Gaussian Bridges

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Summary. We consider Gaussian bridges; in particular their dynamic representations. We prove a Girsanov theorem between the law of Gaussian bridge and the original Gaussian process, which holds with natural assumptions. With some additional conditions we obtain dynamical representation for a Gaussian bridge. We discuss briefly the initial enlargement of filtrations in this context.

Mathematics Subject Classification (2000): 60G15, 60G18, 60G25, 60G44

Keywords and Phrases: Gaussian processes, Brownian bridge, pinned Gaussian processes, tied down Gaussian processes, enlargement of filtration, fractional Brownian motion, fractional Brownian bridge

1 Introduction

Motivation

Let X be a continuous Gaussian process such that $X_0 = 0$ and $\mathbf{E}(X_t) = 0$. Fix $T > 0$ and define the bridge of X $U^{T,0}$ by

$$U_t^{T,0} = X_t - \frac{t}{T} X_T. \tag{1}$$

It is clear that the process $U^{T,0}$ is a Gaussian process. Moreover, it is a bridge in the sense that $U_0^{T,0} = U_T^{T,0} = 0$. If X is a standard Brownian motion, then it is known that the law of the process $U^{T,0}$ defined by (1) is the same as the conditional law of the standard Brownian motion:

$$\mathbf{P} - \mathrm{Law}\left((X_t)_{0\le t\le T} | X_T = 0\right) = \mathbf{P} - \mathrm{Law}\left(\left(U_t^{T,0}\right)_{0\le t\le T}\right).$$

It is well-known that in the case of standard Brownian motion the bridge process $U^{T,0}$ has a representation as solution to the differential equation (2). We refer to the next subsection for more information on Brownian bridge.

We study the properties of the bridge process $U^{T,0}$ of X in the case of arbitrary Gaussian process X. We define the bridge process using the conditional law of X. It turns out that it is quite easy to obtain the analog of (1) for the arbitrary Gaussian process X; see Proposition 4 for the exact result. If the Gaussian process X is a martingale, then it is quite easy to to describe the bridge process $U^{T,0}$ as a solution to a differential equation analogous to (2). But if the process X is a fractional Brownian motion, then the corresponding differential equation contains Volterra operators.

Representations for the Brownian Bridge

Fix $T > 0$ and let $W = (W_t)_{t\in[0,T]}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ starting from $W_0 = \xi$.

Let (T, θ) be a "conditioning". Then the notation $W^{T,\theta}$ means that the process W is conditioned to be θ at time T. That is $W^{T,\theta}$ is a bridge from $(0, \xi)$ to (T, θ).

For the Brownian bridge $W^{T,\theta}$ from $(0, \xi)$ to (T, θ) one finds in the literature the following three equivalent definitions

$$\mathrm{d}Y_t^{T,\theta} = \mathrm{d}W_t + \frac{\theta - Y_t^{T,\theta}}{T-t}\,\mathrm{d}t, \qquad Y_0^{T,\theta} = \xi, \tag{2}$$

$$Y_t^{T,\theta} = \xi + (\theta - \xi)\frac{t}{T} + (T-t)\int_0^t \frac{\mathrm{d}W_s}{T-s}, \tag{3}$$

$$W_t^{T,\theta} = \theta\frac{t}{T} + \left(W_t - \frac{t}{T}W_T\right). \tag{4}$$

The representation (3) is just the solution of the (stochastic or pathwise) differential equation (2). So, the equations (2) and (3) define the same process $Y^{T,\theta}$. The equation (4), however, does not define the same process as the equations (2) and (3). The equality between representations (2)–(3) and (4) is only an equality in law: $\mathrm{Law}(Y^{T,\theta}; \mathbf{P}) = \mathrm{Law}(W^{T,\theta}; \mathbf{P})$. That the processes $Y^{T,\theta}$ and $W^{T,\theta}$ are different is obvious from the fact that the process $Y^{T,\theta}$ is adapted to the filtration of W while the process $W^{T,\theta}$ is not. Indeed, to construct $W_t^{T,\theta}$ by using (4) we need information of the random variable W_T. The fact that the two processes $Y^{T,\theta}$ and $W^{T,\theta}$ have the same law is also obvious, since they have the same covariance and expectation. It is also worth noticing that if the Brownian bridge $Y^{T,\theta}$ is given by the equation (3) then the original Brownian motion W may be recovered from the bridge $W^{T,\theta}$ by using the equation (2). In particular, this means that in this case the filtration of this Brownian bridge is the same as the filtration of the Brownian motion: $\mathbf{F}^{Y^{T,\theta}} = \mathbf{F}^W$.

