On critical Gaussian multiplicative chaos

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Multifractality

In this presentation we consider random measures μ on the unit interval.

A random measure is said to exhibit multifractal scaling if there exists a nonlinear function $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}\mu(I)^q = |I|^{\xi(q)}\mathbb{E}\mu([0,1])^q$$

for some class of intervals $I \subset [0,1]$ and range of $q \in \mathbb{R}$.

Random multifractal measures turn up e.g. in mathematical physics and in many different modelling contexts.

Mandelbrot cascades

Define the *symbolic space* by $\Sigma_n = \{0,1\}^n, \quad \Sigma = \bigcup_{n=0}^{\infty} \Sigma_n$.

Let $\{X_\sigma\}_{\sigma \in \Sigma}$ an i.i.d. family of random variables. For simplicity we assume that the common distribution of the X_σ is $N(0,1)$.

Definition (Mandelbrot cascade measures)

Let $\beta > 0$ be a parameter. For $n \in \mathbb{N}$, define the random measure $\mu_{\beta,n}$ by

$$(\mu_{\beta,n}(I_\sigma))_{\sigma \in \Sigma_n} = \left(2^{-n} \prod_{k=1}^{n} e^{\beta \sum_{k=0}^{n} X_{\sigma_1 \dots \sigma_k} - \frac{\beta^2}{2}(n+1)} \right)_{\sigma \in \Sigma_n} .$$

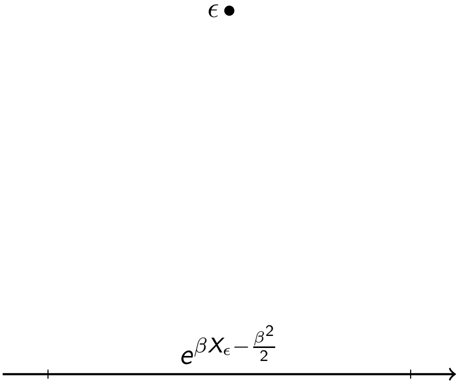
For any interval $I \subset [0,1]$, the sequence $(\mu_{\beta,n}(I))_n$ is a positive martingale. It follows from the martingale convergence theorem that almost surely

$$\mu_{\beta,n} \to \mu_\beta \quad \text{weakly.}$$

The limit measure μ_β is called the Mandelbrot cascade measure.

Mandelbrot cascades

$$\left(\mu_{\beta,n}(I_\sigma)\right)_{\sigma\in\Sigma_n} = \left(2^{-n}\prod_{k=1}^{n} e^{\beta\sum_{k=0}^{n} X_{\sigma_1\ldots\sigma_k}-\frac{\beta^2}{2}(n+1)}\right)_{\sigma\in\Sigma_n}$$



Mandelbrot cascades

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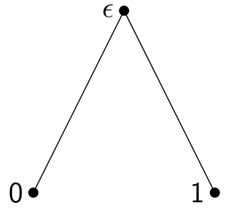

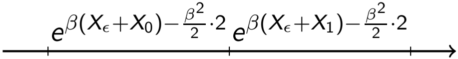

Mandelbrot cascades

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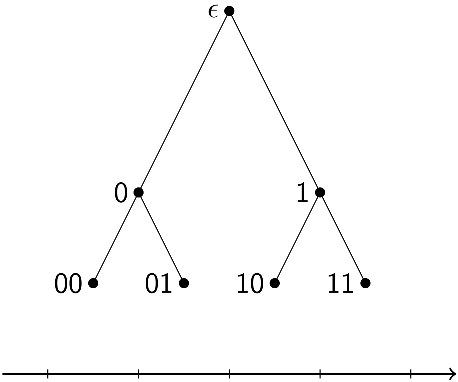

