Indifference pricing in illiquid markets

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Illiquidity

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- Wealth cannot be transferred quite freely in time (there is no numeraire) or between assets (transaction costs etc.).
- In the absence of a numeraire, much of trading consists of exchanging sequences of cash-flows: swaps, insurance contracts, coupon payments, dividends, ...
- Illiquidity effects make hedging costs nonlinear functions of the cash-flows.
- We describe a mathematical model for valuation of sequences of cash-flows in illiquid markets.
- Traditional risk neutral pricing formulas are recovered in perfectly liquid and complete markets.
- Ingredients: stochastic analysis and convex analysis.

Illiquidity

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Outline

1. Market model
2. Asset-liability management
3. Indifference pricing
4. Reserving
5. Duality
6. An example: Valuation of pension liabilities

Market model

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- Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space.
- Consider a market where d assets are traded at $t = 0, \ldots, T$.
- Trading costs are given by an $(\mathcal{F}_t)_{t=0}^T$-adapted sequence $S = (S_t)_{t=0}^T$ of random lower semicontinuous convex functions on $\mathbb{R}^d$ such that $S_t(0) = 0$ almost surely for every $t = 0, \ldots, T$.
- The sequence is adapted if S_t is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable.

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Example 1 (Liquid markets) *If* $s = (s_t)_{t=0}^T$ *is an* $(\mathcal{F}_t)_{t=0}^T$*-adapted* $\mathbb{R}^d$*-valued price process, then the functions*

$$S_t(x,\omega) = s_t(\omega) \cdot x$$

have the above properties.

Example 2 (Jouini and Kallal, 1995) *If* $\overline{s} = (\overline{s}_t)_{t=0}^T$ *and* $\underline{s} = (\underline{s}_t)_{t=0}^T$ *are* $(\mathcal{F}_t)_{t=0}^T$*-adapted real-valued processes with* $\underline{s} \leq \overline{s}$*, then the functions*

$$S_t(x,\omega) = \begin{cases} \overline{s}_t(\omega)x & \text{if } x \geq 0, \\ \underline{s}_t(\omega)x & \text{if } x \leq 0 \end{cases}$$

have the above properties.

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Example 3 (Çetin and Rogers, 2007) *If* $s = (s_t)_{t=0}^T$ *is an* $(\mathcal{F}_t)_{t=0}^T$*-adapted process and* φ *is a lower semicontinuous convex function on* $\mathbb{R}$ *with* $\varphi(0) = 0$*, then the functions*

$$S_t(x, \omega) = x^0 + s_t(\omega)\varphi(x^1)$$

have the above properties.

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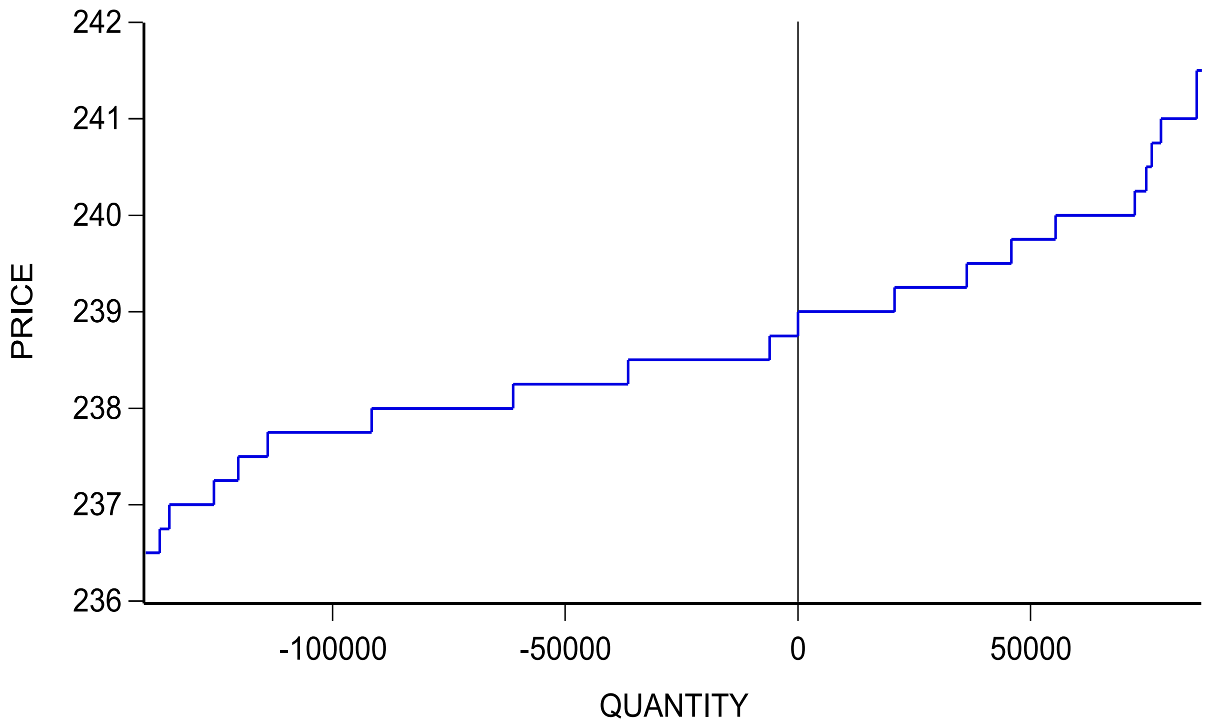
Example 4 (Limit order markets) *The cost of a market order is obtained by integrating the order book.*