The non-adapted representation (4) comes from the orthogonal decomposition of Gaussian variables. Indeed, the conditional law of process $(W_t)_{t\in[0,T]}$ given the variable W_T is Gaussian with

$$\mathbf{E}(W_t|W_T) = \frac{t}{T}(W_T - \xi) + \xi,$$
$$\mathbf{Cov}(W_t, W_s|W_T) = t \wedge s - \frac{ts}{T}.$$

The second-order structure of the Brownian bridge is easily calculated from the representation (4):

$$\mathbf{E}\left(W_t^{T,\theta}\right) = \xi + (\theta - \xi)\frac{t}{T}, \tag{5}$$

$$\mathbf{Cov}\left(W_t^{T,\theta}, W_s^{T,\theta}\right) = t \wedge s - \frac{ts}{T}. \tag{6}$$

Girsanov Theorem and Brownian Bridge

We know that Brownian bridge is defined only up to distribution. Put $\mathbf{P}^{T,\theta} := \mathrm{Law}(W^{T,\theta}; \mathbf{P})$. We have that $\mathbf{P}^{T,\theta} = \mathbf{P}(\cdot|W_T = \theta)$, where $\mathbf{P}$ is the law of the Brownian motion W. Consider now the restrictions of the measures $\mathbf{P}^{T,\theta}$ and $\mathbf{P}$ on the sigma-algebra $\mathcal{F}_t^W$: denote the restriction by $\mathbf{P}_t$ and $\mathbf{P}_t^{T,\theta}$. We know that $\mathbf{P}_t^{T,\theta} \sim \mathbf{P}_t$ for all $t \in [0, T)$, but, of course, $\mathbf{P}_T^{\theta} \perp \mathbf{P}_T$. From (2) we get, by Girsanov theorem, that

$$\frac{d\mathbf{P}_t^{T,\theta}}{d\mathbf{P}_t} = \exp\left(\int_0^t \frac{\theta - Y_s^{T,\theta}}{T-s} dW_s - \frac{1}{2}\int_0^t \left(\frac{\theta - Y_s^{T,\theta}}{T-s}\right)^2 ds\right).$$

This is a key observation for the non-anticipative representation in the general case.

Non-anticipative and Anticipative Representations

Let now $X = (X_t)_{t\in[0,T]}$ be a Gaussian process on $(\Omega, \mathcal{F}, \mathbf{P})$ with $X_0 = \xi$. We want to understand what is the corresponding bridge $X^{T,\theta}$ from $(0, \xi)$ to (T, θ). If one merely replaces the Brownian motion W with the process X in representations (2)–(4) then the “X-bridges” obtained from the first two representations of course coincide. However, the bridge obtained from the last one does not coincide with the first two ones. The following example, communicated to us by M. Lifshits, elaborates this point.

Example 1. Let $(f_n)_{n\geq 1}$ be a sequence of smooth isomorphisms of $[0, T]$ onto itself. Take

$$X_{n,t} := W_{f_n(t)}$$

and set

$$X_{n,t}^{1,T,\theta} := \theta\frac{t}{T} + X_{n,t} - \frac{t}{T}X_{n,T},$$

$$X_{n,t}^{2,T,\theta} := \theta\frac{t}{T} + (T-t)\int_0^t \frac{\mathrm{d}X_{n,s}}{T-s}.$$

Then

$$\begin{aligned}\mathbf{Cov}_{n,1}(s,t) &:= \mathbf{Cov}\left(X_{n,t}^{1,T,\theta}, X_{n,s}^{1,T,\theta}\right) \\ &= f_n(s\wedge t) + st - sf_n(t) - tf_n(s), \\ \mathbf{Cov}_{n,2}(s,t) &:= \mathbf{Cov}\left(X_{n,t}^{2,T,\theta}, X_{n,s}^{2,T,\theta}\right) \\ &= (T-t)(T-s)\int_0^{s\wedge t} \frac{\mathrm{d}f_n(u)}{(T-u)^2}.\end{aligned}$$

