QUASIDISKS AND TWISTING OF THE RIEMANN MAP

István Prause

UNIVERSITY OF HELSINKI

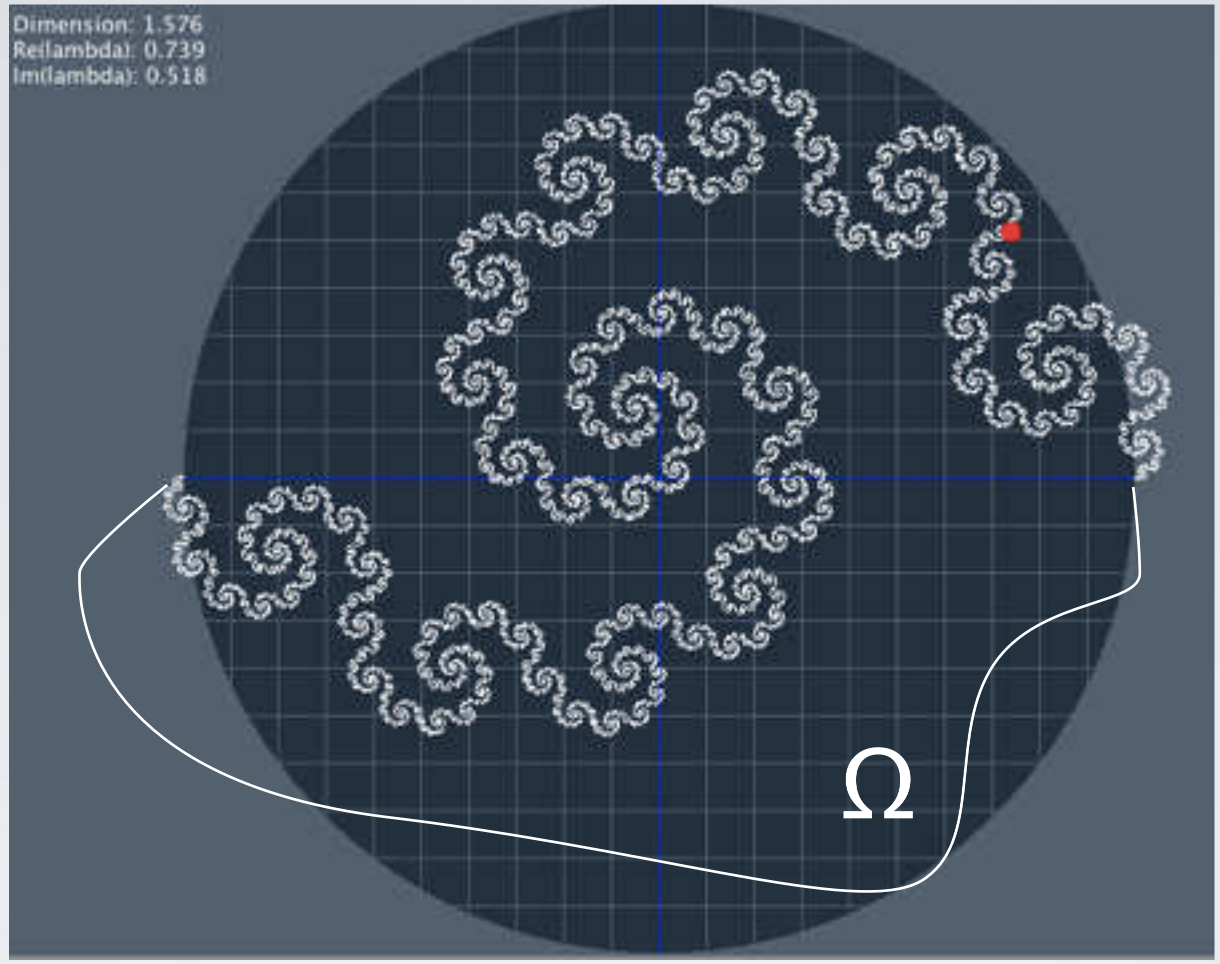
Oaxaca - Sept 2016

QUASIDISKS AND TWISTING

$$\Omega = g(\mathbb{D})$$

$$g : \mathbb{C} \to \mathbb{C}$$

L-quasiconformal



$f \colon \mathbb{D} \to \Omega$

conformal

with

k-qc extension

$$|\partial_{\bar{z}} f(z)| \leqslant k |\partial_z f(z)|$$

$$z \in \mathbb{D}^*$$

