Parameter estimation in a fractional Ornstein-Uhlenbeck model

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talk is based on two closely related joint works with
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- What about fractional Brownian motion?

Surprisingly, the answer is NOT ! and solution of the Langevin equation with fractional Brownian motion noise and resulting process of Lamperti transformation of fractional Brownian motion leads to two different stochastic processes.

More precisely, let B denote a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Consider Langevin equation with fractional Brownian motion as noise, i.e.

$$dU_t^{(H,\xi_0)} = -\theta U_t^{(H,\xi_0)} dt + dB_t, \qquad U_0^{(H,\xi_0)} = \xi_0. \tag{1}$$

Let $\hat{B}$ denote a two sided fractional Brownian motion. The solution of the following SDE ($\xi_0 := \int_{-\infty}^{0} e^{\theta s}\, d\hat{B}_s$) can be expressed as

$$U_t^{(H)} = e^{-\theta t} \int_{-\infty}^{t} e^{\theta s}\, d\hat{B}_s. \tag{2}$$

Definition (Kaarakka & Salminen (2011))

The process $U^{(H)}$ is called stationary fractional Ornstein-Uhlenbeck process of the first kind.

Theorem (Cheridito & Kawaguchi & Maejima (2003))

The covariance function of the stationary Gaussian process $U^{(H)}$ decays like a power function, so it is ergodic and is long range dependent when $H \in (\frac{1}{2}, 1)$, and short range dependent when $H \in (0, \frac{1}{2})$.

Now, consider stationary Gaussian process $X^{(\alpha)}$ by means of Lamperti transformation of fractional Brownian motion B:

$$X_t^{(\alpha)} := e^{-\alpha t} B_{a_t}, \quad t \in R, \quad \alpha > 0,\ a_t = \frac{H}{\alpha} e^{\frac{\alpha t}{H}}.$$

Observation **I** (Kaarakka & Salminen (2011)): Set $Y_t^{(\alpha)} := \int_0^t e^{-\alpha s}\,\mathrm{d}B_{a_s}$, then the process $X^{(\alpha)}$ is the solution of the Langevin equation

$$\mathrm{d}X_t^{(\alpha)} = -\alpha X_t^{(\alpha)}\,\mathrm{d}t + \mathrm{d}Y_t^{(\alpha)}, \quad X_0^{(\alpha)} = B_{a_0} \overset{d}{=} B_{H/\alpha}.$$

Observation **II** (Kaarakka & Salminen(2011)):

$$\{Y_{t/\alpha}^{(\alpha)}\}_{t\geq 0} \overset{\text{law}}{=} \{\alpha^{-H} Y_t^{(1)}\}_{t\geq 0}.$$

Inspired by the scaling property of process $Y^{(1)}$, consider the following Langevin equation with $Y^{(1)}$ as the driving noise:

$$dX_t = -\theta X_t dt + dY_t^{(1)}, \quad dY_t^{(1)} = e^{-t} dB_{a_t}, \ \theta > 0.$$

The solution of the SDE (with initial value $X_0 = \int_{-\infty}^0 e^{(\theta-1)s} \,\mathrm{d}B_{a_s}$) is given by

$$U_t = e^{-\theta t} \int_{-\infty}^{t} e^{(\theta-1)s} \,\mathrm{d}B_{a_s}, \quad \text{and } a_t = He^{\frac{t}{H}}.$$

Definition

The process U is called stationary fractional Ornstein-Uhlenbeck process of the second kind.

Proposition (Cheridito & Kawaguchi & Maejima (2003) and Kaarakka & Salminen (2011))

The covariance function of the stationary process U decays exponentially and has short range dependence. More precisely, when $H \in (\frac{1}{2}, 1)$, then

$$\mathbb{E}(U_t U_0) = \left(\exp\left(- \min\{\theta, \frac{1-H}{H}\}t\right)\right), \quad \text{as } t \to \infty.$$

Consider following Langevin equation with noise process Z:

$$\mathrm{d}X_t = -\theta X_t \,\mathrm{d}t + \mathrm{d}Z_t, \quad t \in [0, T].$$

- When $Z = B$ is fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, i.e. X is fractional Ornstein-Uhlenbeck process of the first kind, Kleptsyna & Le Breton (2002) obtained MLE for estimating parameter θ and prove strong consistency.
- When $Z = B$ is fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, i.e. X is fractional Ornstein-Uhlenbeck process of the first kind, Hu & Nualart (2010) suggest the following LSE

$$\widehat{\theta}_T = -\frac{\int_0^T X_t \,\delta X_t}{\int_0^T X_t^2 \,dt}, \quad \delta \text{ stands for Skorokhod integral}$$

and prove strong consistency and a CLT towards a

When $Z = Y^{(1)}$, i.e. X is fractional Ornstein-Uhlenbeck process of the second kind. If $X_0 = 0, \; H \in (\frac{1}{2}, 1)$ and $\theta > 1$, then

Theorem (Az & Morlanes (2012))

The least squares estimator $\widehat{\theta}_T$ is weakly consistent, i.e.

$$\widehat{\theta}_T = -\frac{\int_0^T X_t \, \delta X_t}{\int_0^T X_t^2 \, dt} \to \theta$$

in probability, as T tends to infinity.