Mandelbrot cascades

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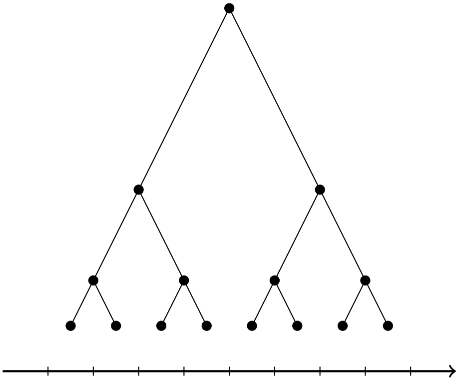

Mandelbrot cascades

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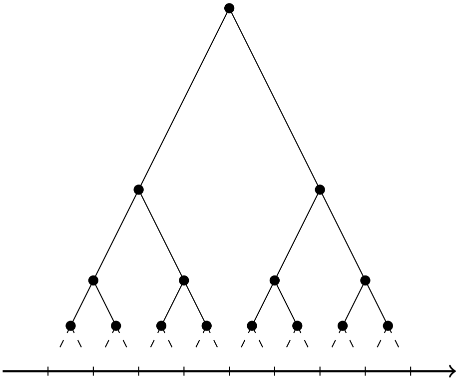


Properties of Mandelbrot cascade measures

Exact scaling

For any $\sigma \in \Sigma_n$

$$\mu_\beta(I_\sigma) \stackrel{d}{=} 2^{-n} e^{\beta(X_\epsilon + X_{\sigma_1} + \cdots + X_{\sigma_1 \ldots \sigma_{n-1}}) - \frac{\beta^2}{2} n} \tilde{\mu}_\beta([0,1]),$$

where $\tilde{\mu}_\beta \stackrel{d}{=} \mu_\beta$ is independent of the X.

Theorem (Kahane, Peyriere '76)

1. *Let* $0 < \beta^2 < 2 \log 2$. *Then almost surely* $\mu_\beta(I) > 0$ *for all intervals* $I \subset [0,1]$. *Moreover,*

$$\mathbb{E}\mu_\beta([0,1])^p < \infty \quad \textit{if and only if } p < \frac{2 \log 2}{\beta^2}.$$

2. *Let* $\beta^2 \geq 2 \log 2$. *Then almost surely the measure* μ_β *is null.*

Critical Mandelbrot cascade measure

Theorem (Aidekon, Shi '11; Webb '11; Barral, Rhodes, Vargas '12)

Let $\beta^2 = 2\log 2$. *Then there exists* μ_β *such that*

$$n^{\frac{1}{2}}\mu_{\beta,n} \to \mu_\beta \quad \textit{weakly in probability.}$$

The measure $\mu_{\sqrt{2\log 2}}$ has the exact scaling property.

Theorem (Buraczewski '09)

There exists a constant $c > 0$ *such that*

$$\mathbb{P}(\mu_{\sqrt{2\log 2}}([0,1]) > \lambda) \sim \frac{c}{\lambda} \quad \textit{as } \lambda \to \infty.$$

Properties of the critical Mandelbrot measure

Theorem (BKNSW '12)

Let $0 < \gamma < \frac{1}{2}$. Almost surely, there exists a number $C(\omega) < \infty$ such that

$$\mu_{\sqrt{2\log 2}}(I) \leq C(\omega)\left(\log\left(1 + |I|^{-1}\right)\right)^{-\gamma}$$

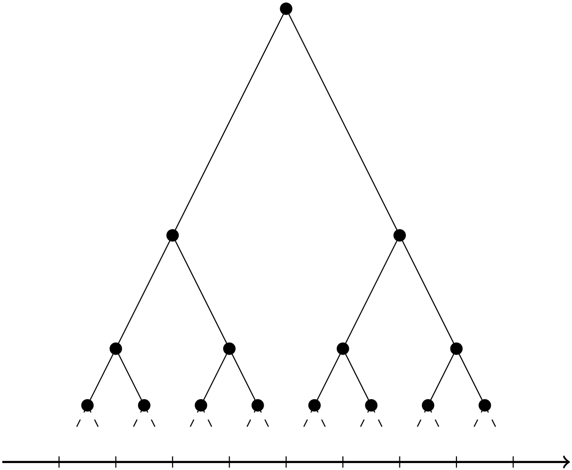
for all intervals $I \subset [0,1]$. Moreover, one cannot take $\gamma > \frac{1}{2}$ above.