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- Portfolio constraints are given by an $(\mathcal{F}_t)_{t=0}^T$-adapted sequence $D = (D_t)_{t=0}^T$ of random closed convex sets in $\mathbb{R}^d$ such that $0 \in D_t$ almost surely for every $t = 0, \dots, T$.
- The sequence is adapted if

$$\{\omega \in \Omega \mid D_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}_t$$

for every open $U \subset \mathbb{R}^d$.

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- Models where $D_t(\omega)$ is independent of (t,ω) have been studied e.g. in [Cvitanić and Karatzas, 1992] and [Jouini and Kallal, 1995].
- In [Napp, 2003],

$$D_t(\omega) = \{x \in \mathbb{R}^d \mid M_t(\omega)x \in K\},$$

where $K \subset \mathbb{R}^L$ is a closed convex cone and M_t is an $\mathcal{F}_t$-measurable matrix.
- General constraints have been studied in [Evstigneev, Schürger and Taksar, 2004] and [Rokhlin, 2005].

Asset-liability management

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Let $c \in \mathcal{M} := \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P)\}$ and consider the problem

$$\text{minimize} \quad E\sum_{t=0}^T v_t(S_t(\Delta x_t) + c_t) \quad \text{over} \quad x \in \mathcal{N}_D$$

- $v_t(\cdot, \omega)$ are convex and nondecreasing with $v_t(0, \omega) = 0$,
- $\mathcal{N}_D$ is the space of $(\mathcal{F}_t)_{t=0}^T$-adapted $\mathbb{R}^d$-valued portfolio processes with $x_{-1} := 0$, $x_t \in D_t$ and $x_T = 0$.

Example 5 *When $v_t = \delta_{\mathbb{R}_-}$ for $t < T$, the problem can be written as*

$$\begin{aligned} &\text{minimize} && Ev_T(S_T(\Delta x_T) + c_T) \quad \text{over} \quad x \in \mathcal{N}_D \\ &\text{subject to} && S_t(\Delta x_t) + c_t \le 0, \quad t = 0, \dots, T-1. \end{aligned}$$

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Example 6 *When*

$$S_t(x,\omega) = x^0 + \tilde{S}_t(\tilde{x},\omega) \quad \textit{and} \quad D_t(\omega) = \mathbb{R} \times \tilde{D}_t(\omega),$$

the problem can be written as

$$\text{minimize} \qquad Ev_T\left(\sum_{t=0}^{T} \tilde{S}_t(\Delta \tilde{x}_t) + \sum_{t=0}^{T} c_t\right) \quad \text{over} \quad x \in \mathcal{N}_D.$$

When $\tilde{S}_t(\tilde{x},\omega) = \tilde{s}_t(\omega) \cdot \tilde{x}$,

$$\sum_{t=0}^{T} \tilde{S}_t(\Delta \tilde{x}_t) = \sum_{t=0}^{T} \tilde{s}_t \cdot \Delta \tilde{x}_t = -\sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}.$$

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We will denote the optimal value by $\varphi(c)$.

