Yukawa's PDE by Killing Brownian Motion

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Abstract

We define the Panharmonic Measure, which is a generalization of the harmonic measure to solutions of the Yukawa Partial Differential Equation. We show that this quantity has some of the important properties of the classical harmonic measure. In particular, there is a natural stochastic definition of the panharmonic measure in terms of the Brownian walk that is killed with independent exponential stopping time.

- R.J. Duffin: Yukawan potential theory, *J. Math. Anal. Appl.* **35** (1971), 105–130.
- O. Kallenberg: *Foundations of Modern Probability*, Springer, 1997.
- A. Rasila and T. Sottinen: Yukawa Potential, Harmonic Measure and Killing Brownian Motion, *Manuscript in preparation* (2012).

The Yukawa PDE

The harmonic measure is a fundamental tool in geometric function theory. We shall define the panharmonic counterpart of this quantity, where the harmonic functions related to the classical harmonic measure are replaced with solutions to the Yukawa equation

$$\Delta u = \mu^2 u, \quad \mu^2 \geq 0.$$

The Yukawa equation first arose from the work of Japanese physicist Hideki Yukawa in particle physics.

Remark (On $\mu^2 \geq 0$)

Note that for $\mu^2 = 0$ we have the Laplace equation. If we replace μ^2 with a constant $\lambda < 0$ we obtain another important PDE, the Helmholtz equation. Our approach does not generalize to the solutions of the Helmholtz equation.

Panharmonic Measure

Panharmonic, or μ-panharmonic measure, is a generalization of the harmonic measure:

Definition (μ-Panharmonic Measures and Functions)

Let $\mu^2 \geq 0$. Then the μ-panharmonic measure on a boundary ∂D with a pole at $x \in D$ is a measure $H_\mu^x(D;\cdot)$ such that any μ-panharmonic function u on D, i.e. functions with $\Delta u = \mu^2 u$ on D, admits the representation

$$u(x) = \int_{\partial D} u(y)\, H_\mu^x(D; \mathrm{d}y).$$

The existence and uniqueness of the panharmonic measure (with regular domain D) will be established by the Kakutani Lemma later.

Feller Diffusions and PDEs (1/3)

Let A be a differential operator. We want to consider the PDE $Au = 0$ from a probabilistic point of view.

Suppose X is a Markov process with the Transition Semigroup

$$P_t f(x) = \mathbb{E}^x \left[f(X_t) \right].$$

Since P is a semigroup, i.e. $P_{t+s} = P_t P_s$, one expects that

$$P_t = e^{tA}$$

for some operator A called the Infinitesimal Generator of X.

We have

$$Af(x) = \frac{\mathrm{d}}{\mathrm{d}t} P_t f(x) \Big|_{t=0} = \lim_{t \to 0+} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t},$$

but the other direction, constructing a Markov process X for a given operator A is the tricky one (and sometimes impossible).

Feller Diffusions and PDEs (2/3)

If A is of the form

$$Af(x) = \sum_{i,j} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x) - c(x) f(x),$$

where c is non-negative, a nice Markov process can be constructed. These Markov processes are called Feller diffusions.

Definition (Regular Domain)

A domain $D \subset \mathbb{R}^d$ is regular for the generator A of the Feller diffusion X if

$$\mathbb{P}^x [\zeta(D) = 0] = 1, \quad \text{for all } x \in \partial D,$$

where $\zeta(D) = \inf\{t > 0; X(t) \in D^c\}$.

Feller Diffusions and PDEs (3/3)

Lemma (Kakutani's Probabilistic Solution to PDEs)

Let D be a regular domain for the Feller diffusion X with the generator A. Then a solution to the Dirichlet problem $Au = 0$ on D and $u = f$ on ∂D is given by

$$u(x) = \mathbb{E}^x\left[f(X(\tau(D))); \tau(D) < \infty\right] = \int_{\partial D} f(y)\, H^x(D; \mathrm{d}y).$$