Theorem (BKNSW '12)

Almost surely, the multifractal spectrum of $\mu_{\sqrt{2\log 2}}$ is given by

$$\begin{aligned} f_{\mu_{\sqrt{2\log 2}}}(\gamma) \quad &:= \quad \dim\left\{x \,\middle|\, \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log 2^{-n}} = \gamma\right\} \\ &= \quad \begin{cases} \gamma\left(1 - \frac{\gamma}{4}\right), & \text{for } 0 \leq \gamma \leq 4 \\ 0, & \text{for } \gamma < 0 \text{ and } \gamma > 4 \end{cases}. \end{aligned}$$

A translation invariant construction



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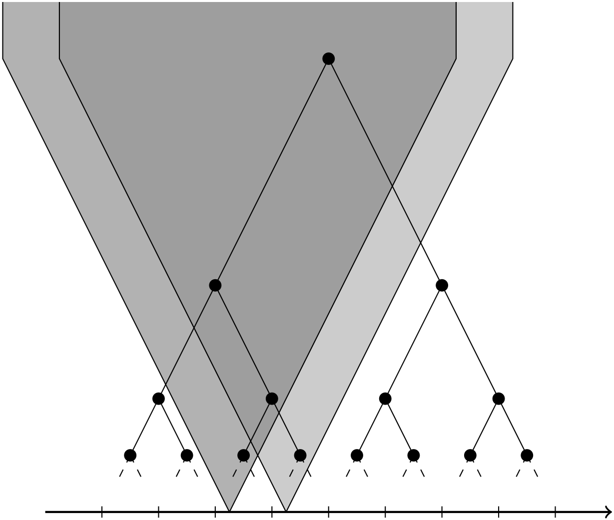

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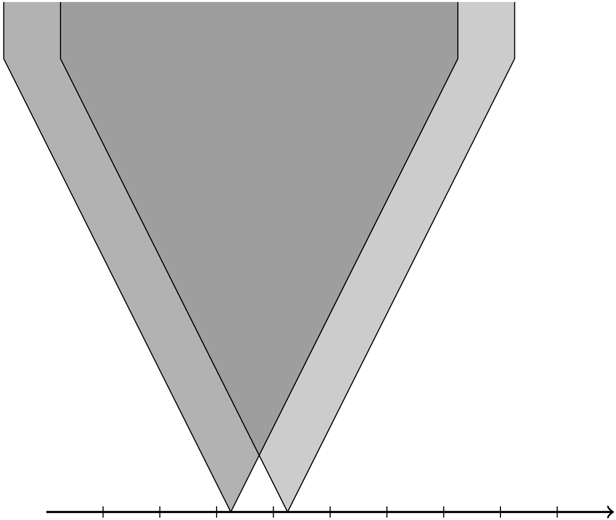


A log-correlated Gaussian field

Let λ be the hyperbolic area measure on $\mathbb{R} \times \mathbb{R}_+$:

$$\lambda(A) = \int_A \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

For $x \in [0,1]$ and $t \in \mathbb{R}_+$, define

$$\mathcal{C}_t(x) = \left\{(x', y') \,\middle|\, y' > \max(2|x'-x|, e^{-t}), |x'-x| < \frac{1}{2}\right\}.$$