- When $v_t = \delta_{\mathbb{R}_-}$ for $t = 0, \dots, T$, we have $\varphi = \delta_{\mathcal{C}}$ where

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \; S_t(\Delta x_t) + c_t \leq 0 \quad \forall t\}.$$

  is the set of claims that can be superhedged for free.
- Conversely,

$$\varphi(c) = \inf\{E \sum_{t=0}^{T} v_t(c_t - d_t) \,|\, d \in \mathcal{C}\}.$$

- If S_t are positively homogeneous and D_t are conical, then $\mathcal{C}$ is a cone.

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Lemma 7 *The value function φ is convex and*

$$\varphi(\bar{c}+c) \leq \varphi(\bar{c}) \quad \forall \bar{c} \in \mathcal{M},\ c \in \mathcal{C}^\infty.$$

where $\mathcal{C}^\infty = \{c \in \mathcal{M} \,|\, \bar{c}+\alpha c \in \mathcal{C} \quad \forall \bar{c} \in \mathcal{C},\ \forall \alpha > 0\}$*.*

- If $\mathcal{C}$ is a cone, then $\mathcal{C}^\infty = \mathcal{C}$.
- We will say that a claim $c \in \mathcal{M}$ is redundant if

$$c \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty),$$

i.e. if $\bar{c}+\alpha c \in \mathcal{C}$ for every $\bar{c} \in \mathcal{C}$ and every $\alpha \in \mathbb{R}$.

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Example 8 (Linear models) *When* $S_t(x) = s_t \cdot x$ *and* $D_t = \mathbb{R}^d$, *a claim* $c \in \mathcal{M}$ *is redundant if there is an* $x \in \mathcal{N}_D$ *such that* $s_t \cdot \Delta x_t + c_t = 0$. *The converse holds under the no-arbitrage condition* $\mathcal{C} \cap \mathcal{M}_+ = \{0\}$.

Example 9 (The classical model) *Assume that* $D_t = \mathbb{R}^d$ *and* $S_t(x) = x_0 + \tilde{s}_t \cdot \tilde{x}$. *A claim* $c \in \mathcal{M}$ *is redundant if* $\sum_{t=0}^T c_t$ *is "attainable at price* 0*" in the sense that*

$$\sum_{t=0}^{T} c_t = \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}$$

for some $x \in \mathcal{N}_D$. *The converse holds under the no-arbitrage condition.*

Indifference pricing

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- A typical situation: an agent receives a sequence $p \in \mathcal{M}$ of premiums in exchange for a sequence $c \in \mathcal{M}$ of claims.
- Examples: swaps, insurance contracts, bonds ...
- Traditionally in mathematical finance,

$$p = (1, 0, \ldots, 0) \quad \text{and} \quad c = (0, \ldots, 0, c_T).$$

- Claims and premiums live in the same space

$$\mathcal{M} = \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R})\}.$$

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- For an agent with liabilities $\bar{c} \in \mathcal{M}$,

$$\pi(\bar{c}; c) := \inf\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + c - \alpha p) \le \varphi(\bar{c})\}$$

  gives the least swap rate that would allow him to enter a swap contract without worsening his risk-return profile.

- Similarly,

$$\pi^b(\bar{c}; c) := \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} - c + \alpha p) \le \varphi(\bar{c})\} = -\pi(\bar{c}; -c)$$

  gives the greatest swap rate he would require for taking the opposite side of the trade.

- This is similar to [Hodges and Neuberger, 1989], where $p = (1, 0, \dots, 0)$, $c = (0, \dots, 0, c_T)$ and $\bar{c} = 0$.

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- Indifference prices can be bounded by

$$\pi_{\sup}(c) = \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}^\infty\},$$
$$\pi_{\inf}(c) = \sup\{\alpha \,|\, \alpha p - c \in \mathcal{C}^\infty\}.$$

- We say that a claim $c \in \mathcal{M}$ is replicable if $c - \alpha p$ is redundant (i.e. belongs to $\in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$) for some $\alpha \in \mathbb{R}$.