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Theorem (Az & Morlanes (2012))

For the least squares estimator $\widehat{\theta}_T$, we have

$$\sqrt{T}\left(\widehat{\theta}_T - \theta\right) \overset{law}{\to} \mathcal{N}(0, \sigma^2(\theta, H)),$$

as T tends to infinity, where $\sigma^2(\theta, H) = \theta^2 C(\theta, H) > 0$.

Since there is not (at the moment) a Malliavin calculus for the driving noise $Y^{(1)}$, we do replace the noise $Y^{(1)}$ with an equivalent (in distribution) noise and do computation in a equivalent model.

- Motivated by the equality (Baudoin & Nualart 2003): for $f > 0$ and enough smooth

$$\{\int_0^t f(s)dB_s\}_{t\in[0,T]} \overset{law}{=} \{B_{\int_0^t f(s)^{\frac{1}{H}}ds}\}_{t\in[0,T]},$$

  we obtain

$$\mathrm{d}X_t = -\theta X_t \,\mathrm{d}t + \mathrm{d}Y_t^{(1)}, \qquad X_0 = 0.$$

Lemma

Let $\tilde{B}_t = B_{t+H} - B_H$ be the shifted fractional Brownian motion. Then there exists a regular Volterra-type kernel $\tilde{L}$, so that for the solution of the following stochastic differential equation

$$\mathrm{d}\tilde{X}_t = -\theta \tilde{X}_t \,\mathrm{d}t + \mathrm{d}\tilde{G}_t, \quad \tilde{X}_0 = 0,$$

we have, $\{X_t\}_{t\in[0,T]} \overset{law}{=} \{\tilde{X}_t\}_{t\in[0,T]}$ where the Gaussian process

$$\tilde{G}_t = \int_0^t \Big(K_H(t,s) + \tilde{L}(t,s) \Big) \,\mathrm{d}\tilde{W}_s \overset{law}{=} Y_t^{(1)}.$$

Using multiple Wiener integral techniques, and the fact that law of random variables in the second Wiener chaos is determined by their moments, we obtain

Lemma

For the least squares estimator $\widehat{\theta}_T$, we have

$$\begin{aligned} \widehat{\theta}_T &= \theta - \frac{\int_0^T X_t \, \delta Y_t^{(1)}}{\int_0^T X_t^2 \, dt} \\ &\overset{law}{=} \theta - \frac{\int_0^T \tilde{X}_t \, \delta \tilde{G}_t}{\int_0^T \tilde{X}_t^2 \, dt}. \end{aligned} \tag{3}$$

Using relation between Skorokhod integral and Paths-wise integral, we obtain

Lemma

The least squares estimator $\widehat{\theta}_T$ can be written as

$$\widehat{\theta}_T \overset{law}{=} -\frac{\frac{1}{2}\tilde{X}_T^2}{\int_0^T \tilde{X}_t^2\,dt} + \frac{\int_0^T \int_s^T D_s^{(\tilde{W})}\tilde{X}_t \left(K_H(dt,s) + \tilde{L}(dt,s)\right)\,ds}{\int_0^T \tilde{X}_t^2\,dt},$$

where $D^{(\tilde{W})}$ denotes the Malliavin derivative operator with respect to Brownian motion $\tilde{W}$.

Using ergodic theorem for stationary processes, we obtain

Lemma

For the processes X and $\tilde{X}$, we have

$$\frac{1}{T}\int_0^T X_t^2\,dt \to \frac{(2H-1)H^{2H}}{\theta}\,B\left((\theta-1)H+1,2H-1\right),$$

almost surely and in L^2, as T tends to infinity, where here $B(x,y)$ denotes the complete Beta function.

- Using a result on the supremum of stationary Gaussian processes by Pickands (1969), we obtain $\lim_{T\to\infty} \frac{\tilde{X}_T^2}{T} = 0$.
- With doing some computation, one can obtain

$$\int_0^T \int_s^T D_s^{(\tilde{W})} \tilde{X}_t \left(K_H(dt, s) + \tilde{L}(dt, s) \right) ds \\ \to (2H-1) H^{2H} B\left((\theta - 1)H + 1, 2H - 1\right).$$

- End of the proof.

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- End of the proof.