Definition

Let W denote the white noise on $\mathbb{R} \times \mathbb{R}_+$ with control measure λ (i.e. $\mathbb{E}W(A)^2 = \lambda(A)$ for $A \subset \mathbb{R} \times \mathbb{R}_+$) and define the field X by

$$X_t(x) = W(\mathcal{C}_t(x)) \quad x \in [0,1], t \in \mathbb{R}_+.$$

For intervals $I \subset [0,1]$ we further denote

$$\mathcal{C}_t(I) = \bigcap_{x \in I} \mathcal{C}_t(x) \quad \text{and} \quad X_t(I) = W(\mathcal{C}_t(I)).$$

Multiplicative chaos

Definition

Let $\beta > 0$ be a parameter. For all $t \in \mathbb{R}_+$ we define the measure $\mu_{\beta,t}$ by

$$\mu_{\beta,t}(I) = \int_I e^{\beta X_t(x) - \frac{\beta^2}{2}\mathbb{E}X_t(x)^2}\,\mathrm{d}x$$

for all intervals $I \subset [0,1]$.

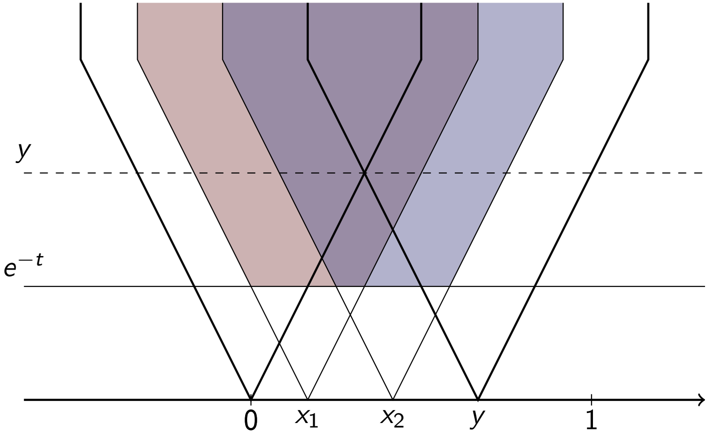
For any interval $I \subset [0,1]$, the process $(\mu_{\beta,t}(I))_t$ is a positive martingale. It follows from the martingale convergence theorem that almost surely

$$\mu_{\beta,t} \to \mu_\beta \quad \text{weakly as } t \to \infty.$$

The limit measure μ_β is called the multiplicative chaos measure.

Exact scaling of $(X_t(x))_{x\in I}$

$$(X_t(x))_{x\in I} \overset{d}{=} \Big(X(I) + \tilde{X}_{t-\log 1/|I|}(x/|I|)\Big)_{x\in I}, \quad \text{where } \tilde{X} \perp X(I), \tilde{X} \overset{d}{=} X$$



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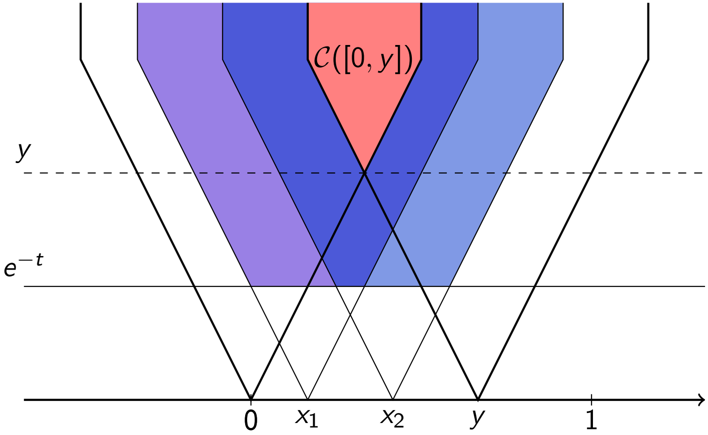

