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INITIAL ENLARGEMENT IN A MARKOV CHAIN MARKET MODEL

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Enlargement of filtrations is a classical topic in the general theory of stochastic processes. This theory has been applied to stochastic finance in order to analyze models with insider information. In this paper we study initial enlargement in a Markov chain market model, introduced by Norberg. In the enlarged filtration, several things can happen: some of the jumps times can be accessible or predictable, but in the original filtration all the jumps times are totally inaccessible. But even if the jumps times change to accessible or predictable, the insider does not necessarily have arbitrage possibilities.

Keywords: Markov chain market model; initial enlargement; jump times; insider information.

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1. Introduction

Enlargement of filtrations is a classical topic in the general theory of stochastic processes [7]. This theory has been applied to stochastic finance in order to analyze models with insider information (see, for example, [1–4]). In this paper we study initial enlargement in a Markov chain market model, introduced by Norberg [9]. In this model the state of economy is modeled by a finite state Markov chain, and the state of economy determines the dynamics for the risky assets.

The ordinary agent has the information described by the filtration generated by an observable process, but the insider has the additional information given by a certain random variable.

We assume that the ordinary agent has no arbitrage possibilities. Then, in the initial enlargement the following things can happen;

- In the original filtration the jump times are totally inaccessible, but in the enlarged filtration there can be accessible and predictable jump times.
- Independently of the possible changes in the properties of jump times, the insider may have arbitrage possibilities, or may not have arbitrage possibilities.

The motivation for this study comes from the jump model example introduced by Kohatsu-Higa [8]. Our results show some additional features in the enlargement theory for processes with jumps.

2. Markov Chain Market Model

2.1. *States of the economy*

We describe the model introduced by R. Norberg. We work with probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The state of the economy is given by a process Y. Next we list the properties of the process Y: $Y = (Y_t)_{t\geq 0}$ is a time-homogeneous Markov process with finite state space $\mathcal{Y} = \{1, \ldots, n\}$ and the paths of Y are right continuous with left-hand limits.

We denote the transition probabilities by $P_t^{ek} = \mathbb{P}(Y_{t+s} = k \,|\, Y_s = e)$, $s \geq 0$, $P_t = \{P_t^{ek}\}$ is the transition matrix and $\Lambda = \{\lambda^{ek}\}$ is the intensity matrix.

The states of the Markov chain determine the dynamics of the risky assets.

2.2. *Dynamics of risky assets*

The market model has $m+1$ assets $S = (S^0, S^1, \ldots, S^m)$. We describe their dynamics with the help of the state process Y.

The counting process

$$N_t^{ek} = \#\{\tau : 0 < \tau \leq t, Y_{\tau-} = e, Y_\tau = k\}$$

counts direct transitions of Y from state e to state k during the time interval $(0, t]$.

The bank account S_t^0 has dynamics

$$S_t^0 = \exp\left(\int_0^t r_u du\right) = \exp\left(\sum_e \int_0^t r^e \mathbf{1}_{\{Y_{u-}=e\}} du\right),$$

where $r_t = r^{Y_t}$, i.e. the short rate depends on the state of the economy.

The rest of the assets have the following dynamics

$$S_t^i = S_0^i \exp\left(\sum_e \left\{\int_0^t \mu^{ie} \mathbf{1}_{\{Y_{u-}=e\}} du + \sum_k \beta^{iek} N_t^{ek}\right\}\right),$$

where $\mu^{ie} \in \mathbb{R}, \beta^{iek} \in \mathbb{R}$. Then the logarithmic discounted prices $L^i = \log(S^i/S^0)$ have dynamics

$$dL_t^i = \sum_e (\mu^{ie} - r^e)\mathbf{1}_{\{Y_{t-}=e\}}dt + \sum_e \sum_k \beta^{iek} dN_t^{ek}.$$

Later we shall work with three different filtrations: with the filtration generated by the Markov chain process Y, which we denote by $\mathbb{F}$, or by $\mathbb{F}^Y$, and with two initially enlarged filtrations, which we denote by $\mathbb{G}$ and $\mathbb{G}^H$. We will specify the filtrations $\mathbb{G}$ and $\mathbb{G}^H$ later. Note that the filtration $\mathbb{F}$ is also generated by the counting processes N^{ek}, $e, k \in \mathcal{Y}$ with $e \neq k$.

2.3. *No-arbitrage criterion in the Norberg model*

From the definition of the model we have that

$$M_t^{ek} = N_t^{ek} - \int_0^t \lambda^{ek}\mathbf{1}(Y_s = e)ds,$$

are mutually orthogonal $(\mathbb{F}, \mathbb{P})$-martingales: indeed we have for $e, k, l, p \in \mathcal{Y}$ when $k \neq p$ or $e \neq l$ that $[M^{ek}, M^{lp}]_T = \sum_{s \leq T} \Delta M_s^{ek} \Delta M_s^{lp} = 0$ a.s., and this implies mutual orthogonality in the sense of [5].

Definition 2.1. The intensity matrices $\Lambda = (\lambda^{ef})$ and $\widetilde{\Lambda} = (\widetilde{\lambda}^{ef})$ are equivalent, when $\forall\, e, f \in \mathcal{Y}$, $\lambda^{ef} > 0 \Leftrightarrow \widetilde{\lambda}^{ef} > 0$.

In order to make the discounted stock price process a martingale, we should have that the new intensity $\widetilde{\lambda}^{ek}$ satisfies for all $e \in \mathcal{Y}$:

$$\mu^{ie} - r^e = -\sum_{k \in \mathcal{Y}^e} \gamma^{iek}\widetilde{\lambda}^{ek}, \quad i = 1, \ldots, m,$$

where $\mathcal{Y}^e = \{k : \lambda^{ek} > 0\}$ is the set of states directly reachable from state e, and $\gamma^{ek} = e^{\beta^{ek}} - 1$. Rewrite this in matrix form as

(NA) $\quad r^e\mathbf{1} - \mu^e = \Gamma^e\widetilde{\lambda}^e,$

where $e = 1, \ldots, n$, $\mathbf{1}$ and $\mu^e = (\mu^{ie})_{i=1,\ldots,m}$ are $1 \times m$ row vectors, $\Gamma^e = (\gamma^{ief})_{i=1,\ldots,m}^{f \in \mathcal{Y}^e}$, $\widetilde{\lambda}^e = (\widetilde{\lambda}^{ef})_{f \in \mathcal{Y}^e}$. We can now summarize the situation:

Proposition 2.1. ([9]) *Assume that we can find* $\widetilde{\Lambda}$, *equivalent to* Λ, *such that* (*NA*) *holds, then defining* $\mathbb{Q}$ *by* $d\mathbb{Q}_t = Z_t d\mathbb{P}_t$ *with the density*

$$Z_t = \exp\left(\sum_{e \in \mathcal{Y}} \sum_{k \in \mathcal{Y}^e} (\log(\widetilde{\lambda}^{ek}) - \log(\lambda^{ek}))N_t^{ek}\right)$$
$$\times \exp\left(\sum_{e \in \mathcal{Y}} \sum_{k \in \mathcal{Y}^e} \int_0^t (\lambda^{ek} - \widetilde{\lambda}^{ek})\mathbf{1}(Y_{s-} = e)ds\right)$$

we obtain a martingale measure for the Norberg market model.

Without loss of generality, we can assume that the state dependent interest rate $r^e = 0$.

We give two basic examples, which we use to illustrate various aspects of the Norberg model in connection to initial enlargement.

Example 2.1. In [8], Kohatsu-Higa introduced the following model for the stock price:

$$L_T = \log(S_T) = L_0 + \beta^+ N_T^+ + \beta^- N^-;$$

here $\beta^+ > 0, \beta^- < 0$, N^+ and N^- are Poisson process with respective intensities λ^+ and λ^-, counting respectively the upward and downward jumps, respectively.

This can be put in the Norberg model as follows. The state space is $\mathcal{Y} = \{1, 2, 3\}$, and there is one stock S.

The parameters are $\beta^+ > 0$ $\beta^- < 0$, and the drift $\mu = \mu^i$, for $i = 1, 2, 3$. Take

$$\frac{dS_t}{S_t} = \gamma^+(dN_t^{12} + dN_t^{23} + dN_t^{31}) + \gamma^-(dN_t^{13} + dN_t^{21} + dN_t^{32}) + \mu dt,$$

where $\gamma^\pm = (\exp(\beta^\pm) - 1)$ and

$$dN_t^{ij} - \mathbf{1}(Y_{t-} = i)\lambda^{ij} dt$$

are martingale increments for $i \neq j$ under the measure P with $\lambda^{ij} > 0$.

Now take $\lambda^{12} = \lambda^{23} = \lambda^{31} = \lambda^+$ and $\lambda^{21} = \lambda^{32} = \lambda^{13} = \lambda^-$. Then the aggregated processes $N_t^+ = (N_t^{12} + N_t^{21} + N_t^{31})$ and $N_t^- = (N_t^{13} + N_t^{21} + N_t^{32})$ have deterministic compensators

$$\sum_{i=1}^{3} \int_0^t \lambda^\pm \mathbf{1}(Y_{s-} = i) ds = \lambda^\pm t,$$

and since $[N^+, N^-] = 0$ a.s., by Watanabe's characterization N^+, N^- are independent Poisson processes (see [6]).

To check the (NA) condition, we find $\widetilde{\lambda}^\pm > 0$ such that

$$\gamma^+ \widetilde{\lambda}^+ + \gamma^- \widetilde{\lambda}^- + \mu = 0,$$

and then we obtain an equivalent risk-neutral measure $\mathbb{Q}$ with intensities $\widetilde{\lambda}^{12} = \widetilde{\lambda}^{23} = \widetilde{\lambda}^{31} = \widetilde{\lambda}^+$, and $\widetilde{\lambda}^{21} = \widetilde{\lambda}^{32} = \widetilde{\lambda}^{13} = \widetilde{\lambda}^-$.