The covariances $\mathbf{Cov}_{n,1}$ and $\mathbf{Cov}_{n,2}$ are not the same in general. Indeed, let $f_n \to \mathbf{1}_{\{1\}}$. Then for all $s,t < 1$ we have that as $n \to \infty$, $\mathbf{Cov}_{n,1}(s,t) \to st$ while $\mathbf{Cov}_{n,2}(s,t) \to 0$.

Structure of the Paper

We will study Gaussian bridges. After the definition of Gaussian bridge we obtain the anticipative representation of the Gaussian bridge, which is a generalisation of the representation (4). Next we give the density between the bridge measure $\mathbf{P}^{T,\theta}$ and the original measure $\mathbf{P}$ and give an abstract version of the non-anticipative representation (3) in the general setup. In the section three we study bridges of Gaussian martingales, and this part is an easy generalisation of the Brownian bridge. In the next sections we study bridges of certain special Gaussian processes: Wiener predictable process, Volterra process and fractional Brownian motion. We end the paper by giving the connection to the enlargement of filtrations theory, where the enlargement is an initial enlargement with the final value of the Gaussian process X_T.

2 Gaussian Bridges in General

2.1 Definition of the X-bridge

The fact that for Brownian motion the Brownian bridge in unique up to law only suggests the following definition in the case of an arbitrary Gaussian process.

Definition 2. *Let X be a Gaussian stochastic process with $X_0 = \xi$. Then the Gaussian process $X^{T,\theta}$ is an X-*bridge *from $(0,\xi)$ to (T,θ) if*

$$\mathrm{Law}\big(X^{T,\theta};\mathbf{P}\big) \;=\; \mathrm{Law}\big(X;\mathbf{P}^{T,\theta}\big), \tag{1}$$

where the measure $\mathbf{P}^{T,\theta}$ on $(\Omega,\mathcal{F})$ is defined by

$$\mathbf{P}^{T,\theta} \;=\; \mathbf{P}(\,\cdot\,|X_T=\theta). \tag{2}$$

Remark 3. The definition above assumes that the process $X^{T,\theta}$ exists in the *original* space $(\Omega,\mathcal{F},\mathbf{P})$. Also, we have

$$1 \;=\; \mathbf{P}(X_T=\theta|X_T=\theta) \;=\; \mathbf{P}^{T,\theta}(X_T=\theta) \;=\; \mathbf{P}(X_T^{T,\theta}=\theta),$$

as we should. Note that in (2) we condition on a set of zero measure. However, we can define (2) as a regular conditional distribution in the Polish space of continuous functions on $[0,T]$ (see Shiryaev [9, pp. 227–228]).

In what follows we denote by μ and R the mean and covariance of X, respectively.

2.2 Anticipative Representation

The anticipative representation corresponding to (4) is easily obtained from the orthogonal decomposition of X with respect to X_T. Indeed, $\mathrm{Law}(X|X_T)$ is Gaussian with

$$\mathbf{E}(X_t|X_T) = \big(X_T-\mu(T)\big)\frac{R(T,t)}{R(T,T)} + \mu(t),$$
$$\mathbf{Cov}(X_t,X_s|X_T) = R(t,s) - \frac{R(T,t)R(T,s)}{R(T,T)}.$$

Thus, we have an anticipative representation for any Gaussian bridge.