Example 10 *In liquid markets with a numeraire and* $p = (1, 0, \ldots, 0)$, *the functions* $\pi_{\sup}$ *and* $\pi_{\inf}$ *coincide with the usual arbitrage bounds (super- and subhedging costs) and a claim* $c \in \mathcal{M}$ *is replicable if* $\sum_{t=0}^T c_t$ *is "attainable". The converse holds under the no-arbitrage condition.*

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Theorem 11 *The function $\pi(\bar{c};\cdot)$ is convex and*

$$\pi(\bar{c}; c + c') \leq \pi(\bar{c}; c) \quad \forall c \in \mathcal{M},\ \forall c' \in \mathcal{C}^\infty.$$

If $\pi(\bar{c}; 0) \geq 0$, then

$$\pi_{\inf}(c) \leq \pi_b(\bar{c}; c) \leq \pi(\bar{c}; c) \leq \pi_{\sup}(c)$$

with equalities throughout if c is replicable.

- Agents with identical views P, preferences v and financial position $\bar{c}$ do not trade with each other.
- Prices of replicable claims are independent of P, v and $\bar{c}$.

Reserving

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- Pricing: What is the least premium we can sell a financial product for without worsening our risk-return profile?
- Reserving/economic capital: What is the least amount of capital needed to cover given liabilities at an acceptable level of risk?
- The latter is an important notion in accounting, financial reporting and supervision of financial institutions.
- Unlike indifference swap rates, the economic capital does not depend on a company's assets.
- It turns out that, in complete markets, economic capital and indifference prices coincide.

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- We define the economic capital for liabilities $c \in \mathcal{M}$ by

$$\pi_0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \leq 0\}$$

where $p^0 = (1, 0, \dots, 0)$.

- The function π_0 may be interpreted much like a risk measure in [Artzner, Delbaen, Eber and Heath, 1999].
- However, we do not assume the existence of a numeraire so π_0 operates on sequences of cash flows and it does not have the "cash invariance" property often required of risk measures.

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Theorem 12 *The economic capital π_0 is convex and nondecreasing with respect to $\mathcal{C}^\infty$. We have $\pi_0 \leq \pi_{\sup}$ and if $\pi_0(0) \geq 0$, then*

$$\pi_{\inf}(c) \leq \pi_0(c) \leq \pi_{\sup}(c)$$

with equalities throughout for replicable c.

- In general, reserves depend on the views P and the risk preferences v.
- In complete markets, however, reserves are independent of P (up to null sets) and v and they coincide with indifference prices.

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- When the claims $c = (c_t)_{t=1}^T$ are deterministic and all wealth is invested in riskless bonds, we get

$$\pi_0(c) = \sum_{t=1}^{T} P_t c_t,$$

where P_t is the price of zero-coupon bond with maturity t.

- If c_t are the expectations of the future claims, this becomes the "best estimate" in Article 77.2 of Solvency II.
- However, riskless yield curves are meant for valuation of deterministic cash-flows, not uncertain ones.
- For example, the "best estimate" of a European call-option is much higher than its market (or Black-Scholes) value.
- The "best estimate" is inherently procyclical.

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- Bond prices can be expressed in terms of zero curves.
- In models with a cash account, the price of a random claim can be expressed in terms of martingale measures.
- Under proportional transaction costs, prices can be expressed in terms of the same dual variables that characterize the no-arbitrage condition.

In illiquid markets,

- we need richer dual objects that encompass both the time value of money and randomness.
- traditional risk neutral pricing formulas are recovered in liquid markets with a numeraire.
- arbitrage has little to do with pricing and the corresponding dual variables.

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Duality

- Let $\mathcal{M}^p = \{c \in \mathcal{M} \,|\, c_t \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R})\}$.
- The bilinear form

$$\langle c, y \rangle := E \sum_{t=0}^{T} c_t y_t$$

puts $\mathcal{M}^1$ and $\mathcal{M}^\infty$ in separating duality.

- Given a convex function f on $\mathcal{M}^1$, its conjugate is defined

$$f^*(y) = \sup_{c \in \mathcal{M}^1} \{\langle c, y \rangle - f(c)\}.$$

- If f is proper and lower semicontinuous, then

$$f(y) = \sup_{y \in \mathcal{M}^\infty} \{\langle c, y \rangle - f^*(y)\}.$$

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Theorem 13 *If v_t are bounded from below by an integrable function and if*

$$\{x \in \mathcal{N}_{D^\infty} \mid S_t^\infty(\Delta x_t) \le 0\}$$

is a linear space, then φ is proper and lower semicontinuous and the infimum is attained for every $c \in \mathcal{M}^1$.