The model is incomplete:

$$\widetilde{\lambda}^- = -(\mu + \gamma^+ \widetilde{\lambda}^+)/\gamma^- > 0$$

is a solution for any fixed μ and large enough $\widetilde{\lambda}^+ > 0$, since $\gamma^- < 0$ and $\gamma^+ > 0$. Hence there are many martingale measures.

Example 2.2. Later we will illustrate what can happen in the initial enlargement using the following model.

- The economy can be in two different states: $\mathcal{Y} = \{1, 2\}$.
- Write $\mu^- = \mu^1$, $\mu^+ = \mu^2$, $\lambda^+ = \lambda^{12}$, $\lambda^- = \lambda^{21}$ and similarly with γ^+, γ^-, N^+ and N^-.
- Assume that $\mu^+ > 0$, $\gamma^+ > 0$, $\mu^- < 0$ and $\gamma^- < 0$.
- We have only one stock and

$$L_T = \log(S_T) = L_0 + \beta^+ N_T^+ + \beta^- N_T^- + \int_0^T \mu^{Y_u} du.$$

It is easy to see that the (NA) condition holds for the assumed parameter values.

Note that here, in contrast to Example 2.1, the processes N^+ and N^- are not independent.

2.4. *An alternative description of the model*

The randomness of the model comes from the finite state Markov processes Y. Alternatively, we can consider the matrix valued counting process $\mathbf{N} = (N^{el})_{e,l \in \mathcal{Y}, e \neq l}$, where N^{el} counts the direct transitions from state e to state l.

The other possibility is to consider a single counting process N, where $N = \sum_{e,l} N^{el}$, and keep track, how the Markov process Y behaves at the jump times of N. More precisely, this information is given by the *scenarios*. A scenario $h = (n; e_0, e_1, \ldots, e_n)$ gives information about the associated Markov chain, here $n \geq 0$ is the number of changes in the economy, e_0 is the initial state, and $e_i \neq e_{i+1}$, $i = 1, \ldots, n-1$, are the states of the economy in the scenario h. For example, $(0; e_0)$ is the scenario, where there are no changes in the economy. Notation: $e^{0:n} = e_0, e_1, \ldots, e_n$. The random scenario $H_T = (N_T; Y_0, Y_{\tau_1}, \ldots, Y_{N_T})$, where τ_k is the kth-jump time of the counting process N, together with the aggregated counting process N defined above, has the same information as the matrix-valued counting process $\mathbf{N}$. The scenarios will be useful for us both in some computations and in the analysis of the initial enlargement.

3. Calculation of the Insider's Compensator: Classical Theory

3.1. *A martingale representation result*

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space with $\mathbb{F} = (F_t)_{t \geq 0}$ any filtration, not necessarily the filtration generated by the counting process $\mathbf{N}$. Next we study how the compensator Λ^{ek} of the counting process N^{ek} is computed in the initially enlarged filtration, $e, k \in \mathcal{Y},\ e \neq k$. We shall simply write N and Λ, instead of N^{ek} and Λ^{ek}.

So, assume that $N = (N_t)_{t \geq 0}$ is a $\mathbb{F}$-adapted counting process with $\mathbb{F}$-compensator $\Lambda = (\Lambda_t)_{t \geq 0}$. We consider, in the next subsection, an enlargement of the filtration $\mathbb{F}$ by a random variable ϑ. To be able to compute the compensator of the process N in this enlarged filtration, we need a few results given below.

We use the notation ${}^pX$ (respectively ${}^oX$) to be the predictable (respectively optional) projection of X and X^p the dual predictable projection (respectively X^o) w.r.t. $(\mathbb{F},\mathbb{P})$, unless otherwise stated.

The next lemma is a version of the martingale representation theorem in our context.

Lemma 3.1. *Let N be a counting process with continuous compensator $\Lambda = N^p$ w.r.t. the filtration $\mathbb{F}$. Denote by $\widetilde{N} = (N - \Lambda)$ the compensated process. Then every $\mathbb{R}^d$ valued $\mathbb{F}$-local martingale $(M_t)_{t\geq 0}$ has the representation*

$$M_t = M_0 + \int_0^t (\widehat{M}_s - M_{s-})d\widetilde{N}_s + U_t,$$

where $\widehat{M}$ is $\mathbb{F}$-predictable, and $(U_t)_{t\geq 0}$ is a $\mathbb{F}$-local martingale with $\langle \widetilde{N}, U\rangle = 0$.

Proof. The proof is essentially a modification of the results in [6], we give it here to clarify the nature of $\widehat{M}$.

Put $\mathbb{R}^d_0 = \mathbb{R}^d \backslash \{0\}$. Let $\mu = \mu^{M,N}$ be the jump measure of the process (M,N). Note that μ is an integer-valued random measure on $E \times \mathbb{R}^+$, where $E = \mathbb{R}^d_0 \times \{0,1\}$ is equipped with the Borel σ-algebra $\mathbb{E}$.

Obviously $\mu^M(\cdot, dt) = \mu(\cdot \times \{0,1\}, dt)$ and $\mu^N(dt) = \mu(\mathbb{R}^d_0 \times \{1\}, dt)$. Let $\nu = \nu^{M,N}$ be the $(\mathbb{F},\mathbb{P})$ compensator of μ.

The uniqueness of the compensator implies that $\nu^M(\cdot, dt) = \nu(\cdot \times \{0,1\}, dt)$ and $\Lambda(dt) = \nu^N(dt) = \nu(\mathbb{R}^d_0 \times \{1\}, dt)$. We introduce the $\mathbb{R}^d$-valued process

$$A_t := \int_0^t \int_{\mathbb{R}^d_0} x\nu(dx \times \{1\}, ds),$$

which is predictable with finite variation. The corresponding $\mathbb{R}^d$-valued predictable random measure $A(dt)$ satisfies $A \ll \Lambda$ on $\mathcal{B}(\mathbb{R}^+)$, with Radon–Nikodym derivative $\rho := \frac{dA}{d\Lambda}$. Define

$$\widetilde{M}_t = \int_0^t \rho_s d\widetilde{N}_s,$$

where $\widetilde{N} = N - \Lambda$.

The martingale M has a decomposition

$$M_t = M_0 + M^c_t + \int_0^t \int_{\mathbb{R}^d_0} x(\mu^M - \nu^M)(dx, ds).$$

Put $U = M - \widetilde{M}$. Then U has a decomposition

$$U_t = M_0 + M^c_t + \int_0^t \int_{\mathbb{R}^d_0} x(\mu^M - \nu^M)(dx, ds) - \widetilde{M}_t.$$

We will show that $\langle U, \widetilde{N}\rangle = 0$.

Obviously $[M^c, \widetilde{N}] = 0$ and so $\langle M^c, \widetilde{N}\rangle = 0$. Next, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}_0^d} x(\mu^M - \nu^M)(dx, ds) \\
&\quad = \int_0^t \int_{\mathbb{R}_0^d} x(\mu - \nu)(dx \times \{0,1\}, ds) \\
&\quad = \int_0^t \int_{\mathbb{R}_0^d} x(\mu - \nu)(dx \times \{0\}, ds) + \int_0^t \int_{\mathbb{R}_0^d} x(\mu - \nu)(dx \times \{1\}, ds) \\
&\quad =: M_t^{d,0} + M_t^{d,1}.
\end{aligned}$$

By construction, $[M^{d,0}, \widetilde{N}] = 0$, and hence $\langle M^{d,0}, \widetilde{N}\rangle = 0$. Finally,

$$\langle M^{d,1}, \widetilde{N}\rangle_t = \int_0^t \int_{\mathbb{R}_0^d} x\nu(dx \times \{1\}, ds) = \int_0^t \rho_s ds,$$

and this gives $\langle M^{d,1} - \widetilde{M}, \widetilde{N}\rangle = 0$. Finally, put $\widehat{M} = M_- + \rho$, and we have the claim. $\square$

Remark 3.1. Note that we have the interpretations

$$d\Lambda_t = \mathbb{P}(\Delta N_t = 1|F_{t-}), \quad dA_t = E_{\mathbb{P}}(\Delta M_t \Delta N_t|F_{t-}).$$

Hence ρ_t has the interpretation

$$\rho_t = \frac{E_{\mathbb{P}}(\Delta M_t \mathbf{1}(\Delta N_t = 1)|F_{t-})}{\mathbb{P}(\Delta N_t = 1|F_{t-})} =: E_{\mathbb{P}}(\Delta M_t|F_{t-}, \Delta N_t = 1),$$

which is well-defined, since ρ is a Radon–Nikodym derivative. Similarly, $\widehat{M}$ has the interpretation

$$\widehat{M}_t = E_{\mathbb{P}}(M_t \,|\, F_{t-}, \Delta N_t = 1).$$

We recall the definitions of the stopped σ-algebras [6] associated to a stopping time τ:

$$\begin{aligned}
F_\tau &:= \{A \in F : A \cap \{\tau \le t\} \in F_t; \forall\, t \ge 0\}, \\
F_{\tau-} &:= \sigma(A \cap \{t < \tau\} : t \ge 0, A \in F_t).
\end{aligned}$$

It follows that $F_{\tau-} \subseteq F_\tau$, and by taking $A = \Omega$ in the definition, τ itself is $F_{\tau-}$-measurable. In simple words, $F_{\tau-}$ contains the information about τ and everything that happened before it, while F_τ also contains the information which comes with τ.

Since the simple left-continuous adapted processes

$$K_t(\omega) = \mathbf{1}_A(\omega)\mathbf{1}(u < t), \quad u \ge 0, \ \ A \in F_u$$

generate the predictable σ-algebra, it follows that

$$F_{\tau-} = \sigma(K_\tau \mathbf{1}(\tau < \infty) : K \text{ is } \mathbb{F}\text{-predictable}).$$

Lemma 3.2. *Let τ be a stopping time in a right-continuous filtration $\mathbb{F} = (F_t)$ completed by the $\mathbb{P}$-null sets. Denote $n_t = \mathbf{1}(\tau \le t)$ and $\tilde{n} = (n - \varrho)$ with $\varrho = n^p$. Then for all $\mathbb{F}$-martingales u*

$$\langle \tilde{n}, u \rangle = 0 \Rightarrow [\tilde{n}, u] = [n, u] = 0,$$

if and only if $F_\tau = F_{\tau-}$.