Remark

Note that if one can replace Skorokhod integral with path-wise Riemann-Stieltjes integral in the formula of least squares estimator, then we can obtain the new estimator

$$\widehat{\theta'_T} := -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = -\frac{X_T^2}{2\int_0^T X_t^2 dt} \to 0 \quad a.s.$$

Proposition (Nualart & Ortiz-Laorre (2008))

Let $\{F_n\}_{n\geq 1}$ be a sequence of random variables in the q-th Wiener chaos, $q \geq 2$, such that $\lim_{n\to\infty} \mathbb{E}(F_n^2) = \sigma^2$. Then the following statements are equivalent:

- *F_n converges in distribution to $\mathcal{N}(0, \sigma^2)$ as n tends to infinity.*
- *$\|DF_n\|^2_{\mathcal{H}}$ converges in $L^2(\Omega)$ to $q\sigma^2$ as n tends to infinity.*

- Taking into account that

$$\sqrt{T}\left(\widehat{\theta}_T - \theta\right) \overset{law}{=} -\frac{\sqrt{T}\, I_2^{\tilde{G}}\left(\frac{1}{2}e^{-\theta|t-s|}\right)}{\int_0^T \tilde{X}_t^2\, dt} = -\frac{F_T}{\frac{1}{T}\int_0^T \tilde{X}_t^2\, dt},$$

where F_T stands for the double stochastic integral

$$F_T = \frac{1}{\sqrt{T}} I_2^{\tilde{G}}\left(\frac{1}{2}e^{-\theta|t-s|}\right),$$

and applying the above proposition, we obtain the CLT.

- Motivation comes from the following result.

$$\frac{1}{T}\int_0^T X_t^2\,dt \overset{a.s.}{\to} \Psi(\theta) := \frac{(2H-1)H^{2H}}{\theta} B\left((\theta-1)H+1, 2H-1\right).$$

Lemma (Az & Viitasaari (2012))

As a function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Psi(\theta)$ is bijective, and hence invertible.

Remark

Note that (up to our knowledge) there exists not an explicit formula for the inverse. Hence, the inverse must be computed numerically.

We start with the following CLT.

Theorem (Az & Viitasaari (2012))

For the fractional Ornstein-Uhlenbeck process of the second kind X, $(X_0 = 0)$, we have

$$\sqrt{T}\left(\int_0^T X_u^2 du - \Psi(\theta)\right) \stackrel{law}{\to} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = \sigma^2(H, \theta) > 0$.

The shape of the variance σ^2 is rather complicated and is given by a triplet integral.

Assume we observe X_t at discrete points $\{t_1 < t_2 < \cdots < t_k, k = 1, \ldots, N\}$. Put $T_N = N\Delta_N$ and

$$\hat{\mu}_{2,N} = \frac{1}{T_N}\sum_{k=1}^{N} X_{t_k}^2 \Delta t_k.$$

Theorem (Az & Viitasaari (2012))

Assume we have $T_N \to \infty$ and $N\Delta_N^p \to 0$, $p > 1$. Then,

$$\hat{\theta}_N := \Psi^{-1}\left(\hat{\mu}_{2,N}\right) \to \theta$$

almost surely as N tends to infinity, where Ψ^{-1} is the inverse of function $\Psi(\theta)$. Moreover

$$\sqrt{T_N}(\hat{\theta}_N - \theta) \overset{law}{\to} \mathcal{N}(0, \sigma_\theta^2), \quad \sigma_\theta^2 = \frac{\sigma^2}{[\Psi'(\theta)]^2}.$$

- First using some rough estimations, we obtain

$$\sqrt{T_N}\left|\hat{\mu}_{2,N} - \frac{1}{T_N}\int_0^{T_N} X_t^2 dt\right| \to 0, \quad a.s.$$

- Using Taylor's Theorem: set $\mu = \Psi(\theta)$,

$$\begin{aligned}\sqrt{T_N}\left(\hat{\theta}_N - \theta\right) &= \frac{d\Psi^{-1}}{d\mu}(\mu)\sqrt{T_N}\left(\hat{\mu}_{2,N} - \Psi(\theta)\right) \\ &\quad + R_1(\hat{\mu}_{2,N})\sqrt{T_N}\left(\hat{\mu}_{2,N} - \Psi(\theta)\right) \\ &= \frac{d\Psi^{-1}}{d\mu}(\mu)\sqrt{T_N}\left(\frac{1}{T_N}\int_0^{T_N} X_t^2 dt - \Psi(\theta)\right) \\ &\quad + \frac{d\Psi^{-1}}{d\mu}(\mu)\sqrt{T_N}\left(\hat{\mu}_{2,N} - \frac{1}{T_N}\int_0^{T_N} X_t^2 dt\right) \\ &\quad + R_1(\hat{\mu}_{2,N})\sqrt{T_N}\left(\hat{\mu}_{2,N} - \Psi(\theta)\right),\end{aligned}$$

and conclude the result.

Another estimator: Case when Hurst parameter H is unknown

Under Progress !

- Is LSE strongly consistent in the setup of fractional Ornstein-Uhlenbeck of the second kind?
- What about MLE in the setup of fractional Ornstein-Uhlenbeck of the second kind? Does it exist? If yes, what about Consistency and CLT ?
- What about parameter estimation beyond linear drift coefficient? i.e. $dX_t = \theta b(X_t)dt + dY^{(1)}$ for a suitable class of non-linear functions b.
- To what extend for the driving noise in the Langevin equation, one can prove consistency of LSE?

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Thank you for your attention !