Here $\tau(D)$ is the first hitting time *of X to the set ∂D*

$$\tau(D) = \inf\left\{t \geq 0; X(t) \in \partial D\right\}$$

and $H^x(D; \cdot)$ is the hitting measure

$$H^x(D; \mathrm{d}y) = \mathbb{P}^x\left[\tau(D) < \infty, X(\tau(D)) \in \mathrm{d}y\right].$$

Yukawa and Killed Brownian motion (1/5)

In Yukawa by Killing Theorem below we will construct a Feller diffusion X associated with the Yukawa's PDE by killing the Brownian motion. Before the construction, we introduce some notations:

Let $W = (W_1(t), \ldots, W_d(t); t \geq 0)$ be the standard Brownian motion on $\mathbb{R}^d$ and let $\tau(D) = \inf\{t > 0; W(t) \in \partial D\}$ be the first hitting time of the Brownian motion on the boundary of D. Let

$$\begin{aligned} \phi(x) &= (2\pi)^{-d/2} e^{-\frac{1}{2}\|x\|^2}, \\ h^x(D; \mathrm{d}t) &= \mathbb{P}^x\left[\tau(D) \in \mathrm{d}t\right] \\ \gamma_\mu(t) &= 1 - e^{-\frac{\mu^2}{2}t}. \end{aligned}$$

ϕ is the standard Gaussian distribution on $\mathbb{R}^d$ and thus

$$\phi\left(\frac{x-y}{\sqrt{t}}\right)\mathrm{d}y = \mathbb{P}^x\left[W(t) \in \mathrm{d}y\right].$$

Yukawa and Killed Brownian motion (2/5)

$h^x(D;\cdot)$ is the hitting time distribution of Brownian motion at the boundary of D.

γ_μ is the cumulative distribution function of an exponential random variable with mean $2/\mu^2$.

By the Kakutani Lemma we have the following disintegration representation for Harmonic measure on regular domains:

$$\begin{aligned} H^x(D;\mathrm{d}y) &= \mathbb{P}^x\left[W(\tau(D)) \in \mathrm{d}y, \tau(D) < \infty\right] \\ &= \int_{t>0} \mathbb{P}^x\left[W(t) \in \mathrm{d}y\right] \mathbb{P}^x\left[\tau(D) \in \mathrm{d}t\right] \\ &= \int_{t>0} \phi\left(\frac{x-y}{\sqrt{t}}\right) h^x(D;\mathrm{d}t). \end{aligned}$$

Yukawa and Killed Brownian motion (3/5)

Theorem (Yukawa by Killing)

Let W be the standard Brownian motion and let Y_μ be an independent exponentially distributed random variable with mean $2/\mu^2$. Define an exponentially killed Brownian motion W_μ by

$$W_\mu(t) = W(t)1_{\{Y_\mu > t\}} + \dagger 1_{\{Y_\mu \le t\}},$$

where $\dagger$ is a coffin state[1]. Then the infinitesimal generator of W_μ is the (non-normalized) Yukawa operator

$$A = \frac{1}{2}\Delta - \frac{1}{2}\mu^2.$$

[1] By convention $f(\dagger) = 0$ for all functions f.

Yukawa and Killed Brownian motion (4/5)

Proof.

Let P^μ be the semigroup of W_μ and let $P = P^0$. Then

$$\begin{aligned} P^\mu_t f(x) &= \mathbb{E}^x\left[f(W_\mu(t))\right] = \mathbb{E}^x\left[f\left(W(t)\right)1_{\{Y_\mu > t\}}\right] \\ &= \mathbb{E}^x\left[f\left(W(t)\right)\right]\mathbb{P}\left[Y_\mu > t\right] = P_t f(x)\, e^{-\frac{1}{2}\mu^2 t}. \end{aligned}$$

Now, the generator of the Brownian motion W is $\frac{1}{2}\Delta$. So,

$$\begin{aligned} Af(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(P^\mu_t f(x)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(P_t f(x) e^{-\frac{1}{2}\mu^2 t}\right)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}P_t f(x) e^{-\frac{1}{2}\mu^2 t}\Big|_{t=0} + P_t f(x)\frac{\mathrm{d}}{\mathrm{d}t}e^{-\frac{1}{2}\mu^2 t}\Big|_{t=0} \\ &= \frac{1}{2}\Delta f(x) - \frac{1}{2}\mu^2 f(x). \end{aligned}$$