Exact scaling of $(X_t(x))_{x\in I}$

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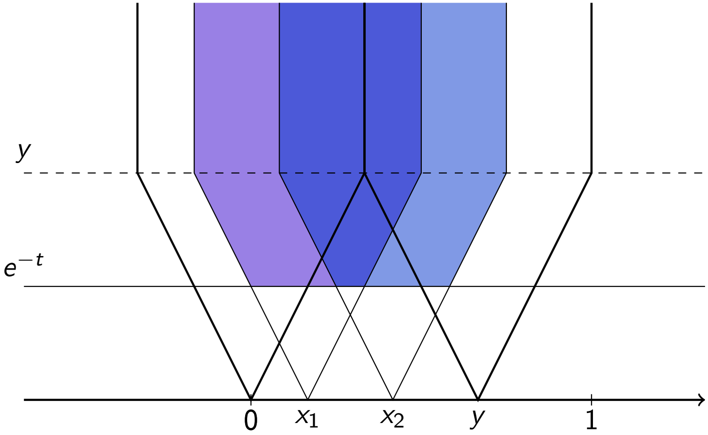


Exact scaling of μ_β

Observation

For all intervals $I \subset [0,1]$,

$$\mu_\beta(I) \overset{d}{=} |I|\, e^{\beta X(I) - \frac{\beta^2}{2}\mathbb{E}X(I)^2}\, \tilde{\mu}_\beta([0,1]), \quad \textit{where } \tilde{\mu}_\beta \perp X(I),\ \tilde{\mu}_\beta \overset{d}{=} \mu.$$

Proof.

By the exact scale invariance property of the field X,

$$\begin{aligned} \mu_\beta(I) &= \lim_{t\to\infty} \int_I e^{\beta X_t(x) - \frac{\beta^2}{2}\mathbb{E}X_t(x)^2}\, \mathrm{d}x \\ &\overset{d}{=} e^{\beta X(I) - \frac{\beta^2}{2}\mathbb{E}X(I)^2} \lim_{t\to\infty} \int_I e^{\beta \tilde{X}_{t-\log 1/|I|}(x/|I|) - \frac{\beta^2}{2}\mathbb{E}\tilde{X}_{t-\log 1/|I|}(x/|I|)^2}\, \mathrm{d}x \\ &\overset{d}{=} |I|\, e^{\beta X(I) - \frac{\beta^2}{2}\mathbb{E}X(I)^2}\, \tilde{\mu}_\beta([0,1]). \end{aligned}$$

□

Nondegeneracy of multiplicative chaos

Theorem (Kahane '85)

1. *Let* $0 < \beta^2 < 2$*. Then almost surely* $\mu_\beta(I) > 0$ *for all intervals* $I \subset [0,1]$*. Moreover,*

$$\mathbb{E}\mu_\beta([0,1])^p < \infty \quad \textit{if and only if } p < \frac{2}{\beta^2}.$$

2. *Let* $\beta^2 \geq 2$*. Then almost surely the measure* μ_β *is null.*

Structure of μ_β, $\beta < \sqrt{2}$

Multiplicative chaos measures are in many respects similar to Mandelbrot cascades.

The main difference in the analysis of chaos and cascade measures is that there are many more dependences between the variables in the exact scaling relations

$$\left(\mu_\beta(I_\sigma)\right)_{\sigma\in\Sigma_n} \overset{d}{=} \left(|I_\sigma|\, e^{\beta X(I_\sigma) - \frac{\beta^2}{2}\mathbb{E}X(I_\sigma)^2}\, \tilde{\mu}_\beta^{(\sigma)}([0,1])\right)_{\sigma\in\Sigma_n}$$

for the chaos measures than for the cascade measures.

Derivative martingale

The following object is one of the keys to understanding the critical case $\beta = \sqrt{2}$.

Definition

Define the measures μ_t' on the unit interval by

$$\mu_t'(I) = \frac{d}{d\beta}\bigg|_{\beta=\sqrt{2}} \mu_{\beta,t}(I) = \int_I \left(\sqrt{2}t - X_t(x)\right) e^{\sqrt{2}X_t(x) - \mathbb{E}X_t(x)^2}\,\mathrm{d}x.$$

The process $(\mu_t([0,1]))_t$ is called the derivative martingale.