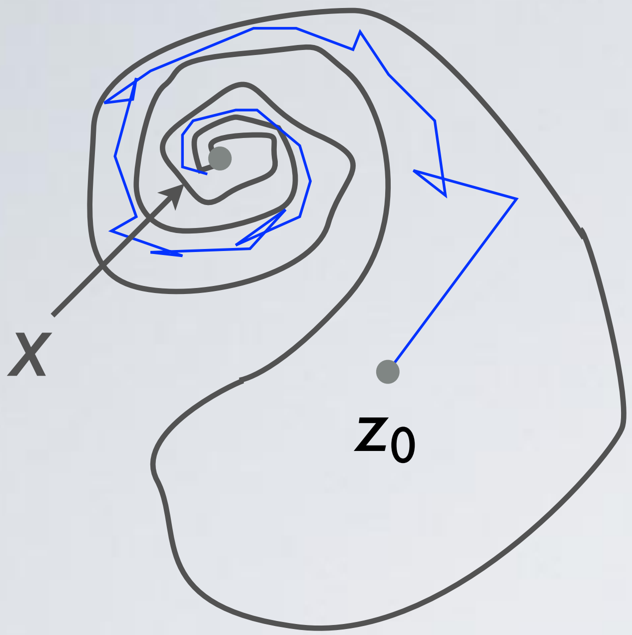
$$k = \frac{K-1}{K+1} = \frac{L^2-1}{L^2+1}$$

Questions:

- how much spiralling ?
- how fast and on how big set ?
- integrability of complex powers $(f')^t$

(in terms of k)

RATE OF ROTATION

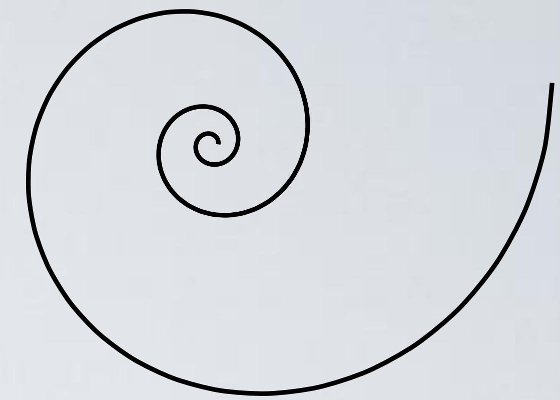


winding at x on scale r

$$\sim \gamma \log r$$

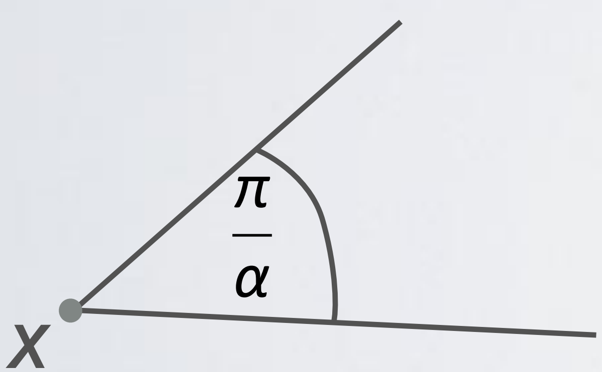
(measured from z_0)