Proposition 4. *Let X be a Gaussian process with mean μ and covariance R. Then the X-bridge $X^{T,\theta}$ from $(0,\mu(0))$ to (T,θ) admits a representation*

$$\begin{aligned} X_t^{T,\theta} &= \theta\frac{R(T,t)}{R(T,T)} + X_t^{T,0} \\ &= \theta\frac{R(T,t)}{R(T,T)} + \left(X_t - \frac{R(T,t)}{R(T,T)}X_T\right). \end{aligned} \tag{3}$$

Moreover,

$$\mathbf{E}\Big(X_t^{T,\theta}\Big) = \big(\theta-\mu(T)\big)\frac{R(T,t)}{R(T,T)} + \mu(t), \tag{4}$$

$$\mathbf{Cov}\Big(X_t^{T,\theta},X_s^{T,\theta}\Big) = R(t,s) - \frac{R(T,t)R(T,s)}{R(T,T)}. \tag{5}$$

Example 5. Let X be a centered fractional Brownian motion. The bridge process $Z_t := X_t - \frac{t}{T}X_T$ is a H- self similar process, but it is not a 'fractional Brownian bridge' in the sense of Definition 2.

The correct fractional Brownian bridge in the sense of the Definition 2 is

$$X_t^{T,\theta} = X_t - \frac{t^{2H} + T^{2H} - |t-T|^{2H}}{2T^{2H}} X_T.$$

X-bridge and Drift

Let ${}^aW$ is a Brownian motion with drift $a \in \mathbb{R}$, i.e. $W_t := {}^aW_t - at$ is a standard Brownian motion starting from ξ. Then from (4) it easy to see that the Brownian bridge is invariant under this drift: ${}^aW^{T,\theta} = W^{T,\theta}$.

Consider now a general centered Gaussian process X, and let μ be a deterministic function with $\mu(0) = 0$. Define ${}^\mu X$ by ${}^\mu X_t := X_t + \mu(t)$. Transform ${}^\mu X$ to ${}^\mu X^{T,\theta}$ by (3). Then ${}^\mu X$ is a Gaussian process with the same covariance R as X and with mean μ. When does ${}^\mu X^{T,\theta}$ define the same bridge as $X^{T,\theta}$ in the sense of Definition 2? From (3) it follows that an invariant mean function μ must satisfy the equation

$$\mu(t) \quad = \quad \frac{R(T,t)}{R(T,T)}\mu(T).$$

So, such an invariant mean μ may depend on the time of the conditioning T. Indeed,

$$\mu(t) \quad = \quad \mu_T(t) \quad = \quad aR(T,t)$$

for some $a \in \mathbb{R}$. In particular, we see that μ is independent of T if and only if

$$R(t,s) \quad = \quad f(t \wedge s)$$

for some function f. But this means that X has independent increments, or in other words that $X - \mathbf{E}(X)$ is a martingale.

X-bridge and Self-similarity

The Brownian motion W starting from $W_0 = 0$ is 1/2-self-similar. i.e.

$$\mathrm{Law}\Big((W_t)_{t\in[0,T]}\ ; \mathbf{P}\Big) \quad = \quad \mathrm{Law}\bigg(\Big(T^{1/2}W_\tau\Big)_{\tau\in[0,1]}\ ; \mathbf{P}\bigg).$$

Consequently, we have for the Brownian bridge the scaling property

$$\mathrm{Law}\bigg(\Big(W_t^{T,\theta}\Big)_{t\in[0,T]}\ ; \mathbf{P}\bigg) \quad = \quad \mathrm{Law}\bigg(\Big(T^{1/2}W_\tau^{1,\theta T^{-1/2}}\Big)_{\tau\in[0,1]}\ ; \mathbf{P}\bigg).$$

From (3) it is easy to see that if the process $X = (X_t)_{t\in[0,T]}$ is H-self-similar, i.e.

$$\mathrm{Law}\Big((X_t)_{t\in[0,T]}\ ;\mathbf{P}\Big) \;=\; \mathrm{Law}\Big(\big(T^H X_\tau\big)_{\tau\in[0,1]}\ ;\mathbf{P}\Big)$$

then the corresponding bridge satisfies the scaling property

$$\mathrm{Law}\bigg(\Big(X_t^{T,\theta}\Big)_{t\in[0,T]}\ ;\mathbf{P}\bigg) \;=\; \mathrm{Law}\bigg(\Big(T^H X_\tau^{1,\theta T^{-H}}\Big)_{\tau\in[0,1]}\ ;\mathbf{P}\bigg).$$

So, we may represent the bridge $X^{T,\theta}$ as

$$\begin{aligned} X_t^{T,\theta} &= X_s^{1,\theta T^{-H}} \\ &= \theta\frac{R(1,\tau)}{R(1,1)} + T^H X_\tau - \frac{R(1,\tau)}{R(1,1)} T^H X_1, \end{aligned}$$

where $\tau = t/T \in [0,1]$.