Convergence of the derivative martingale

Theorem (Duplantier, Rhodes, Sheffield, Vargas '12)

As $t \to \infty$, almost surely

$$\mu'_t \to \mu' \quad \textit{weakly}$$

for a positive random measure μ'.

Properties of μ'

1. Almost surely $\mu'(I) > 0$ for all intervals $I \subset [0, 1]$.
2. μ' is exactly scale invariant for the parameter value $\beta = \sqrt{2}$.
3. $\mathbb{E}\mu'(I) = \infty$ for any interval $I \subset [0, 1]$.

Exact scale invariance

Proof of the exact scale invariance of μ'.

Denote $X_t^I(x) = \tilde{X}_{t-\log 1/|I|}(x/|I|)$. Then

$$
\begin{aligned}
\mu'(I) &= \lim_{t\to\infty} \int_I \left(\sqrt{2}t - X_t(x)\right) e^{\sqrt{2}X_t(x) - \mathbb{E}X_t(x)^2}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \int_I \left(\sqrt{2}\mathbb{E}X(I)^2 - X(I)\right) e^{\sqrt{2}X_t(x) - \mathbb{E}X_t(x)^2}\,\mathrm{d}x \\
&\quad + \lim_{t\to\infty} \int_I \left(\sqrt{2}(t - \mathbb{E}X(I)^2) - X_t^I(x)\right) e^{\sqrt{2}X_t(x) - \mathbb{E}X_t(x)^2}\,\mathrm{d}x \\
&= e^{\sqrt{2}X(I) - \mathbb{E}X(I)^2} \lim_{t\to\infty} \int_I \left(\sqrt{2}(t - \mathbb{E}X(I)^2) - X_t^I(x)\right) \\
&\qquad\qquad \cdot e^{\sqrt{2}X_t^I(x) - \mathbb{E}X_t^I(x)^2}\,\mathrm{d}x \\
&\overset{d}{=} |I| e^{\sqrt{2}X(I) - \mathbb{E}X(I)^2} \tilde{\mu}'([0,1]).
\end{aligned}
$$

□

Deterministically normalized critical measure

Theorem (Duplantier, Rhodes, Sheffield, Vargas '12)

There exists a deterministic constant $c > 0$ such that

$$\frac{t^{\frac{1}{2}}\mu_{\sqrt{2},t}([0,1])}{\mu'([0,1])} \to c \quad \textit{in probability.}$$

Using this deterministic normalization and a comparison to the cascade case using Kahane's convexity inequality gives the following result on moments.

Corollary (Duplantier, Rhodes, Sheffield, Vargas '12)

$\mathbb{E}\mu'([0,1])^h < \infty$ *for all* $h \in (0,1)$.

Structure of μ'

Theorem (BKNSW '12)

There exists a constant $c > 0$ such that

$$\mathbb{P}\left(\mu'([0,1]) > \lambda\right) \sim \frac{c}{\lambda} \quad as\ \lambda \to \infty.$$

Theorem (BKNSW '12)

Let $0 < \gamma < \frac{1}{2}$. Almost surely, there exists a number $C(\omega) < \infty$ such that

$$\mu'(I) \leq C(\omega)\left(\log\left(1 + |I|^{-1}\right)\right)^{-\gamma}$$

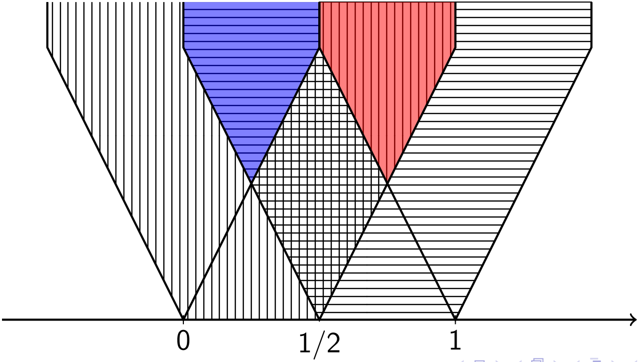
for all intervals $I \subset [0,1]$.