$$\gamma \in \mathbb{R}$$



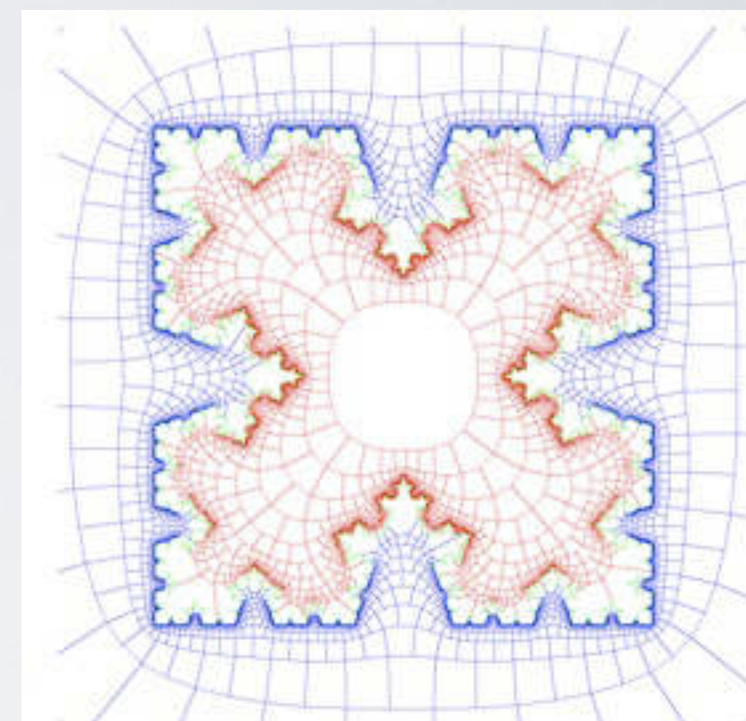
$$z \mapsto z|z|^{i\gamma}$$

scaling exponent



$$\omega B(x, r) \sim r^{\alpha}$$

$$\alpha > 0$$



Courtesy of D. Marshall

multifractal spectrum

$$F(\alpha, \gamma) = \dim\{x \in \partial\Omega : \omega \sim r^{\alpha},\ \arg \sim \gamma \log r\}$$

BEURLING'S ESTIMATE

$$\omega B(x, r) \lesssim r^{\frac{1}{2}} \qquad \alpha \geqslant \frac{1}{2}$$

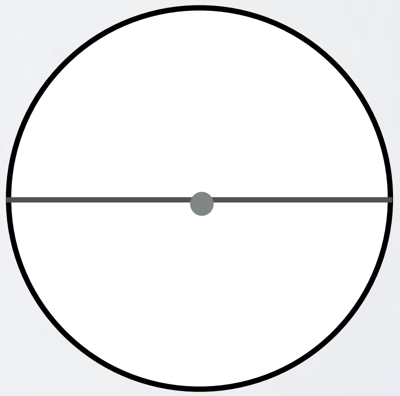
in presence of spiralling

$$\alpha \geqslant \frac{1}{2}(1 + \gamma^2) \qquad \mathrm{Re}\,\frac{1}{\sigma} \geqslant 1/2$$

$$\sigma = \frac{1 + i\gamma}{\alpha} \in \mathbb{C}$$

"holomorphic"

$$|\sigma - 1| \leqslant 1$$

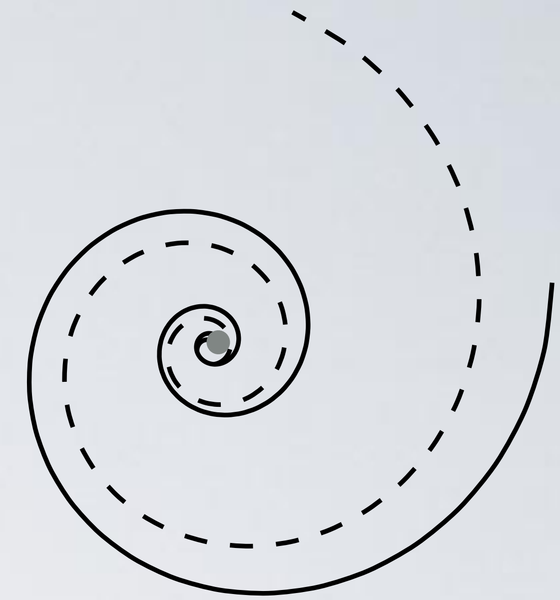


"angle"

EXAMPLES

$$z \mapsto z^{\sigma}$$

$$\sigma = \frac{1 + i\gamma}{\alpha}$$



univalence: $$\frac{1+\gamma^2}{\alpha} \leqslant 2$$

$$|\sigma - 1| \leqslant 1$$

$|\sigma_0 - 1| = k$ → k-qc extension

$$z^{\sigma_0} \hookrightarrow z^{\sigma}, \quad \sigma - 1 \in \mathbb{D}$$

INTEGRAL MEANS SPECTRUM

$f\colon \mathbb{D} \to \Omega$ (bounded) conformal map

$$\beta_f(t) = \inf\left\{\beta : \int |f'(re^{i\theta})^t| d\theta = O((1-r)^{-\beta})\right\}, \quad t \in \mathbb{C}$$

$$B(t) = \sup_{\Omega} \beta_f(t)$$

$$B_k(t) = \sup\{\beta_f(t) : f \text{ has a } k\text{-qc extension}\}$$

contributions

/ \

isolated singularities — t big

fractal boundary — t small

PHASE TRANSITION

Carleson-Makarov, Binder

pointwise bounds → $B(t) \leqslant |t|, \quad B_k(t) \leqslant k|t|$

pointwise examples → $B(t) \geqslant |t| - 1, \; B_k(t) \geqslant k|t| - 1$

"phase transition"

For each $\theta \in [0, 2\pi)$

There exists $T_\theta > 0$ $\quad B(te^{i\theta}) = t - 1, \quad t \geqslant T_\theta$

BRENNAN'S CONJECTURE

$$B(-2) = 1 \quad \longleftrightarrow \quad T_\pi = 2$$

"circular" Brennan conjecture: $B(t) = 1$, $|t| = 2$ $\longleftrightarrow$ $T_\theta \leqslant 2$

Becker-Pommerenke, Binder

radial invariance ?