Proof. Let u_t be a $\mathbb{F}$-martingale with $\langle u, \tilde{n} \rangle = 0$.

The random variable $\Delta[n, u]_\tau = \Delta u_\tau$ is F_τ-measurable. By the assumption $F_\tau = F_{\tau-}$, there is a predictable process k_t such that $\mathbf{1}(\tau < \infty)\Delta u_\tau = \mathbf{1}(\tau < \infty) k_\tau$, which means

$$[u, n]_t = \int_0^t k_s dn_s = (k \cdot n)_t \ .$$

In the notation of Lemma 3.1, $k_t = (\widehat{u}_t - u_{t-})$. Note that

$$[u, \tilde{n}] = [u, n] - [u, \varrho] = (k \cdot n) - [u, \varrho] = (k \cdot \varrho) + (k \cdot \tilde{n}) - [u, \varrho]$$

is a local martingale since by assumption $\langle u, \tilde{n} \rangle = 0$. Since

$$[u, \varrho]_t = \int_0^t \Delta \varrho_s du_s \tag{3.1}$$

is also a local martingale, the predictable process $(k \cdot \varrho)$ is a local martingale with finite variation, therefore

$$(k \cdot \varrho) = 0 \quad \text{and hence also } (k \cdot n) = [u, n] = [u, \tilde{n}] = 0.$$

Next we show that if

$$[u, n] = 0 \quad \text{for all } \mathbb{F}\text{-martingales } u \text{ with } \langle u, \tilde{n} \rangle = 0,$$

then necessarily $F_{\tau-} = F_\tau$. If this is not the case, there is $A \in (F_\tau \backslash F_{\tau-})$ with $\mathbb{P}(A) > 0$, and we find a bounded and F_τ-measurable random variable

$$X(\omega) := \mathbf{1}_A(\omega) - \mathbb{P}(A | \mathcal{F}_{\tau-})(\omega) \not\equiv 0,$$

with $E_\mathbb{P}(X | \mathcal{F}_{\tau-})(\omega) = 0$. We show first that $u_t(\omega) := X(\omega) n_t(\omega)$ is a $\mathbb{F}$-martingale:

- u_t is $\mathbb{F}$-adapted since X is F_τ-measurable.
- For $s \le t$ and $A \in F_s$, $(n_t - n_s)\mathbf{1}_A = n_t(1 - n_s)\mathbf{1}_A$ is $F_{\tau-}$-measurable, since $(1 - n_s)\mathbf{1}_A$ is $F_{\tau-}$-measurable by definition and τ is $F_{\tau-}$-measurable. The martingale property follows:

$$E_\mathbb{P}((u_t - u_s)\mathbf{1}_A) = E_\mathbb{P}(X(n_t - n_s)\mathbf{1}_A) = E_\mathbb{P}(E_\mathbb{P}(X \mid \mathcal{F}_{\tau-})(n_t - n_s)\mathbf{1}_A) = 0.$$

Note also that

$$0 \not\equiv u_t = [u, n]_t = [u, \tilde{n}]_t + [u, \varrho]_t,$$

where $[u, \varrho]$ is a local martingale by (3.1). We see that $[u, \tilde{n}]$ is a local martingale which implies $\langle u, \tilde{n} \rangle = 0$. □

Assumption 1. The jump times τ_k of N satisfy $F_{\tau_k} = F_{\tau_k -}$, with continuous $\mathbb{F}$-compensator.

Corollary 3.1. *Under Assumption* 1, *let $\eta(\omega)$ be a $\mathcal{F}$-measurable $\mathbb{R}^d$-valued random variable. If f is a bounded measurable function, then the optional projection ${}^o f(\eta)$ of $f(\eta)$ is a $\mathbb{F}$-martingale. Hence ${}^o f(\eta)$ has a representation*

(i) $\quad {}^o f(\eta)_t = E_{\mathbb{P}}(f(\eta)) + \int_0^t (\widehat{{}^o f(\eta)}_s - {}^o f(\eta)_{s-}) d\widetilde{N}_s + U_t(f),$

$\quad$ *$U(f)$ is a $\mathbb{F}$-martingale with* $[\widetilde{N}, U(f)] = [N, U(f)] = 0,$

(ii) $\quad {}^p f(\eta)_s = {}^o f(\eta)_{s-}.$

3.2. *Compensator after initial enlargement*

To compute the compensator of N in the initially enlarged filtration $\mathbb{G}$, where $G_t = \cap_{u>t}(F_t \vee \sigma(\vartheta))$, we used the approach initiated in [4], and developed further in [1]:

Consider the measurable product space $(\Omega \times \mathbb{R}^m, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m))$ denoted by $(\bar{\Omega}, \bar{\mathcal{F}})$. Define the map

$$\Phi : (\Omega, \mathcal{F}) \to (\bar{\Omega}, \bar{\mathcal{F}}),$$
$$\omega \mapsto (\omega, \vartheta(\omega)).$$

We denote by $\bar{\mathbb{P}}$ the image of the measure $\mathbb{P}$ under Φ, i.e. $\bar{\mathbb{P}} = \mathbb{P}_\Phi$. Endow the space $\bar{\Omega}$ with the $\bar{\mathbb{P}}$-completed filtration $\bar{\mathbb{F}} = (\bar{F}_t)_{t \geq 0}$ where

$$\bar{F}_t = \bigcap_{u>t} (F_u \otimes \mathcal{B}(\mathbb{R}^m)) \vee \bar{N}, \quad \bar{N} = \{\bar{A} \subseteq \bar{\Omega} : \bar{\mathbb{P}}(\bar{A}) = 0\}.$$

We will consider the initially enlarged filtration $\mathbb{G} = (G_t)_{t \in [0,T]}$ with $G_t = \bigcap_{u>t}(F_u \vee \sigma(\vartheta))$, where $\vartheta \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is an m-dimensional random variable.

Consider also the filtered spaces

$$(\bar{\Omega} \times \mathbb{R}_+, \bar{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+), \bar{\mathbb{P}})$$

and

$$(\Omega \times \mathbb{R}_+, \mathbb{G} \otimes \mathcal{B}(\mathbb{R}_+), \mathbb{P}).$$

Recall the following facts from [1]; let $\bar{X}$ be a stochastic process defined on $(\bar{\Omega} \times \mathbb{R}_+, \bar{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^+))$:

- If $\bar{X}$ is $\bar{\mathbb{F}}$-predictable (respectively, $\bar{\mathbb{F}}$-optional), then $X = \bar{X} \circ \Phi$ is $\mathbb{G}$-predictable (respectively, $\mathbb{G}$-optional).
- If $\bar{\tau}$ is a $\bar{\mathbb{F}}$-stopping time, then $\tau = \bar{\tau} \circ \Phi$ is a $\mathbb{G}$-stopping time.
- If $\bar{M}$ is $(\bar{\mathbb{F}}, \bar{\mathbb{P}})$-local martingale, then $M = \bar{M} \circ \Phi$ is a $(\mathbb{G}, \mathbb{P})$-local martingale.

For example, let $\bar{X}$ be a simple $\bar{\mathbb{F}}$-predictable process:

$$\bar{X}(\omega, \ell, u) = \mathbf{1}_A(\omega)\mathbf{1}_B(\ell)\mathbf{1}_{(s,t]}(u),$$

where $s < t \leq T$, $A \in F_s$ and $B \in \mathcal{B}(\mathbb{R}^m)$. Then we can write $X = \bar{X} \circ \Phi$ as

$$X = \bar{X} \circ \Phi = \mathbf{1}_A \mathbf{1}_B(\vartheta) 1_{(s,t]}, \tag{3.2}$$

which is a $\mathbb{G}$-predictable process.

Extend N to $\bar{\Omega} \times \mathbb{R}_+$ by $\bar{N}$, where $\bar{N}(\omega, \ell, u) = N(\omega, u)$.

Let $\bar{\pi}$ be the measure generated by $\bar{N}$ on $(\bar{F} \otimes \mathcal{B}(\mathbb{R}_+))$:

$$\bar{\pi}(\bar{Y}) := E_{\bar{\mathbb{P}}}\left(\int_0^\infty \bar{Y}_u d\bar{N}_u\right) = E_{\mathbb{P}}(\mathbf{1}_C \mathbf{1}_B(\vartheta)(N_t - N_s)),$$

where

$$\bar{Y}(\omega, \ell, u) = \mathbf{1}_C(\omega)\mathbf{1}_B(\ell)\mathbf{1}_{(s,t]}(u).$$

Since $\bar{N}$ is optional with respect to the history $\bar{\mathbb{F}}$, the measure $\bar{\pi}$ is also optional: for any bounded non-negative $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}_+)$ measurable process $\bar{Y}$ we have

$$\bar{\pi}(\bar{Y}) = \bar{\pi}({}^{\bar{\mathbb{F}},o}\bar{Y})$$

(see [5] for more details).