Theorem (BKNSW '12)

Almost surely, there exists a set E_0 of full μ'-measure that has Hausdorff dimension 0.

Characterizing the distribution of $Y = \mu'([0,1])$

$$
\begin{aligned}
Y &= \mu'([0,1/2]) + \mu'([1/2,1]) \\
&=: \frac{1}{2} e^{\sqrt{2}X([0,1/2]) - \mathbb{E}X([0,1/2])^2} Y_0 + \frac{1}{2} e^{\sqrt{2}X([1/2,1]) - \mathbb{E}X([1/2,1])^2} Y_1 \\
&=: W_0 Y_0 + W_1 Y_1, \quad \text{where } W_0 \perp Y_0,\ W_1 \perp Y_1,\ Y \overset{d}{=} Y_0 \overset{d}{=} Y_1
\end{aligned}
$$



A Poisson equation

Consider the function $F_{\alpha,\beta}(x) = \mathbb{E} Y \mathbf{1}_{Y \in]\alpha e^x, \beta e^x]}$.

Using the distributional equation $Y = W_0 Y_1 + W_1 Y_1$ one may verify that $F_{\alpha,\beta}$ satisfies the Poisson equation

$$F_{\alpha,\beta}(x) = \mathbb{E} F_{\alpha,\beta}(x + S) + \psi_{\alpha,\beta}(x),$$

where $S \sim N(0, 2\log 2)$ and the function $\psi_{\alpha,\beta}(x)$ is given by

$$\psi_{\alpha,\beta}(x) = 2\mathbb{E} W_0 Y_0 \left(\mathbf{1}_{\{\alpha e^x - W_1 Y_1 < W_0 Y_0 \leq \beta e^x - W_1 Y_1\}} - \mathbf{1}_{\{\alpha e^x < W_0 Y_0 \leq \beta e^x\}} \right).$$

The asymptotics of $F_{\alpha,\beta}$ may be determined by using Fourier analysis to study the solutions of the Poisson equation above. The asymptotics of $F_{\alpha,\beta}$ in turn allow one to determine the asymptotics of $\mathbb{P}(Y > \lambda)$.

Dimension of the carrier

Let $f : \mathbb{N} \to \mathbb{R}^+$ be a nonincreasing function tending to 0 at infinity and consider the sets

$$E_n^f = \{x : \mu'(I_n(x)) \leq f(n)\}.$$

Suppose one has

$$\sum_n \mu'(E_n^f) < \infty \quad \text{almost surely.}$$

Then the Borel-Cantelli Lemma implies that

$$\mu'(E_0) := \mu'\left(\left\{x : \mu'(I_n(x)) \geq f(n) \text{ for all but } < \infty \text{ many } n\right\}\right) = \mu'([0,1]).$$

If $f(n)$ decays slower than exponentially in n, then a simple covering argument shows that the set E_0 defined above has Hausdorff dimension 0.

Dimension of the carrier

Write

$$\mu\left(E_n^f\right) = \int_0^1 \mathbf{1}_{\{\mu(I_n(x)) \le f(n)\}} \,\mathrm{d}\mu(x) = \sum_{\sigma \in \Sigma_n} \mu(I_\sigma) \mathbf{1}_{\{\mu(I_n(x)) \le f(n)\}}$$

and consider the expectation of the last sum.

The exact scaling property of μ' and the result on the asymptotics of $\mathbb{P}(\mu'([0,1]) > \lambda)$ allow one to show that for $f(n) = \exp(-c\sqrt{n \log n})$ for sufficiently large $c > 0$ one has

$$\mathbb{E} \sum_n \mu(E_n^f) < \infty.$$

This implies the claim on the dimension of the carrier of μ'.