$B(2) = 1$ trivial $\qquad \int_{\mathbb{D}} |f'|^2 = |f(\mathbb{D})| < \infty$

quasiconformal variant

$$B_k(t) = 1$$

$$|t| = \frac{2}{k}$$

MAJORANT PRINCIPLE

$$B(t) = 1$$
$$|t| = 2$$

$\longrightarrow$

$$B_k(t) = 1$$
$$|t| = \frac{2}{k}$$

$$f \hookrightarrow f_\lambda, \quad \lambda \in \mathbb{D}$$

maximum principle

$$\lambda \mapsto \int_{|z|=r} \left| {f_\lambda}'(z)^{2/\lambda} \right| |dz|$$

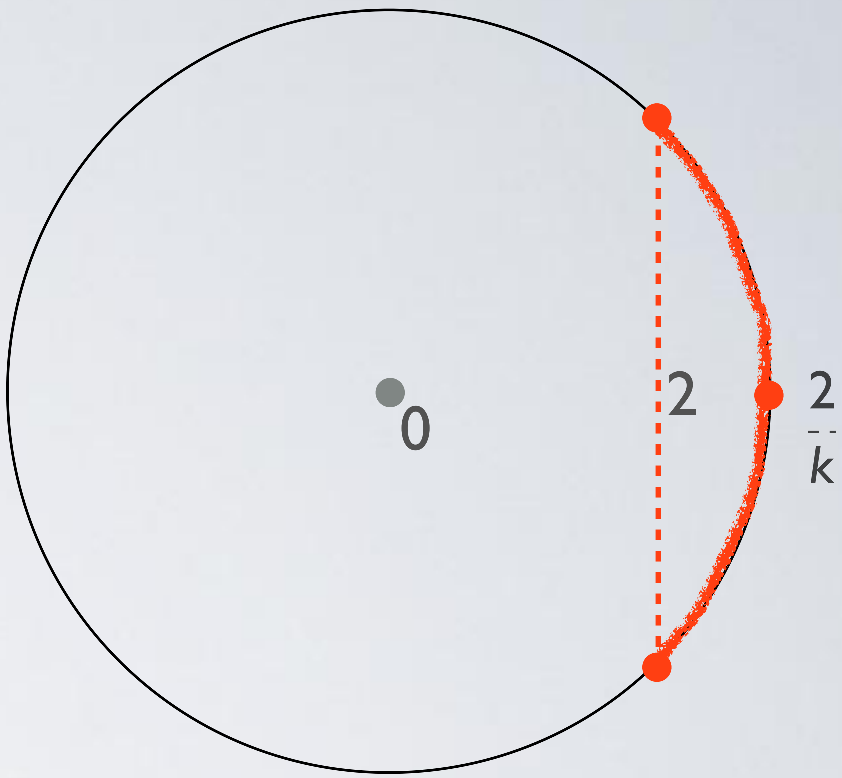
subharmonic

Lower bounds: Examples with $|\sigma - 1| = k$

MAIN RESULT

some radial invariance

$$B_k(t) = 1, \quad |t| = \frac{2}{k} \;\&\; \operatorname{Re} t \geqslant 2$$



Thm: $f\colon \mathbb{D} \to \Omega$ conformal with k-qc extension

$$\int_{\mathbb{D}} \left|(f')^t\right| < \infty \qquad |t| < \frac{2}{k},\ \operatorname{Re} t \geqslant 2$$

Prause-Smirnov: $B_k(2/k) = 1$

Hedenmalm: $B_k(t) \leqslant (1 + 7k)^2 \dfrac{k^2|t|^2}{4}$

holomorphic amplification of "B(2)=1"