□

Yukawa and Killed Brownian motion (5/5)

Corollary (Panharmonic Kakutani)

Let D be a regular domain and let f be a bounded function on ∂D. Let u_μ be μ-panharhomonic on D with $u_\mu|_{\partial D} = f$. Then

$$\begin{aligned} u_\mu(x) &= \mathbb{E}_x\left[f\left(W(\tau(D)\right)1_{\{\tau(D)<\infty\}}1_{\{Y_\mu>\tau(D)\}}\right] \\ &= \int_{\partial D} f(y) \int_{t>0} \gamma_\mu(t)\phi\left(\frac{x-y}{\sqrt{t}}\right) h^x(D;\mathrm{d}t)\,\mathrm{d}y \\ &=: \int_{\partial D} f(y)\, H^x_\mu(D;\mathrm{d}y). \end{aligned}$$

Panharmonic–Harmonic Equivalence (1/7)

The probabilistic interpretation provided by Panharmonic Kakutani Corollary implies, at least intuitively, that the harmonic measure and the panharmonic ones are equivalent.

Indeed, the harmonic measure counts the Brownian particles on the boundary and the panharmonic measures count the killed Brownian particles on the boundary. But the killing happens with independent exponential random variables. So, if the Brownian motion can reach the boundary with positive probability, so can the killed Brownian motion; and vice versa. Also, it is intuitively clear that it does now matter, as far as the equivalence is concerned, what is the starting point of the Brownian motion, killed or not.

The Panequivalence Theorem below makes the loose points expressed above rigorous:

Panharmonic–Harmonic Equivalence (2/7)

Theorem (Panequivalence)

Let D be a regular domain. Then all the panharmonic measures $H^x_\mu(D;\cdot)$, $\mu \geq 0, x \in D$, are equivalent and their Radon–Nikodym derivatives are

$$\Gamma^{x,w}_{\mu,\lambda}(y) := \frac{\mathrm{d}H^x_\mu(D;\cdot)}{\mathrm{d}H^w_\lambda(D;\cdot)}(y) = \frac{\Gamma^x_\mu(D;y)}{\Gamma^w_\lambda(D;y)}\Gamma^{x,w}(D;y),$$

$$\Gamma^x_\mu(D;y) := \frac{\mathrm{d}H^x_\mu(D;\cdot)}{\mathrm{d}H^x(D;\cdot)}(y) = \frac{\int_{t>0}\gamma_\mu(t)\phi\left(\frac{x-y}{\sqrt{t}}\right)h^x(D;\mathrm{d}t)}{\int_{t>0}\phi\left(\frac{x-y}{\sqrt{t}}\right)h^x(D;\mathrm{d}t)},$$

$$\Gamma^{x,w}(D;y) := \frac{H^x(D;\cdot)}{H^w(D;\cdot)}(y) = \frac{\int_{t>0}\phi\left(\frac{x-y}{\sqrt{t}}\right)h^x(D;\mathrm{d}t)}{\int_{t>0}\phi\left(\frac{w-y}{\sqrt{t}}\right)h^w(D;\mathrm{d}t)}.$$

Panharmonic–Harmonic Equivalence (3/7)

Proof.

Let us first disintegrate the harmonic measure $H^x(D;\cdot)$. By Kakutani Lemma, the measure $H^x(D;\cdot)$ the is distribution of a Brownian motion W started at x when it hits the boundary ∂D. So, disintegrating on the time of the hitting we obtain,

$$\begin{aligned}
H^x(D;\mathrm{d}y) &= \mathbb{P}^x\left[W(\tau(D))\in\mathrm{d}y,\tau(D)<\infty\right] \\
&= \int_{t>0}\mathbb{P}^x\left[W(\tau(D))\in\mathrm{d}y,\tau(D)\in\mathrm{d}t\right] \\
&= \int_{t>0}\mathbb{P}^x\left[W(t)\in\mathrm{d}y,\tau(D)\in\mathrm{d}t\right] \\
&= \int_{t>0}\mathbb{P}^x\left[W(t)\in\mathrm{d}y\right]\mathbb{P}^x\left[\tau(D)\in\mathrm{d}t\right].
\end{aligned}$$

□

Panharmonic–Harmonic Equivalence (4/7)

Proof continues.