Denote by π the measure generated by N on $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$. Then for optional $\bar{Y}$, put $Y = \bar{Y} \circ \Phi$, and we have $\pi(Y) = \pi({}^oY)$ (see [1]). Apply this to X of the form (3.2) with $A \in F_s$, and we get

$$\bar{\pi}(\bar{X}) = \pi(X) = \pi({}^oX) = E_{\mathbb{P}}(\mathbf{1}_A\{{}^o(\mathbf{1}_B(\vartheta))_t N_t - {}^o(\mathbf{1}_B(\vartheta))_s N_s\}).$$

We can now continue using Corollary 3.1, the continuity of Λ, Lemma 3.2 under Assumption 1, and integration by parts to obtain

$$\begin{aligned}
&E_{\mathbb{P}}(\mathbf{1}_A\{{}^o(\mathbf{1}_B(\vartheta))_t N_t - {}^o(\mathbf{1}_B(\vartheta))_s N_s\}) \\
&\quad = E_{\mathbb{P}}\left(\mathbf{1}_A \int_s^t {}^o(\mathbf{1}_B(\vartheta))_{u-} dN_u\right) + E_{\mathbb{P}}\left(\mathbf{1}_A \int_s^t N_{u-} d^o(\mathbf{1}_B(\vartheta))_u\right) \\
&\qquad + E_{\mathbb{P}}(\mathbf{1}_A\{[N, {}^o(\mathbf{1}_B(\vartheta))]_t - [N, {}^o(1_B(\vartheta))]_s\}) \\
&\quad = E_{\mathbb{P}}\left(\mathbf{1}_A \int_s^t \widehat{{}^o\mathbf{1}_B(\vartheta)}_u dN_u\right) = E_{\mathbb{P}}\left(\mathbf{1}_A \int_s^t (\widehat{{}^o\mathbf{1}_B(\vartheta)})_u d\Lambda_u\right).
\end{aligned}$$

On the other hand, consider the counting process $\mathbf{1}_B(\vartheta)N$, which is adapted to $\mathbb{G}^\vartheta$, and we know that it has a dual predictable projection with respect to $\mathbb{F}$:

$$E_{\mathbb{P}}\left(\mathbf{1}_A \int_s^t d(\mathbf{1}_B(\vartheta)N)_u\right) = E_{\mathbb{P}}\left(\mathbf{1}_A \int_s^t d(\mathbf{1}_B(\vartheta)N)_u^p\right).$$

This means by the uniqueness of the dual predictable projection that

$$(\mathbf{1}_B(\vartheta)N)_t^p = \int_0^t (\widehat{{}^o\mathbf{1}_B(\vartheta)})_u d\Lambda_u.$$

We use the notation $\bar{\theta}$ for the measure

$$\bar{\theta}(\bar{X}) = \bar{\theta}(C \times B \times (s,t]) = E_{\mathbb{P}}\left(\mathbf{1}_C \int_s^t (\widehat{{}^o\mathbf{1}_B(\vartheta)})_u d\Lambda_u\right)$$

extended to the σ-algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^+)$.

Note that $\bar{\pi}$ coincides with $\bar{\theta}$ on the predictable σ-algebra $\mathcal{P}(\bar{\mathbb{F}})$.

Next, define a measure $\widetilde{\theta}(d\omega, d\ell, dt)$ by

$$\widetilde{\theta}(\bar{X}) = \widetilde{\theta}(C \times B \times (s,t]) := E_{\mathbb{P}}\left(1_C \int_s^t {}^p(\mathbf{1}_B(\vartheta))_u d\Lambda_u\right) \tag{3.3}$$

extended to the σ-algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^+)$.

Next we compare the measures $\bar{\theta}$ and $\widetilde{\theta}$ in the smaller σ-algebra $\mathcal{P}(\bar{\mathbb{F}})$, and use Radon–Nikodym theorem to obtain a $\bar{\mathbb{F}}$-predictable density process.

Theorem 3.1. *Assume that $\bar{\theta} \ll \widetilde{\theta}$ on the predictable σ-algebra $\mathcal{P}(\bar{\mathbb{F}})$ generated by the sets $A \times B \times (s,t]$ with $A \in F_s$, and denote the Radon–Nikodym derivative by*

$$\bar{U}(\omega, \ell, t) = \frac{d\bar{\theta}}{d\widetilde{\theta}}(\omega, \ell, t)|_{\mathcal{P}(\bar{\mathbb{F}})},$$

which is $\bar{\mathbb{F}}$-predictable. Put $Z(\vartheta, t) = (\bar{U} \circ \Phi)_t$ and then $Z(\omega, \vartheta(\omega), \cdot)$ is $\mathbb{G}$-predictable. Then we have that

$$E_{\mathbb{P}}(\mathbf{1}_A \mathbf{1}_B(\vartheta)(N_t - N_s)) = E_{\mathbb{P}}\left(\mathbf{1}_A \mathbf{1}_B(\vartheta) \int_s^t Z(\vartheta, u) d\Lambda_u\right)$$

and hence

$$N_t - \int_0^t Z(\vartheta, u) d\Lambda_u$$

is a martingale in the $\mathbb{G}$-filtration.

Proof. The process $\bar{U}$ is $\bar{\mathbb{F}}$-predictable by the Radon–Nikodym theorem, and using results of [1] we have that Z is $\mathbb{G}$-predictable.

Now, let $A \in F_s$, $B \in \mathcal{B}(\mathbb{R}^m)$, and with $0 \le s \le t$,

$$
\begin{aligned}
E_{\mathbb{P}}(\mathbf{1}_A \mathbf{1}_B(\vartheta)(N_t - N_s)) &= \bar{\pi}(A \times B \times (s,t]) \\
\text{since } \bar{\pi}|\mathcal{P}(\bar{\mathbb{F}}) = \bar{\theta}|\mathcal{P}(\bar{\mathbb{F}}) &= \int_{\Omega\times\mathbb{R}^m\times[0,\infty)} \mathbf{1}_A(\omega)\mathbf{1}_B(\ell)\mathbf{1}_{(s,t]}(u)\bar{\theta}(d\omega, d\ell, du) \\
\text{assumption } \bar{\theta} \ll \widetilde{\theta} &= \int_{\Omega\times\mathbb{R}^m\times[0,\infty)} \mathbf{1}_A(\omega)\mathbf{1}_B(\ell)\mathbf{1}_{(s,t]}(u)\bar{U}(\omega,\ell,u)\widetilde{\theta}(d\omega, d\ell, du) \\
\text{by (3.3)} &= E_{\mathbb{P}}\left(\mathbf{1}_A \int_s^t {}^p(\mathbf{1}_B(\vartheta)Z(\vartheta,\cdot))_u d\Lambda_u\right) \\
&= E_{\mathbb{P}}\left(\mathbf{1}_A \int_s^t \mathbf{1}_B(\vartheta)Z(\vartheta,u) d\Lambda_u\right),
\end{aligned}
$$

where the last equality follows from the property of predictable projection [5, Theorem V.5.16, 2)]. This proves the main claim. □

Remark 3.2. Using Corollary 3.1, we get

$$E_{\mathbb{P}}(f(\vartheta)|G_t) = {}^o f(\vartheta)_t = {}^o f(\vartheta)_{t-} + (\widehat{{}^o f(\vartheta)}_t - {}^o f(\vartheta)_{t-})\Delta N_t;$$

and this in turn gives

$$E_{\mathbb{P}}(f(\vartheta)\,|\,G_t)\Delta N_t = (\widehat{{}^o f(\vartheta)})_t \Delta N_t.$$

Therefore we have the interpretation

$$\widehat{{}^o f(\vartheta)}_t = E_{\mathbb{P}}(f(\vartheta)\,|\,F_{t-}, \Delta N_t = 1) = \frac{E_{\mathbb{P}}(f(\vartheta)\Delta N_t\,|\,F_{t-})}{E_{\mathbb{P}}(\Delta N_t\,|\,F_{t-})}.$$

Remark 3.3. We give an interpretation of the condition $\bar{\theta} \ll \widetilde{\theta}$.

First, consider the formal disintegration of measure

$$\bar{\theta}(d\omega, d\ell, dt) = \mathbb{P}(d\omega)\bar{\nu}(d\ell, dt; \omega).$$

Here we can interpret

$$
\begin{aligned}
\bar{\nu}(d\ell, dt) &= \mathbb{P}(\vartheta \in d\ell, N(dt) = 1\,|\,F_{t-}) \\
&= \mathbb{P}(\vartheta \in d\ell\,|\,F_{t-})\mathbb{P}(N(dt) = 1\,|\,F_{t-}, \vartheta \in d\ell) \\
&=: \mathbb{P}(\vartheta \in d\ell\,|\,F_{t-})\Lambda^{\ell}(dt).
\end{aligned}
$$

On the other hand, we also have the disintegration

$$\widetilde{\theta}(d\omega, d\ell, dt) = \widetilde{\nu}(d\ell, dt; \omega)\mathbb{P}(d\omega),$$

and from (3.3) we have

$$\widetilde{\nu}(d\ell, dt) = \mathbb{P}(\vartheta \in d\ell\,|\,F_{t-})\Lambda(dt).$$

Now, if $\bar{\theta} \ll \widetilde{\theta}$ then

$$\bar{U}(\omega,\ell,t) = \frac{d\bar{\theta}}{d\widetilde{\theta}}(\omega,\ell,t)\Big|_{\mathcal{P}(\bar{\mathbb{F}})} = \frac{d\bar{\nu}}{d\widetilde{\nu}}(\ell,t) = \frac{d\mathbb{P}(\vartheta \in \cdot|F_{t-}, \Delta N_t = 1)}{d\mathbb{P}(\vartheta \in \cdot|F_{t-})}(\ell,\omega)$$
$$= \frac{d\Lambda_t^\ell}{d\Lambda_t}(t,\omega).$$

Moreover, we have the connection

$$\Lambda_t^\ell = \int_0^t Z(\ell,s)d\Lambda_s.$$

Remark 3.4. When the absolute continuity condition fails, the Lebesgue decomposition on $\mathcal{P}(\bar{\mathbb{F}})$

$$\bar{\theta}(d\omega,d\ell,dt) = U(\omega,\ell,t)\widetilde{\theta}(d\omega,d\ell,dt) + \mathbf{1}(U(\omega,\ell,t)=\infty)\bar{\theta}(d\omega,d\ell,dt)$$

corresponds to the Lebesgue decomposition of the $\mathbb{G}$-compensator

$$\Lambda^\vartheta(dt) = Z(\vartheta,t)\Lambda(dt) + \mathbf{1}(Z(\vartheta,t)=\infty)\Lambda^\vartheta(dt).$$

About the singular part of Λ^ϑ, at this level of generality we cannot say much more than this:

Proposition 3.1. *In the $\mathbb{G}$-filtration the jumps of N are decomposed into two classes*, *$\mathbb{G}$-accessible and $\mathbb{G}$-totally inaccessible* [5, *Chapter* IV]. *The next conditions are equivalent*:

- *$\mathbb{P}$-almost surely*
$$\int_0^t \mathbf{1}(Z(\vartheta,s)=\infty)\Lambda^\vartheta(ds) = \sum_{s\le t}\mathbf{1}(Z(\vartheta,s)=\infty)\Delta\Lambda_s^\vartheta, \tag{3.4}$$
i.e. the singular part of the $\mathbb{G}$-compensator is a pure jump process.
- *The $\mathbb{G}$-compensator of the $\mathbb{G}$-totally inaccessible part of N is absolutely continuous w.r.t. Λ and* (3.4) *is the $\mathbb{G}$-compensator of the $\mathbb{G}$-accessible jumps of N.*

Remark 3.5. We will see that in our initially enlarged Markov chain market model we are in the situation described in Proposition 3.1.