HOLOMORPHIC MOTION IN $\mathbb{D}^2$

$$R > 1 \qquad\qquad |\mu| \leqslant \chi_{\{|z| \geqslant R\}}$$

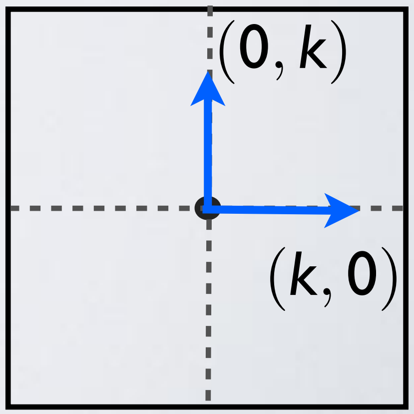
$$\mu_{\lambda,\eta}(z) = \begin{cases} \lambda\mu(z) \text{ for } |z| > 1, \\ \overline{\eta\mu(1/\bar{z})} \text{ for } |z| < 1. \end{cases} \qquad (\lambda, \eta) \in \mathbb{D}^2$$

$$\bar{\partial} f_{\lambda,\eta} = \mu_{\lambda,\eta}\, \partial f_{\lambda,\eta} \qquad \{0, 1, \infty\} \text{ -normalization}$$

$$f'(z) \rightsquigarrow \frac{z f'(z)}{f(z)}$$

$$\varphi_{\lambda,\eta}(z) = \frac{z f_{\lambda,\eta}{}'(z)}{f_{\lambda,\eta}(z)}, \qquad z \in \mathbb{S}^1$$

$$f_{\lambda,\eta}(z) = \frac{1}{\overline{f_{\bar{\eta},\bar{\lambda}}(1/\bar{z})}} \longrightarrow \varphi_{\lambda,\eta} = \overline{\varphi_{\bar{\eta},\bar{\lambda}}}$$



COMPLEX INTERPOLATION

Lemma: (Ω, σ) measure space $\quad p_0 > 0$

$\{\varphi_{\lambda,\eta}\}_{\mathbb{D}^2}$ non-vanishing analytic family of measurable functions

$$\varphi_{0,0} \equiv 1 \qquad \varphi_{\lambda,\eta} = \overline{\varphi_{\bar{\eta},\bar{\lambda}}} \qquad \|\varphi_{\lambda,\eta}\|_{p_0} \leqslant 1$$

$$\longrightarrow \int_\Omega \left|\varphi^t_{\lambda,0}\right| d\sigma \leqslant 1 \qquad |t| = \frac{p_0}{|\lambda|} \quad \& \quad \mathrm{Re}\, t \geqslant p_0$$

Apply for $\quad \varphi_{\lambda,\eta}(z) = \dfrac{z\, f_{\lambda,\eta}{}'(z)}{f_{\lambda,\eta}(z)}, \quad z \in \mathbb{S}^1$

$p_0 = 2 \qquad |f_{\lambda,\eta}(1/R < |z| < R)| \lesssim 1 \qquad d\sigma = c\,(R-1)|dz|$

$$\longrightarrow \int_{|z|=1} \left|(f')^t\right| \lesssim \frac{1}{(R-1)^{1+\varepsilon}}$$

CONVEXITY ARGUMENT

$$p_0 = 1 \qquad \|\varphi_{0,0}\|_1 \leqslant 1 \quad \longrightarrow \quad \sigma(\Omega) \leqslant 1$$

$$\int_\Omega \rho = 1 \qquad h = \int_\Omega \rho \log \rho > 0 \qquad \psi(\lambda, \eta) = \frac{1}{h} \int_\Omega \rho \log \varphi_{\lambda,\eta}$$

$$\operatorname{Re} \psi(\lambda, \eta) - 1 = \frac{1}{h} \int_\Omega \rho \log \frac{|\varphi_{\lambda,\eta}|}{\rho} \leqslant \frac{1}{h} \log \left(\int_\Omega |\varphi_{\lambda,\eta}| \right) \leqslant 0$$

$$\log \left(\int |\varphi^t_{\lambda,0}| \right) = \sup_\rho \int \rho \log \frac{|\varphi^t_{\lambda,0}|}{\rho} = \sup_\rho \{ h \left(\operatorname{Re}(t\, \psi(\lambda, 0)) - 1 \right) \}$$

Q: For which $t \in \mathbb{C}$ $\operatorname{Re}(t\, \psi(\lambda, 0)) \leqslant 1$?

$$\psi : \mathbb{D}^2 \to \{ \operatorname{Re} < 1 \} \qquad \psi(0,0) = 0 \qquad \psi(\lambda, \eta) = \overline{\psi(\bar\eta, \bar\lambda)}$$