Example 14 *In the classical perfectly liquid market model, the linearity condition is the no-arbitrage condition. We then recover the fundamental lemma from [Schachermayer, 1992].*

Example 15 *The linearity condition holds if there exists a componentwise strictly positive market price process and if infinite short selling is prohibited.*

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Lemma 16 *The conjugate of the value function φ can be expressed as*

$$\varphi^*(y) = E\sum_{t=0}^{T} v_t^*(y_t) + \sigma_{\mathcal{C}}(y),$$

where $\sigma_{\mathcal{C}}(y) = \sup_{c\in\mathcal{C}}\langle c, y\rangle$.

With little work, the above results yield illiquid extensions

- duality frameworks for utility maximization and optimal consumption
- martingale representations of arbitrage bounds and indifference prices,
- fundamental theorem of asset pricing.

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The following example is from

Hilli, Kivu and Pennanen, Cash-flow based valuation of pension liabilities, European Actuarial Journal, 2011.

- The aim is to calculate reserves for the pension insurance portfolio of the Finnish private sector occupational pension system.
- The yearly claims c_t consist of aggregate old age, disability and unemployment pension benefits earned by the end of 2008.
- The claims depend on mortality and the price- and wage-inflation, etc.

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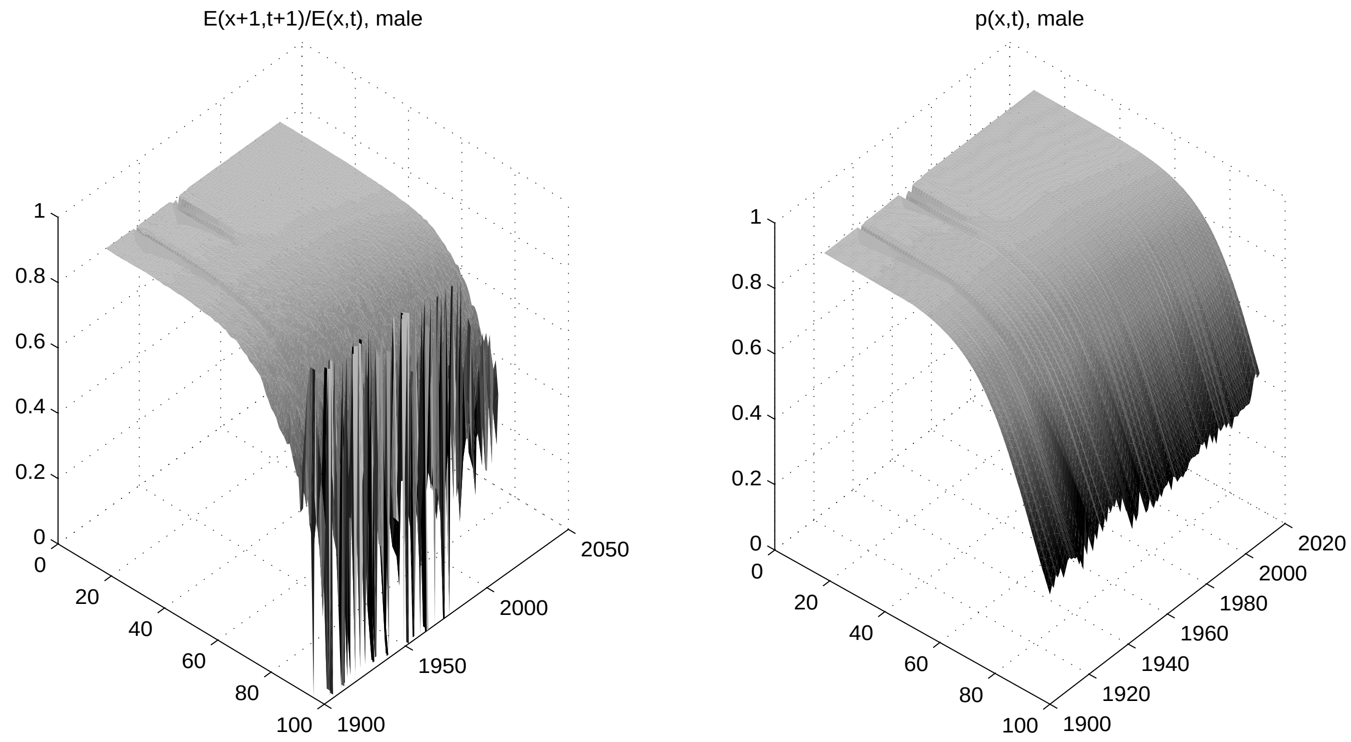