4. Scenarios and Support of the Predictive Distribution

4.1. *Shrinkage*

We start with a useful lemma, which helps to compute compensators. We assume now that the filtration $\mathbb{F}$ is the filtration of the Markov process Y, and the random variable in the initial enlargement is the logarithm of final value of the stock: $\vartheta = \log(S_T)$: $F_t = \sigma\{Y_s : s \le t\}$ and $G_t = \cap_{u>t} F_u \vee \sigma(\vartheta)$. In addition to the random variable ϑ we enlarge the filtration $\mathbb{F}$ with the realized scenario $\zeta = H_T$, where

$H_T = (N_T; Y_0, Y_{\tau_1}, \dots, Y_{\tau_{N_T}})$ (see Sec. 2.4 for more details). Note that the random variable ζ can take only countably many values.

Lemma 4.1. *Assume that $(\vartheta, \zeta) \in F_T^Y$, and the random variable ζ takes in a countable set, say $\zeta(\omega) \in \mathbb{Z}$ without loss of generality.*

Let $\mathbb{G}^\zeta = (G_t^\zeta)_{t\in[0,T]}$ with $G_t^\zeta = \cap_{u>t} G_u \vee \sigma(\zeta)$ be a bigger filtration than $\mathbb{G}$. Then we have the filtration shrinkage *formula*

$$\Lambda_t^\vartheta = (\Lambda^{\vartheta,\zeta})_t^{\mathbb{G},p} = \sum_z \int_0^t \mathbb{P}(\zeta = z \,|\, F_{s-}, \vartheta)\Lambda^{\vartheta,\zeta=z}(ds),$$

where the $\mathbb{G}$-predictable processes $\{\Lambda_t^{\vartheta,\zeta=z}(\omega) : z \in \mathbb{Z}$ and $\mathbb{P}(\zeta = z) > 0\}$ gives the disintegration of the $\mathbb{G}^\zeta$ compensator.

$$\Lambda_t^{\vartheta,\zeta} = \sum_{z:\mathbb{P}(\zeta=z)>0} \int_0^s \mathbf{1}_{\{\zeta=z\}}\Lambda^{\vartheta,\zeta=z}(ds).$$

Proof. For more general results of this type, see [10]. We prove the result in this simple case. Let $s \le t$ and $A \in G_s$. We have

$$\begin{aligned}
E_{\mathbb{P}}(1_A(N_t - N_s)) &= E_{\mathbb{P}}\left(\sum_z \mathbf{1}_{A\cap\{\zeta=z\}}(N_t - N_s)\right) \\
&= E_{\mathbb{P}}\left(\sum_z \mathbf{1}_{A\cap\{\zeta=z\}}(\Lambda_t^{\vartheta,\zeta} - \Lambda_s^{\vartheta,\zeta})\right),
\end{aligned}$$

where the sum is taken over the values z with $\mathbb{P}(\zeta = z) > 0$. But on the set $\{\zeta = z\}$ we have the identity $\mathbf{1}\{\zeta = z\}\Lambda_u^{\vartheta,\zeta} = \mathbf{1}\{\zeta = z\}\Lambda_u^{\vartheta,z}$. We obtain

$$\begin{aligned}
E_{\mathbb{P}}\left(\sum_z \mathbf{1}_{A\cap\{\zeta=z\}}(\Lambda_t^{\vartheta,\zeta} - \Lambda_s^{\vartheta,\zeta})\right) &= E_{\mathbb{P}}\left(\sum_z \mathbf{1}_{A\cap\{\zeta=z\}}(\Lambda_t^{\vartheta,\zeta=z} - \Lambda_s^{\vartheta,\zeta=z})\right) \\
E_{\mathbb{P}}\left(\sum_z \int_s^t {}^{\mathbb{G},p}(\mathbf{1}_{A\cap\{\zeta=z\}})_u d\Lambda_u^{\vartheta,\zeta=z}\right) &= E_{\mathbb{P}}\left(\sum_z \mathbf{1}_A \int_s^t {}^{\mathbb{G},p}(\mathbf{1}_{\{\zeta=z\}})_u d\Lambda_u^{\vartheta,\zeta=z}\right) \\
&= E_{\mathbb{P}}\left(\sum_z \mathbf{1}_A \int_s^t \mathbb{P}(\zeta = z \,|\, G_{u-}) d\Lambda_u^{\vartheta,\zeta=z}\right),
\end{aligned}$$

since $\Lambda^{\vartheta,\zeta=z}$ is $\mathbb{G}$-predictable, $A \in G_s$, and by the definition of predictable projection. □

Remark 4.1. Lemma 4.1 gives a way to compute the compensator Λ^ϑ by using an additional countable enlargement. We have also

$$\begin{aligned}
\Lambda^\vartheta \ll \Lambda &\Leftrightarrow \Lambda^{\vartheta,\zeta} \ll \Lambda \\
&\Leftrightarrow \Lambda^{\vartheta,z} \ll \Lambda \quad \forall z \in \mathbb{Z} \quad \text{and} \quad \mathbb{P}(\zeta = z) > 0.
\end{aligned}$$

4.2. *More on scenarios*

4.2.1. *Random scenarios*

We have already introduced the notion of *scenario* in Sec. 2.4. Now we will assume that in addition to the final value, the insider has at his disposal the information, which states the Markov process Y visited before the time T.

Let Ξ_{e_0} be the set of all possible scenarios starting from e_0 and let Ξ be the set of all possible scenarios:

$$\Xi_{e_0} = \{h = (n; e^{0:n}) : n \in \mathbb{N}, \lambda^{e_i e_{i+1}} > 0, i = 0, \dots, n-1\},$$

and $\Xi = \cup_{e_0 \in \mathcal{Y}} \Xi_{e_0}$. Note that the set Ξ is numerable.

Recall that τ_i is the ith jump time of the economy and then H_t is the random scenario

$$H_t(\omega) = (N_t(\omega) : Y_0, Y_{\tau_1}, \dots, Y_{\tau_{N_t}})$$

and H_T is the random scenario $H_T = (N_T : Y_0, Y_{\tau_1}, \dots, Y_{\tau_{N_T}})$.

4.2.2. *Operations with scenarios*

To analyze the scenarios dynamically we need the following operations with them. Let $h = (n; e^{0:n})$ and $\widetilde{h} = (m; \widetilde{e}^{0:m})$ be two scenarios. Put

$$h_{(k)} = (n \wedge k; e^{0:n \wedge k}), h^{(k)} = ((n-k)^+; e^{n \wedge k:n}),$$

and $h \vee \widetilde{h} = (n + m; e^{0:n}, \widetilde{e}^{0:m})$; here we assume that $e^n = \widetilde{e}^0$. With these notations $h^{(0)} = h = h_{(k)} \vee h^{(k)} = h_{(n)} \vee h^{(n)}$.

Let $h = (n; e^{0:n})$ be a fixed scenario, and put

$$\Pi_{e_0,t}(h) := \mathbb{P}(H_t = h \,|\, Y_0 = e_0).$$

Note that for every $h \in \Xi_{e_0}$ we have that $\Pi_{e_0,T}(h) > 0$.

We have $\Pi_{e_0,T}((0; e_0)) = \exp(-\lambda^{e_0} T)$ and with $h = (n; e^{0:n})$, when $n \geq 1$, we have the recursion:

$$\Pi_{e_0,T}(h) = \int_0^T \lambda^{e_0,e_1} \exp(-\lambda^{e_0} t) \Pi_{e_1,T-t}(h^{(1)}) dt.$$

To summarize what we have achieved by now:

$\Pi_{e_0,T}(h) > 0$ if and only if $h \in \Xi_{e_0}$ and we have the implications

$$\Pi_{e_0,T}(h) > 0 \Rightarrow \Pi_{e_0,t}(h) > 0 \quad \text{for all } t > 0;$$

this means that if a fixed scenario h has positive probability on the interval $[0, T]$, it has a positive probability on every sub-interval $[0, t]$, too. Finally, using the identity $h = (h_{(k)} \vee h^{(k)})$, we have the following implications for all $t \in (0, T)$:

$$\Pi_{e_0,T}(h) > 0 \Rightarrow \Pi_{e_0,t}(h_{(k)}) > 0, \quad \text{and}$$

$$\Pi_{e_0,T}(h) > 0 \Rightarrow \Pi_{e_k,T-t}(h^{(k)}) > 0.$$

4.3. *Joint distribution of L_t and H_t*

Recall that $L_t = \log(S_t)$, where S is the discounted stock vector, and H_t is the random scenario $H_t = (N_t; Y_0, \dots, Y_{N_t})$. We denote their joint distribution by Q.