STRETCHING vs ROTATION

Astala-Iwaniec-Prause-Saksman

harmonic dependence *"conjugate harmonic"*

stretching	rotation
quasiconformal	**bilipschitz**
Grötzsch problem	John's problem
Hölder exponent	rate of spiralling
$\log J(z,f) \in BMO$	$\arg f_z \in BMO$
higher integrability	exponential integrability
multifractal spectrum	

NEVANLINNA-PICK INTERPOLATION

$$\lambda_i \in \mathbb{D} \qquad w_i \in \mathbb{D} \qquad i = 1 \ldots n$$

There exists holomorphic

$$\phi : \mathbb{D} \to \mathbb{D} \qquad \phi(\lambda_i) = w_i$$

$\Longleftrightarrow$

$$\Gamma = \left[\frac{1 - w_i \bar{w}_j}{1 - \lambda_i \bar{\lambda}_j} \right]_{n \times n} \geqslant 0$$

positive semidefinite

$$[1 - w_i \bar{w}_j] = [1 - \lambda_i \bar{\lambda}_j] \cdot \Gamma$$

Schur product (entrywise)

Extremals: finite Blaschke products

VON NEUMANN'S INEQUALITY

$$\phi \in H^\infty(\mathbb{D})$$

$$T \colon \mathcal{H} \to \mathcal{H} \qquad \longrightarrow \qquad \|\phi(T)\| \leqslant \|\phi\|_\infty$$

$$\|T\| < 1$$

(may assume $\|\phi\|_\infty = 1$)

$$\frac{1 - \overline{\phi(w)}\phi(z)}{1 - \bar{w}z} = K(z, w) = \sum \overline{k_j(w)}\, k_j(z)$$

$$I - \phi(T)^*\phi(T) = \sum k_j(T)^*(I - T^*T)k_j(T) \geqslant 0$$

Andô's inequality: $\phi \in H^\infty(\mathbb{D}^2)$ $\quad T_1 T_2 = T_2 T_1$ contractions

$$\longrightarrow \quad \|\phi(T_1, T_2)\| \leqslant \|\phi\|_\infty$$

fails for more variables

NEVANLINNA-PICK ON BIDISK

Agler

$(\lambda_i, \eta_i) \in \mathbb{D}^2 \qquad w_i \in \mathbb{D} \qquad i = 1 \ldots n$

There exists holomorphic

$\phi : \mathbb{D}^2 \to \mathbb{D} \qquad \phi(\lambda_i, \eta_i) = w_i$

$\Longleftrightarrow$

$$[1 - w_i \bar{w}_j] = [1 - \lambda_i \bar{\lambda}_j] \cdot \Gamma + [1 - \eta_i \bar{\eta}_j] \cdot \Delta$$

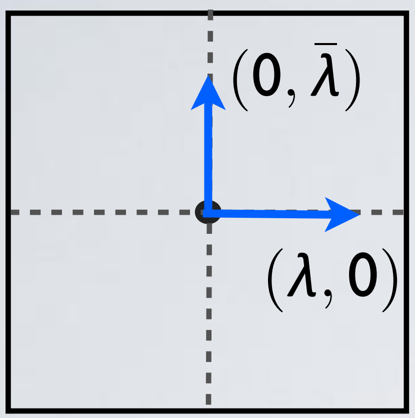
$\Gamma, \Delta \geqslant 0$

positive semidefinite matrices

Agler-McCarthy: Extremals are rational inner functions (need not be unique)

A THREE-POINT PROBLEM

$\phi : \mathbb{D}^2 \to \mathbb{D}$ or $\{\text{Re} > 0\}$



fix $\lambda \in \mathbb{D}$

$$\phi(0,0) = 0$$
$$\phi(\lambda, 0) = w$$
$$\phi(0, \bar{\lambda}) = \bar{w}$$

$w = ?$

(in terms of $|\lambda| = k$)

Agler-McCarthy: Unique solution for an *extremal* & *non-degenerate* 3-point problem. It is a degree 2 rational inner function. (Γ and Δ are rank-one)

$$\phi(\lambda, \eta) = \frac{\lambda \eta \overline{p(1/\bar{\lambda}, 1/\bar{\eta})}}{p(\lambda, \eta)}$$