Figure 1: Survival rates of Finnish males

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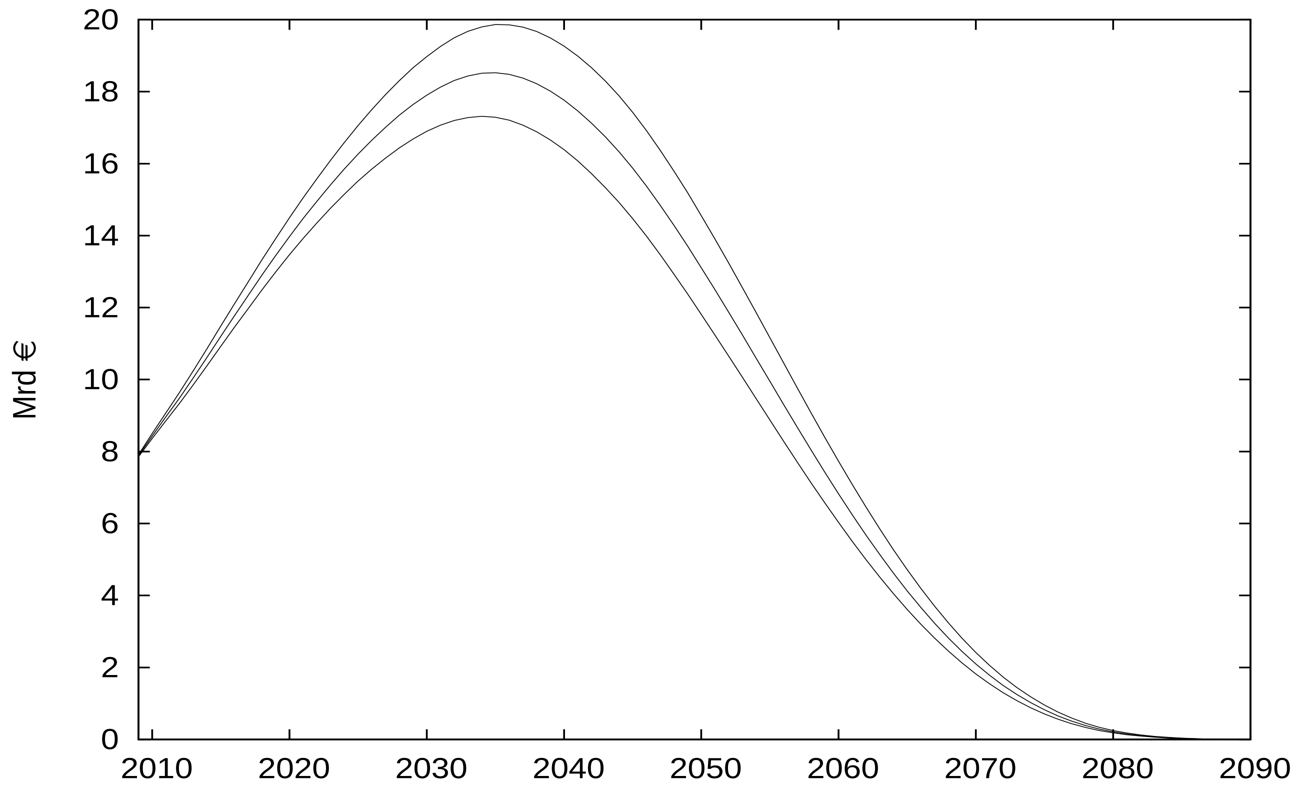
Figure 2: Yearly claims



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- The traded assets consist of five equity indices and two bond indices.
- Yearly bond returns are modeled by

$$R_t = \exp(Y_t \Delta t - D \Delta Y_t),$$

where Y is the yield to maturity and D the duration.

- Market risk factors are modeled together with the liability risk factors (mortality, price- and wage-inflation) by a stochastic difference equation of the form

$$\Delta \xi_t = A \xi_{t-1} + b + \varepsilon_t,$$

where ξ is the vector of (transformed) risk factors.

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- We find the minimum reserve

$$\pi_0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \le 0\}$$

  by line search and numerical optimization.
- For given α, the optimum value $\varphi(c - \alpha p^0)$ of the ALM-problem is approximated by the Galerkin method.
- The Galerkin method optimizes over convex combinations of a given set of "basis strategies".
- In this study, we used buy and hold, fixed proportion and constant proportion portfolio insurance rules with varying parameters.