Put $Q_{e_0,t}(d\ell, (0; e_0)) := \mathbb{P}(L_t \in d\ell, H_t = (0; e_0))$. Note first that

$$Q_{e_0,t}(d\ell, (0; e_0)) = \exp(-\lambda^{e_0} t)\delta_{\mu^{e_0} t}(d\ell).$$

After this we can proceed recursively

$$\begin{aligned} Q_{e_0,t}(d\ell, h) &= \mathbb{P}(L \in d\ell, H_t = h) \\ &= \int_0^t \lambda^{e_0,e_1} \exp(-\lambda^{e_0} u) Q_{e_1,t-u}(d\ell - \mu^{e_0} u - \beta^{e_0,e_1}, h^{(1)})\, du. \end{aligned}$$

From the joint distribution $Q_{e_0,T}(d\ell, h)$ we obtain the marginal distribution

$$Q_{e_0,T}(d\ell) = \sum_{h \in \Xi_{e_0}} Q_{e_0,T}(d\ell, h).$$

4.4. *Support of the conditional measure $Q_{e_0,t}(\cdot|h)$*

When $h \in \Xi_{e_0}$, the conditional probability $Q_{e_0,t}(d\ell|h) = \frac{Q_{e_0,t}(d\ell,h)}{\Pi_{e_0,t}(h)}$ is well defined, since $\Pi_{e_0,t}(h) > 0$ for $0 < t \le T$.

Fix $h = (n; e^{0:n})$ and put $\beta^{\overline{0:n}} = \beta^{e_0 e_1} + \dots + \beta^{e_{n-1} e_n} \in \mathbb{R}^m$. The support of the conditional measure $Q_{e_0,T}(d\ell \,|\, h)$ is obviously the convex hull of the set

$$\{L_0 + \beta^{\overline{0:n}} + \mu^{e_i} T : i = 0, \dots, n\}.$$

We denote this convex hull by $\mathcal{A}_T(h)$. Fix $0 < s < T$, and consider the convex hull $\mathcal{A}_{T-s}(h^{(N_s)})$ of the random set

$$\{L_s + \beta^{\overline{N_s:n}} + \mu^{e_i}(T - s) : i = N_s, \dots, n\}.$$

Then, either $Q_{e_0,T}(d\ell \,|\, h)$ is a point mass, which happens if and only if $\mu^{e_i} = \mu^{e_0}$ for all $i = 1, \dots, n$, or $Q_{e_0,T}(d\ell \,|\, h)$ is equivalent to Lebesgue measure on its support.

Moreover, we have for every ω in the canonical space, $0 < s < t < T$, $h \in \Xi$:

$$\mathbb{R}^m \supset \mathcal{A}_T(h) \supseteq \mathcal{A}_{T-s}(h^{(N_s)}) \supseteq \mathcal{A}_{T-t}(h^{(N_t)}),$$

and by summing over the scenarios $h \in \Xi$ we get

$$\mathbb{R}^m \supset \operatorname{supp} Q_{Y_s,T-s}(\cdot - L_s) \supseteq \operatorname{supp} Q_{Y_t,T-t}(\cdot - L_t).$$

Since these predictive distributions are equivalent to Lebesgue measure on their support, the relation

$$\operatorname{supp} Q_{Y_s,T-s}(\cdot - L_s | h^{(N_s)}) \supseteq \operatorname{supp} Q_{Y_t,T-t}(\cdot - L_t \,|\, h^{(N_t)})$$

does not imply that

$$Q_{Y_s,T-s}(\cdot - L_s | h^{(N_s)}) \gg Q_{Y_t,T-t}(\cdot - L_t \,|\, h^{(N_t)}).$$

This implication is true only in the case that the supports of these predictive distributions have the same dimension.

More formally, put

$$
\begin{aligned}
D_T(h) &= \dim \mathcal{A}_T(h) \\
&= \max\{p : \exists\, x_0, x_1, \ldots, x_p \in \mathcal{A}_T(h) \text{ with } (x_k - x_0) \text{ linearly independent}\},
\end{aligned}
$$

where $D_T(h) = 0$, if the set $\mathcal{A}_T(h)$ consists of one point. Note that the value of $D_T(h)$ does not depend on T, and we simply write $D(h)$.

Fix now $h = (n : e^{0:n})$ and assume that $n \geq 1$. Recall that $h^{(k)}$ is the remaining scenario after k changes in the economy. Obviously we have

$$
D(h) \geq D(h^{(1)}) \geq D(h^{(2)}) \geq \cdots \geq D(h^{(n)}) = 0.
$$

Clearly, if $D(h) = 0$, then $D(h) = D(h^{(1)}) = \cdots = D(h^{(n)}) = 0$.

Example 4.1. Returning to Example 2.1 of Kohatsu-Higa, we have for $0 < s < T$ that

$$
\mathcal{A}_T(h) = \{L_0 + \beta^{\overline{0:n}} + \mu^{0:n} T\} = \mathcal{A}_{T-s}(h^{(N_s)}),
$$

hence we have $D(h) = D(h^{(1)}) = \cdots = D(h^{(n)}) = 0$.

Example 4.2. Take $h = (n; e^{0:n})$ with $n \geq 1$. Then $n = n^+ + n^-$, where $n^- = \lfloor \frac{n}{2} \rfloor$ and

$$
D(h) = \cdots = D(h^{(n-1)}) = 1 > D(h^{(n)}) = 0.
$$

4.5. *Scenarios and final value*

In order to analyze, how the properties of the jump times may change with the additional information, we need more definitions.

For given $T > 0, \ell \in \mathbb{R}^m$, consider those scenarios $h \in \Xi_{e_0}$, $h = (n; e^{0:n})$, $n \in \mathbb{N}$, such that $\ell \in \mathcal{A}_T(h)$, that is for some $\Delta t_j > 0, j = 0, 1, \ldots, n$

$$
\sum_{j=0}^{n} \mu^{e_j} \Delta t_j = \ell - L_0 - \beta^{\overline{0:n}}, \quad \text{and} \quad \Delta t_0 + \Delta t_1 + \cdots + \Delta t_n = T. \tag{4.1}
$$

Note that for given (T, ℓ, h), the solution vector $(\Delta t_j : 0 \leq j \leq n)$ possibly does not exist, and when it exists, it is not always unique.

Consider the projection $C_0(h)$ and random times $\underline{\mathcal{T}}(T, \ell, h)$ and $\overline{\mathcal{T}}(T, \ell, h)$:

$$
\begin{aligned}
&C_0(h) := \{\Delta t_0 > 0 : \exists\, (\Delta t_0, \ldots, \Delta t_n) \in \mathbb{R}_+^{n+1} \text{ solving } (4.1)\}, \\
&\underline{\mathcal{T}}(T, \ell, h) := \inf C_0(h), \quad \overline{\mathcal{T}}(T, \ell, h) := \sup C_0(h).
\end{aligned} \tag{4.2}
$$

When $C_0(h) \neq \emptyset$, since the solutions of (4.1) form a convex set, we have

- either $\underline{\mathcal{T}}(T, \ell, h) < \overline{\mathcal{T}}(T, \ell, h)$ and $C_0(h) = (\underline{\mathcal{T}}(T, \ell, h), \overline{\mathcal{T}}(T, \ell, h))$,
- or $\underline{\mathcal{T}}(T, \ell, h) = \overline{\mathcal{T}}(T, \ell, h)$ and Δt_0 is *determined* by (T, ℓ) and the scenario $h = (n; e^{0:n})$.

Proposition 4.1. *We have the following characterization*:

$$\Delta t_0 \text{ is determined by } (T,\ell,h) \Leftrightarrow \ell \in \mathcal{A}_T(h) \text{ and } D(h^{(1)})+1 = D(h).$$

Proof. The constrained linear problem (4.1) is rewritten as

$$\sum_{i=0}^{n-1}(\mu_i-\mu_n)\Delta t_i = (\ell - L_0 - \overline{\beta^{0:n}} - \mu_n T), \Delta t_i > 0, \quad \sum_{i=0}^{n-1}\Delta t_i < T. \tag{4.3}$$

Consider the constrained linear systems

$$(A):\ \sum_{i=0}^{n-1} A_{ji}t_i = y_j, 1 \le j \le m, t \in C \subseteq \mathbb{R}^n,$$

$$(A'):\ \sum_{i=1}^{n-1} A_{ji}t'_i = y'_j, 1 \le j \le m, t' \in C' \subseteq \mathbb{R}^{n-1},$$

where C and C' are open simplexes (cf. Sec. 4.1).

(A) corresponds to (4.3) and (A') corresponds to the situation after the first transition $e_0 \to e_1$. Denote

$$A' := (A_{ji})_{1\le i\le (n-1), 1\le j\le m}.$$

The images AC and $A'C'$ are open in $\mathbb{R}^m$, and their dimension coincides with the ranks $\dim(\mathrm{Im}(A)) = D(h)$ and $\dim(\mathrm{Im}(A')) = D(h^{(1)})$ respectively.

We have the linear isomorphisms

$$\mathrm{Im}(A) \simeq (\mathbb{R}^n/\mathrm{Ker}(A)), \quad \mathrm{Im}(A') \simeq (\mathbb{R}^{n-1}/\mathrm{Ker}(A')),$$

where $\mathrm{Ker}(A)$ denotes the null space and we take the algebraic quotient. This implies

$$\dim(\mathrm{Im}(A)) = n - \dim(\mathrm{Ker}(A)), \quad \dim(\mathrm{Im}(A')) = n - 1 - \dim(\mathrm{Ker}(A')).$$

Either

(1) the column vector $A_{\bullet 0}$ is linearly independent from the columns $(A_{\bullet 1},\ldots, A_{\bullet n-1})$

$$\Leftrightarrow \dim(\mathrm{Im}(A)) = \dim(\mathrm{Im}(A')) + 1 \Leftrightarrow \dim(\mathrm{Ker}(A)) = \dim(\mathrm{Ker}(A')),$$

(2) or $\dim(\mathrm{Im}(A)) = \dim(\mathrm{Im}(A'))$,

$$\Leftrightarrow \dim(\mathrm{Ker}(A)) = \dim(\mathrm{Ker}(A')) + 1.$$

In case (1), the dimension of the null space does not change after adding the column $A_{\bullet 0}$ to the matrix A'. If $(t_0, t_1, \ldots, t_{n-1})$ is a solution of the homogeneous system associated to (A),

$$(A^*):\ \sum_{i=0}^{n-1} A_{ji}t_i = 0, \quad 1 \le j \le m,$$

then necessarily $t_0 = 0$ and $\sum_{i=1}^{n-1} A_{ji}t_i = 0$, $\forall\, j$. This means that all solutions of (A) begin with the same coordinate t_0.