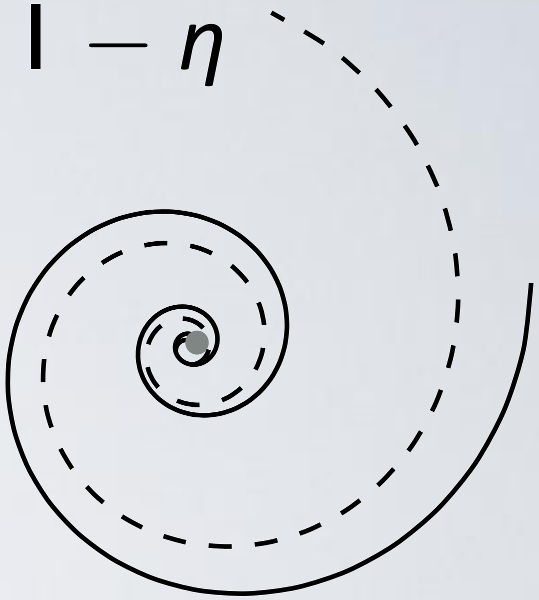
p linear polynomial does not vanish on $\mathbb{D}^2$

WELDING OF RADIAL STRETCHINGS

$$\mu(z) = \frac{z}{\bar{z}} \qquad \mu_{\lambda,\eta} = \begin{cases} \lambda\mu \text{ in } \mathbb{C}_+, \\ \eta\mu \text{ in } \mathbb{C}_-. \end{cases}$$

$$f_{\lambda,\lambda}(z) = \frac{z}{|z|}\,|z|^{\frac{1+\lambda}{1-\lambda}} \qquad \sigma_+ = \frac{1+\lambda}{1-\lambda} \quad \sigma_- = \frac{1+\eta}{1-\eta}$$

$$f_{\lambda,\eta}(z) = \left(\frac{z}{|z|}\,|z|^{\sigma_\pm}\right)^{\sigma/\sigma_\pm} \text{ for } z \in \mathbb{C}_\pm$$



$$\frac{1}{\sigma} = \frac{1}{2}\left(\frac{1}{\sigma_+} + \frac{1}{\sigma_-}\right)$$

$$\frac{1}{\sigma(\lambda,\eta)} = \frac{1}{2}\left(\frac{1-\lambda}{1+\lambda} + \frac{1-\eta}{1+\eta}\right)$$

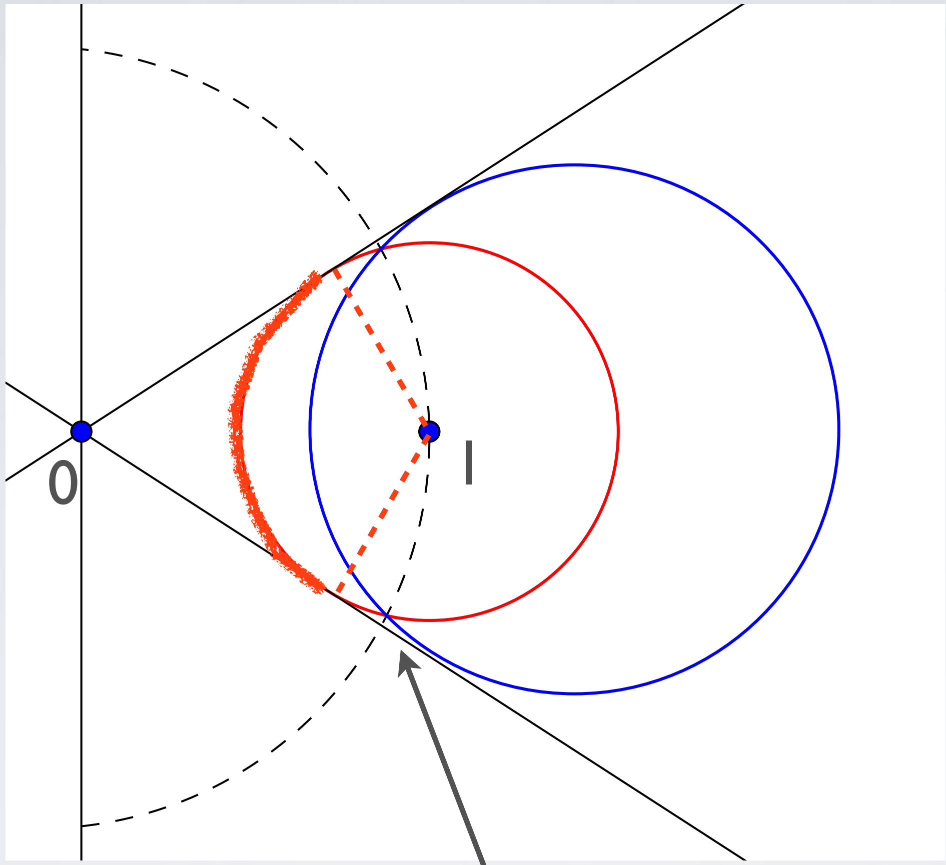
SOLUTION OF THE 3-POINT PROBLEM

$|\sigma - 1| \leqslant k$

(consistent with circular Brennan)