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	Confidence level				
	95%	90%	85%	80%	66%
Best basis	296	284	273	261	239
Optimized	288	271	254	236	202

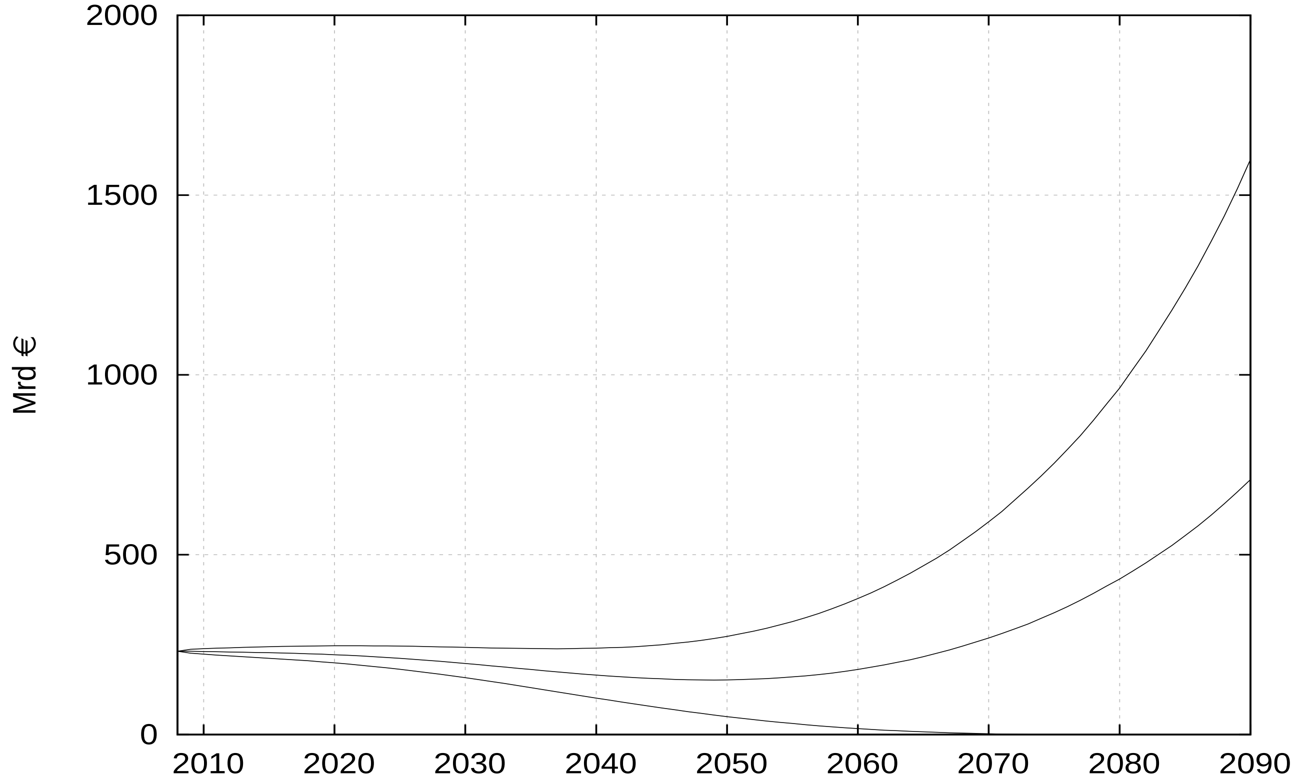
Table 1: Liability values with varying risk tolerances

	Confidence level				
	95%	90%	85%	80%	66%
Best basis	24.3	25.4	26.4	27.6	30.1
Optimized	25.0	26.6	28.3	30.5	35.6

Table 2: Corresponding funding ratios

An example

Figure 3: The development of 34%, 50%- and 66%-quantiles of net wealth when $\pi_0(c)$ is defined with $\mathcal{V} = V@R_{66\%}$.



Summary

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- Reserving and pricing both come down to asset-liability management.
- Much of classical asset pricing theory can be extended to convex models of illiquid markets.
- The adequacy of reserves, prices and investment strategies is subjective.
- When there is no numeraire, the timing of payments matters.
- Dual representations involve stochastic term structures that capture uncertainty as well as time value of money.