In case (2) the homogeneous system (A^*) admits solutions with $t_0 \neq 0$ and t_0 is not uniquely determined by (A). $\square$

Example 4.3. Consider Example 4.1, fix h and take $\ell \in \mathcal{A}_T(h)$. It is easy to see that (T, ℓ, h) never determines Δt_0. On the other hand, in Example 4.2, take $h = (1; e^1, e^2)$, and $\ell \in \mathcal{A}_T(h)$. Then (T, ℓ, h) determines Δt_0 and we have

$$\Delta t_0 = \frac{\ell - L_0 - \beta^+ - \mu^+ T}{\mu^- - \mu^+}.$$

More generally, in this example, for any h with n changes in the economy, and $\ell \in \mathcal{A}_T(h)$, the last jump time $\sum_{k=0}^{n-1} \Delta t_k$ is known, if we know the value of $\sum_{k=0}^{n-2} \Delta t_k$.

5. Computation of the Insiders Compensator

Our program has two parts: (i) Obtain information about the compensator of N with respect to the filtration $\mathbb{G}$. (ii) Check the (NA) criteria in the enlarged filtration $\mathbb{G}$.

The idea is to enlarge the filtration $\mathbb{G}$ with the information of the random scenario H_T, and then use filtration shrinkage to obtain the compensator with respect to $\mathbb{G}$.

5.1. *Classification of the jump times in an extended filtration*

We work with the filtration $\mathbb{G}^{\mathbb{H}}$, where $G_t^{\mathbb{H}} = \bigcap_{u>t} G_u \vee \sigma(H_T)$.

The following proposition is a summary of the results of the previous section.

Proposition 5.1. *Consider the kth jump time τ_k. Fix a history h on the set $\{\omega : H_T(\omega) = h\}$,*

(a) *either $D(h_T^{(k-1)}) = D(h_T^{(k)})$, so that $\forall\, s \in (\tau_{k-1}, \tau_k]$,*

$$Q_{e_k, T-s}(d\ell - (\mu^{e_{k-1}} - r^{e_{k-1}})(T-s) - \beta^{e_{k-1}, e_k}, h^{(k)}) \ll Q_{e_{k-1}, T-s}(d\ell, h^{(k-1)})$$

with

$$\begin{aligned}
&\mathbf{1}(\tau_{k-1} < s \le \tau_k)\Lambda^{L_T, h}(ds) \\
&\quad = \mathbf{1}(\tau_{k-1} < s \le \tau_k) q_{(e_k, T-s, h)}(L_T - L_{s-} - \beta^{e_{k-1} e_k})\lambda^{e_{k-1}, e_k} ds,
\end{aligned}$$

where

$$q_{(e_k, T-s, h)}(\ell) := \frac{dQ_{e_k, T-s}(\cdot - (\mu^{e_{k-1}} - r^{e_{k-1}})(T-s) - \beta^{e_{k-1}, e_k}, h^{(k)})}{dQ_{e_{k-1}, T-s}(\cdot, h^{(k-1)})}(\ell)$$

is supported by the random interval $(\underline{\tau}_k(h), \overline{\tau}_k(h)]$, and by using (4.2) we define the $\mathbb{G}$-predictable times

$$\underline{\tau}_k(h) := \tau_{k-1} + \underline{T}(T - \tau_{k-1}, L_T - L_{\tau_{k-1}}, h_T^{(k-1)}),$$

$$\overline{\tau}_k(h) := \tau_{k-1} + \overline{T}(T - \tau_{k-1}, L_T - L_{\tau_{k-1}}, h_T^{(k-1)}),$$

which are also $(\mathbb{G}_{\tau_{(k-1)}})$*-measurable and satisfy*

$$\tau_{k-1} \le \underline{\tau}_k(h) \le \tau_k \le \overline{\tau}_k(h) \quad \text{on } \{\omega : H_T(\omega) = h\};$$

(b) *or* $D(h_T^{(k-1)}) = D(h_T^{(k)}) + 1$, *so that* $\tau_k = \underline{\tau}_k(h) = \overline{\tau}_k(h)$.

When we sum over all scenarios h *we obtain*

(A) *when* $D(H_T^{(k-1)}) = D(H_T^{(k)})$, τ_k *has* $\mathbb{G}^{\mathbb{H}}$*-compensator absolutely continuous w.r.t.* Λ *and*

$$\tau_{k-1} \le \underline{\tau}_k(H_T) \le \tau_k \le \overline{\tau}_k(H_T),$$

(B) *otherwise* $\tau_k = \underline{\tau}_k(H_T) = \overline{\tau}_k(H_T)$,

where $\underline{\tau}_k(H_T), \overline{\tau}_k(H_T)$ *are* $\mathbb{G}^{\mathbb{H}}$*-predictable times.*

Corollary 5.1. $\Lambda^{L_T,\mathbb{H}}$ *is absolutely continuous w.r.t.* Λ *if and only if* $D(H_T) = 0$.

Now, we apply the countable filtration shrinkage argument to the $\mathbb{G}^{\mathbb{H}}$-compensator of N to obtain the $\mathbb{G}$-compensator.

Proposition 5.2. *The* $\mathbb{G}$*-compensator of* τ_k *is given by*

$$\begin{aligned}
&\int_0^t \mathbf{1}(\tau_{k-1} < s \le \tau_k)\Lambda^L(ds) \\
&\quad = \sum_{h \in \mathcal{D}_k} \int_0^t \mathbb{P}(H_T = h \,|\, L_T, F_{s-})\mathbf{1}(\tau_{k-1} < s \le \tau_k) d_s(\mathbf{1}(\overline{\tau}_k(h) \le s)) \\
&\qquad + \sum_{h \in \Xi \setminus \mathcal{D}_k} \int_0^t \mathbb{P}(H_T = h \,|\, L_T, F_{s-})\mathbf{1}(\tau_{k-1} < s \le \tau_k)\Lambda^{L,h}(ds),
\end{aligned}$$

where

$$\mathcal{D}_k = \{h \in \Xi : D(h^{(k-1)}) = D(h^{(k)}) + 1\}$$

is the set of scenarios for which τ_k *is determined by* L_T *and* H_T *at time* τ_{k-1}, *and* $\Lambda^{L,h}(ds) \ll \Lambda(ds)$ *for* $s \in (\tau_{k-1}, \tau_k]$ *and* $h \in \Xi \setminus \mathcal{D}_k$.

This gives the decomposition of τ_k *into* $\mathbb{G}$*-accessible and* $\mathbb{G}$*-totally inaccessible parts.*

Note also that the predictable times $\{\overline{\tau}_k(h) : h \in \mathcal{D}_k\}$ are not necessarily distinct. Let $\mathcal{D}_k^* \subseteq \mathcal{D}_k$ be a choice of distinct representatives w.r.t. the equivalence relation

$$h \overset{k}{\sim} h' \Leftrightarrow \overline{\tau}_k(h) = \overline{\tau}_k(h'), \quad h, h' \in \mathcal{D}_k.$$

By re-summation, the compensator $\mathbb{G}$-accessible part of the stopping time τ_k is rewritten as

$$\sum_{h\in\mathcal{D}_k^*}\int_0^t \left\{ \sum_{h'\in\mathcal{D}_k:\overline{\tau}_k(h')=\overline{\tau}_k(h)} \mathbb{P}(H_T = h' \mid L_T, F_{s-}) \right\} d_s(\mathbf{1}(\overline{\tau}_k(h)\leq s)),$$

where the $\mathbb{G}$-predictable jump times $\{\overline{\tau}_k(h) : h \in D_k^*\}$ are distinct.

Example 5.1. Concerning Example 2.2, there are two possibilities. The final value $L_T = \ell$ does not uniquely determine the scenario $H_T = h$. In this case the compensator is totally inaccessible in the filtration $\mathbb{G}$. But with special parameter values $\mu^\pm$ and $\beta^\pm$, H_T is uniquely determined by $L_T = \ell$, and then for the insider the last jump is predictable.

6. Insider's Free Lunch with Vanishing Risk

From the general theory it follows that the property *No free lunch with vanishing risk* (NFLVR) in the insider filtration $\mathbb{G}$ is equivalent to the existence of a measure $\mathbb{Q}^L \sim \mathbb{P}$ under which the discounted stock process $(\widetilde{S}_t)_{t\geq 0}$ is a $\mathbb{F}^\vartheta$-martingale. This leads to conditions concerning the accessible and totally inaccessible parts of the jumps of $(L_t)_{t\geq 0}$. We also see that, for arbitrage considerations, we do not need to fully compute the compensators in the insider filtration: it is enough to compute the random sets $\mathcal{D}_k$ at each jump time τ_{k-1}.

For $\mathcal{A} \subseteq \mathcal{Y}^e$ we consider the system of equations

$$\Gamma^{e,\mathcal{A}}\widetilde{\lambda}^{e,\mathcal{A}} = -\mu^e, \tag{6.1}$$

and the homogeneous system

$$\Gamma^{e,\mathcal{A}}\widetilde{\lambda}^{e,\mathcal{A}} = \mathbf{0}, \tag{6.2}$$

where $\mu^e = \left(\mu^{ie}\right)_{i=1;\ldots,m}$,

$$\Gamma^{e,\mathcal{A}} := (\gamma^{ief} : i = 1,\ldots,m, f \in \mathcal{A}),$$
$$\widetilde{\lambda}^{e,\mathcal{A}} = (\lambda^{e,f} : f \in \mathcal{A}),$$

with the constraints

$$\widetilde{\lambda}^{e,f} > 0 \text{ strictly for } f \in \mathcal{A}.$$

This means that respectively $(-\mu^e)$ and $\mathbf{0}$ are in the interior of the the convex cone generated by the columns of the matrix $\Gamma^{e,\mathcal{A}}$.

