. *Remark.* Sublinearity means that we have the pointwise estimates

$$|T(f+g)| \le |Tf| + |Tg| \qquad \text{and} \qquad |T(\lambda f)| = |\lambda||Tf|,\ \lambda \in \mathbb{C}.$$

Marcinkiewicz interpolation theorem is an easy but very useful interpolation theorem. The good points are:

(1) We can assume only $L^q \to L^{q,\infty}$ type estimates at the endpoints $q \in \{p_0, p_1\}$ but conclude strong $L^p \to L^p$ estimates for $p_0 < p < p_1$.
(2) T does not need to be linear – for example, T can be some maximal function.

This theorem has a rather simple proof using the important identity

$$\int_X |f|^p \,\mathrm{d}\mu = p\int_0^\infty \lambda^{p-1}\mu(\{x \in X : |f(x)| > \lambda\})\,\mathrm{d}\lambda, \qquad 0 < p < \infty.$$

The weak point of the Marcinkiewicz interpolation theorem is that we cannot interpolate estimates like $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$, but rather need to have $p_0 = q_0$ and $p_1 = q_1$.

A.3. **Theorem** (Riesz-Thorin interpolation theorem)**.** *Let (X, μ) and (Y, ν) be measure spaces. Let T be a **linear** operator defined on the set of all simple functions on X and taking values in the set of measurable functions on Y. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and assume that*

$$\|Tf\|_{L^{q_0}(Y)} \le A_0\|f\|_{L^{p_0}(X)},$$
$$\|Tf\|_{L^{q_1}(Y)} \le A_1\|f\|_{L^{p_1}(X)},$$

for all simple functions f on X. Let $\theta \in (0, 1)$ and define p_θ, q_θ via

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$\|Tf\|_{L^{q_\theta}(Y)} \le A_0^{1-\theta}A_1^{\theta}\|f\|_{L^{p_\theta}(X)}, \qquad f \in L^{p_\theta}(X).$$

This is mainly useful when we need to interpolate estimates like $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$, where $p_0 \ne q_0$ and $p_1 \ne q_1$. However, note that even in the case $p_0 = q_0$ and $p_1 = q_1$ the conclusion of the Riesz–Thorin theorem is somewhat stronger compared to Marcinkiewicz interpolation theorem: the estimate only involves $A_0^{1-\theta}A_1^{\theta}$ and does not have the additional constant

$$2\Big(\frac{p}{p-p_0} + \frac{p}{p_1-p}\Big)^{1/p}$$

in front. This is not important to us. Riesz–Thorin requires a linear operator and we cannot allow $L^{p,\infty}$ type estimates, and so Marcinkiewicz interpolation theorem is often more useful. Riesz–Thorin is trickier to prove and requires complex analysis (Hadamard's three lines lemma). For both proofs consult Grafakos [2].

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