Then, by using the self-similarity of the Brownian motion, we obtain

$$\begin{aligned} H^x(D;\mathrm{d}y) &= \int_{t>0} \mathbb{P}^x\left[W(t)\in \mathrm{d}y\right]\mathbb{P}^x\left[\tau(D)\in \mathrm{d}t\right] \\ &= \int_{t>0} \mathbb{P}^0\left[W(1)\in \frac{\mathrm{d}y-x}{\sqrt{t}}\right]\mathbb{P}^x\left[\tau(D)\in \mathrm{d}t\right] \\ &= \left(\int_{t>0}\phi\left(\frac{x-y}{\sqrt{t}}\right)h^x(D;\mathrm{d}t)\right)\mathrm{d}y. \end{aligned}$$

Next, let us disintegrate the μ-panharmonic measure in the same way by using Kakutani Lemma and disintegrating on the time of hitting: □

Panharmonic–Harmonic Equivalence (5/7)

Proof continues.

$$
\begin{aligned}
H^x_\mu(D;\mathrm{d}y) &= \mathbb{P}^x\left[W_\mu(\tau_\mu(D)) \in \mathrm{d}y, \tau_\mu(D) < \infty\right] \\
&= \int_{t>0} \mathbb{P}^x\left[W_\mu(\tau_\mu(D)) \in \mathrm{d}y, \tau_\mu(D) \in \mathrm{d}t\right] \\
&= \int_{t>0} \mathbb{P}^x\left[W(t) \in \mathrm{d}y, \tau(D) \in \mathrm{d}t, Y_\mu \le t\right] \\
&= \int_{t>0} \mathbb{P}^x\left[W(t) \in \mathrm{d}y\right] \mathbb{P}[Y_\mu \le t] \mathbb{P}^x\left[\tau(D) \in \mathrm{d}t\right] \\
&= \int_{t>0} \mathbb{P}^0\left[W(1) \in \frac{\mathrm{d}y - x}{\sqrt{t}}\right] \gamma_\mu(t) \mathbb{P}^x\left[\tau(D) \in \mathrm{d}t\right] \\
&= \left(\int_{t>0} \phi\left(\frac{x-y}{\sqrt{t}}\right) h^x(D;\mathrm{d}t)\, \gamma_\mu(t)\right) \mathrm{d}y.
\end{aligned}
$$

Panharmonic–Harmonic Equivalence (6/7)

Proof continues.

The second formula is now clear

The third formula is pretty easy to see. Indeed, as

$$H^x(D; \mathrm{d}y) = \left(\int_{t>0} \phi\left(\frac{x-y}{\sqrt{t}}\right) h^x(D; \mathrm{d}t) \right) \mathrm{d}y,$$

we see that the measures $H^x(D; \cdot)$ admit densities with respect to the Lebesgue measure. Hence the third formula is simply the ratio of these densities.

Finally, the first formula is simply the chain rule for Radon–Nikodym derivatives. □

Panharmonic–Harmonic Equivalence (7/7)

Corollary (Panequivalence)

Let $0 \le \nu^2 \le \mu^2 < \infty$. *Let* D *be a regular domain and let* u_μ *by* μ*-panharmonic and* u_ν *be* ν*-panharmonic function on* D. *Then*

1. *if* $u_\nu \le u_\mu$ *on* ∂D *then* $u_\mu \le u_\nu$ *on* D,
2. *if* D *is regular for some* $\mu^2 \ge 0$ *then it is regular for all* $\mu^2 \ge 0$,
3. *if* D *is such that the Dirichlet problem can be solved for some* $\mu^2 \ge 0$ *then it can be solved for all* $\mu^2 \ge 0$.

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Thank You for listening!

Any questions, comments or (constructive) critizism?