After the $(k-1)$th jump time, let $Y_{\tau_{k-1}}(\omega) = e$, and define

$$\Xi_k(i,j) := \{h = (n; e_0,\ldots,e_n) : n \geq k, e_{k-1} = i, e_k = j\} \subseteq \Xi,$$
$$\hat{\mathcal{Y}}_k^{(e)}(\omega) := \{f : \exists\, h \in (\Xi_k(e,f)\backslash\mathcal{D}_k) \text{ and } h^{(k-1)} = H^{(k-1)}(\omega)\}.$$

Similarly, for a $\mathbb{G}$-predictable time $\overline{\tau}_k(h) = \underline{\tau}_k(h), h \in \mathcal{D}_k$, we define

$$\check{\mathcal{Y}}^{(e)}_{k,h}(\omega) := \{f : \exists\, h' \in \Xi_k(e,f) \cap \mathcal{D}_k \text{ with } \overline{\tau}_k(h) = \underline{\tau}_k(h) = \underline{\tau}_k(h') = \overline{\tau}_k(h') \\ \text{and } h^{(k-1)} = H^{(k-1)}(\omega)\},$$

where we have the interpretation $\hat{\mathcal{Y}}^{(Y_{\tau_{k-1}})}_k(\omega)$ is the set of states reachable by one $\mathbb{G}$-totally inaccessible jump after time τ_{k-1}, and $\check{\mathcal{Y}}^{(Y_{\tau_{k-1}})}_{k,h}$ is the set of states reachable after time τ_{k-1} by one accessible transition at the $\mathbb{G}$-predictable time $\overline{\tau}_k(h), h \in \mathcal{D}_k$.

Note that these random sets are determined at time τ_{k-1} in the insider filtration $\mathbb{G}$.

Theorem 6.1. *NFLVR is equivalent to the following condition:*

(1) (*Totally inaccessible jump part*): *$\mathbb{P}$-almost surely, for all k the constrained linear system* (6.1) *with $\mathcal{A} = \hat{\mathcal{Y}}^{(Y_{\tau_{k-1}})}_k(\omega)$ has strictly positive solutions.*
(2) (*Accessible jump part*): *$\mathbb{P}$-almost surely, for all k and all $\mathbb{G}$-predictable times $\overline{\tau}_k(h), h \in \mathcal{D}_k$, the homogeneous constrained linear system* (6.2) *with $\mathcal{A} = \check{\mathcal{Y}}^{(Y_{\tau_{k-1}})}_{k,h}(\omega)$ has strictly positive solutions.*

Proof. Any choice of positive solutions of the linear systems (6.1) and (6.2), for all k and all $h \in \mathcal{D}_k$ corresponds in the standard way to a $\mathbb{G}$-martingale measure $\mathbb{Q}$. □

Corollary 6.1. *Define*

$$\tau' := \min\{\tau_{k-1} : k \geq 1,\ \textit{the NFLVR-condition } (1) \textit{ fails}\}$$
$$\tau'' := \inf\{\overline{\tau}_k(h) : k \geq 0, h \in \mathcal{D}_k,\ \textit{and NFLVR-condition } (2) \textit{ fails}\}$$
$$\tau^{\mathrm{FLVR}} := (\tau' \wedge \tau'' \wedge T).$$

There are not arbitrage possibilities for a $\mathbb{G}$-insider who is restricted to trade in the interval $[0, \tau^{\mathrm{FLVR}})$, equivalently, there is an equivalent $\mathbb{G}$-martingale measure for the stopped process $(\widetilde{S}_{t \wedge \tau^{\mathrm{FLVR}}})$.

We describe an arbitrage strategy, for an insider which is allowed to trade after the $\mathbb{G}$-stopping times τ' or τ'', when $\mathbb{P}(\tau^{\mathrm{FLVR}} < \infty) > 0$.

(1) For simplicity assume that $\tau'(\omega) = \tau_{k-1}(\omega)$ and the next jump time τ_k is $\mathbb{G}$-totally inaccessible.
Let $Y_{\tau_{k-1}}(\omega) = e$. Since the nonhomogeneous system (6.1) has not strictly positive solutions, by the separating hyperplane theorem there is a vector

$\xi = (\xi_1, \dots, \xi_m)$ such that for

$$\sum_i \sum_{f \in \mathcal{A}} \xi_i \gamma^{ief} \lambda^{ef} + \sum_i \mu^{ie} \xi_i > 0 \tag{6.3}$$

for all vectors $(\lambda^{ef} : f \in \mathcal{A}) > 0$. This implies that

$$\sum_i \mu^{ie} \xi_i > 0, \quad \text{and} \quad \sum_i \xi_i \gamma^{ief} > 0 \quad \forall f \in \mathcal{A}.$$

In a discounted world, at any time $s \in [\tau_{k-1}, \tau_k)$, the insider starts to play and borrows $V = (\xi \cdot S_{\tau'})$ from the bank at zero interest rate, to buy a portfolio of stocks S with weights $(\xi_1, \dots, \xi_m)$. The insider sells its portfolio at any time $t \in (s, \tau_k]$ and pays back its debt to the bank. Whether $\{\tau_k = t\}$ or $\{\tau_k > t\}$, from condition (6.3) we see that the insider makes a positive profit.

(2) Next we discuss the insider's strategy at the time τ''. Assume that $\tau_{k-1}(\omega) < \tau''(\omega) \le \tau_k(\omega)$ and let $Y_{\tau_{k-1}}(\omega) = e$.
Since the homogeneous system (6.2) has not strictly positive solutions, by the separating hyperplane theorem there is a vector $\xi \in \mathbb{R}^m$ such that

$$\sum_i \sum_{f \in \mathcal{A}} \xi_i \gamma^{ief} \lambda^{ef} > 0$$

for all vectors $(\lambda^{ef} : f \in \mathcal{A}) > 0$. This implies

$$\sum_i \xi_i \gamma^{ief} > 0 \quad \forall f \in \mathcal{A}.$$

Since τ'' is $\mathcal{F}_{\tau_{k-1}}$-measurable, the insider chooses some $\varepsilon > 0$ small enough so that $\tau_{k-1}(\omega) < \tau''(\omega) - \varepsilon < \tau''(\omega) \le \tau_k(\omega)$, buys the portfolio $(\xi \cdot S_s)$ at time $s = (\tau'' - \varepsilon)$, and sells the portfolio at time τ'' after the jump, making a profit

$$\sum_i \sum_f \xi_i \gamma^{ief} \Delta N^{ef}_{\tau''} - \varepsilon \sum_i \xi_i \mu^{ie}.$$

Since ε is arbitrarily small the insider has a free lunch with vanishing risk, regardless of the sign of $(\xi \cdot \mu^e)$.

Example 6.1. Consider Example 2.1. Using the notation from Remark 3.3 we must compute

$$\begin{aligned}\bar{\nu}(d\ell, dt) &= \mathbb{P}(\vartheta \in d\ell, N(dt) = 1 \,|\, F_{t-}) \\ &= \mathbb{P}(\nu \in d\ell \,|\, F_{t-}) \mathbb{P}(N(dt) = 1 \,|\, F_{t-}, \vartheta \in d\ell)\end{aligned}$$

and

$$\widetilde{\nu}(d\ell, dt) = \mathbb{P}(\vartheta \in d\ell \,|\, F_{t-}) \Lambda(dt) = \mathbb{P}(\vartheta \in d\ell \,|\, F_{t-}) \mathbb{P}(N(dt) = 1 \,|\, F_{t-}).$$

Recall from Example 2.1 that in fact we have here two independent Poisson processes N^+ (respectively, N^-) counting the positive (respectively, negative) jumps. We have $\mathbb{P}(N^+(dt) = 1 \,|\, F_{t-}) = \lambda^+ dt$ and $\mathbb{P}(N^-(dt) = 1 \,|\, F_{t-}) = \lambda^- dt$.

To compute the conditional probability $\mathbb{P}(N^+(dt) = 1 \,|\, F_{t-}, \vartheta \in d\ell)$ recall that $\vartheta = L_0 + \mu T + \beta^+ N_T^+ + \beta^- N_T^-$. Assume that β^+ and β^- are such that with fixed $\vartheta = \ell$ the equation

$$\ell = \log(S_0) + \mu T + \beta^+ n^+ + \beta^- n^- \tag{6.4}$$

has a unique solution (n^+, n^-). We have then that

$$\mathbb{P}(N^+(dt) = 1 \,|\, F_{t-}, \vartheta \in d\ell) = \mathbb{P}(N^+(dt) = 1 \,|\, F_{t-}, N_T^+ = n^+).$$

Recall that for a Poisson process N the compensator of N in the filtration $\mathbb{F} \wedge \sigma(N_T)$ is

$$\mathbb{P}(N(dt) = 1 \,|\, F_{t-}, N_T) = \frac{N_T - N_{t-}}{T - t} dt$$

(see, for example, [4]). This gives

$$\mathbb{P}(N^+(dt) = 1 \,|\, F_{t-}, \vartheta \in d\ell) = \mathbb{P}(N^+(dt) = 1 \,|\, F_{t-}, N_T^+) = \frac{N_T^+ - N_{t-}^+}{T - t} dt.$$

In this case there is always arbitrage, after the last jump of N^+ or N^-.

Note that in the special case of $\frac{\beta^+}{-\beta^-} = \frac{k_1}{k_2}$ for some $k_1, k_2 \in \mathbb{N}$, Eq. (6.4) does not uniquely determine the pair (n^+, n^-), and then there might also be no-arbitrage. We refer to [8] for a detailed analysis of this model using the expected utility of the insider.

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