$\left|\frac{1}{\sigma} - 1\right| \leqslant k$

('non-physical example')



$p(\lambda,\eta) = 2i + c\lambda + \bar{c}\eta$

$$\phi(\lambda,\eta) = \frac{\lambda\eta\,\overline{p(1/\bar{\lambda}, 1/\bar{\eta})}}{p(\lambda,\eta)}$$

$p(\lambda,\eta) = 2 + c\lambda + \bar{c}\eta$

$|c| \leqslant 1$

degenerate to 2-pt

$$d_{\mathbb{D}^2}((\lambda,0),(0,\bar{\lambda})) = d_{\mathbb{D}}(w,\bar{w})$$

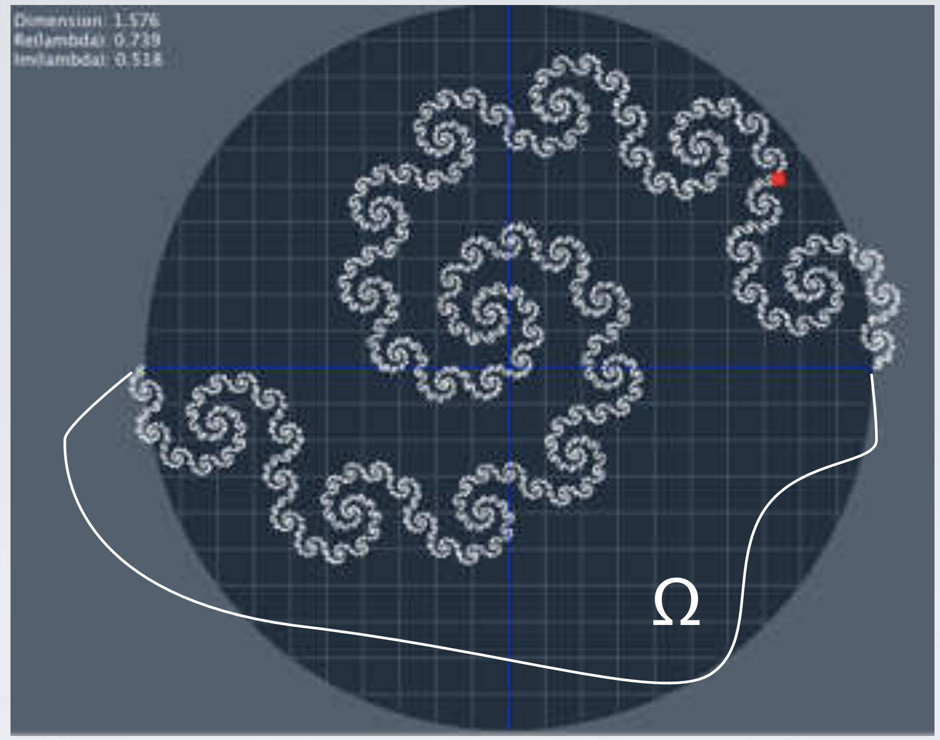
$|t| = \dfrac{p_0}{|\lambda|}$ & $\mathrm{Re}\, t \geqslant p_0$ → $\mathrm{Re}(t\,\psi(\lambda,0)) \leqslant 1$ → $\|\varphi^t_{\lambda,0}\|_1 \leqslant 1$

tangents control pure rotation ⟺ $\mathrm{Re}\, t = p_0 = 2$

TWISTING FOR A QUASIDISK

twisting of boundary

⤳ twisting of f'



$$\int_{\mathbb{D}} \left|(f')^{2+is}\right| < \infty$$

$$|s| < \frac{2\sqrt{1-k^2}}{k}$$

$$F(\gamma) = \dim\{x \in \partial\Omega : \gamma(x) = \gamma\} \leqslant 2 - \frac{2\sqrt{1-k^2}}{k}|\gamma|$$