2.3 Density Between the Bridge Measure $\mathbf{P}^{T,\theta}$ and P

When we look for analogies for the non-anticipative, or dynamic, representation (3) and the corresponding differential equation (2), then the main idea is to work with the prediction martingale of X and to use the Girsanov's theorem.

We introduce some notation. Let $X^{T,\theta}$ and $\mathbf{P}^{T,\theta}$ be as in (1) and (2). Let $\hat{X}_{T|\,\cdot} = (\hat{X}_{T|\,t})_{t\in[0,T]}$ be the prediction martingale of X. I.e.

$$\hat{X}_{T|t} \;:=\; \mathbf{E}\big(X_T|\mathcal{F}_t^X\big).$$

For the incremental bracket of the Gaussian martingale $\hat{X}_{T|\,\cdot}$ we use the shorthand notation

$$\begin{aligned} \langle \hat{X}_{T|\,\cdot}\rangle_{T,t} &:= \langle \hat{X}_{T|\,\cdot}\rangle_T - \langle \hat{X}_{T|\,\cdot}\rangle_t \\ &:= \langle \hat{X}_{T|\,\cdot}, \hat{X}_{T|\,\cdot}\rangle_T - \langle \hat{X}_{T|\,\cdot}, \hat{X}_{T|\,\cdot}\rangle_t. \end{aligned}$$

(Note that since $\hat{X}_{t|\,\cdot}$ is a Gaussian martingale it has independent increments, and consequently its bracket $\langle \hat{X}_{T|\,\cdot}\rangle$ is deterministic.) Denote

$$\mathbf{P}_t \;:=\; \mathbf{P}|\mathcal{F}_t^X \qquad \text{and} \qquad \mathbf{P}_t^{T,\theta} \;:=\; \mathbf{P}^{T,\theta}|\mathcal{F}_t^X.$$

Let α_T^t denote the regular conditional law of X_T given the information $\mathcal{F}_t^X$ and let $\alpha_T = \alpha_T^0$ be the law of X_T. So, if p denotes the Gaussian density

$$p(\theta;\mu,\sigma^2) \;=\; \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2},$$

it is easy enough to see that

$$\alpha_T^t(\mathrm{d}\theta) = p\left(\theta; \hat{X}_{T|t}, \langle \hat{X}_{T|\cdot} \rangle_{T,t}\right) \mathrm{d}\theta,$$
$$\alpha_T(\mathrm{d}\theta) = p\left(\theta; \mu(T), \langle \hat{X}_{T|\cdot} \rangle_{T}\right) \mathrm{d}\theta.$$

Now, by using Bayes' rule we have that

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}_t^{T,\theta}}{\mathrm{d}\mathbf{P}_t} &= \frac{\mathrm{d}\alpha_T^t}{\mathrm{d}\alpha_T}(\theta) \\ &= \frac{p\left(\theta; \hat{X}_{T|t}, \langle \hat{X}_{T|\cdot} \rangle_{T,t}\right)}{p\left(\theta; \mu(T), \langle \hat{X}_{T|\cdot} \rangle_{T}\right)} \\ &= \sqrt{\frac{\langle \hat{X}_{T|\cdot} \rangle_{T}}{\langle \hat{X}_{T|\cdot} \rangle_{T,t}}} \exp\left(-\frac{1}{2} \frac{\left(\theta - \hat{X}_{T|t}\right)^2}{\langle \hat{X}_{T|\cdot} \rangle_{T,t}} + \frac{1}{2} \frac{\left(\theta - \mu(T)\right)^2}{\langle \hat{X}_{T|\cdot} \rangle_{T}} \right). \end{aligned} \tag{6}$$

Since we want to use the Girsanov's theorem later we need to assume that the prediction martingale $\hat{X}_{T|\cdot}$ is continuous. Another way of stating this assumption is the following:

(A0) The history of X is continuous, i.e. $\mathcal{F}_{t-}^X = \mathcal{F}_{t+}^X$; here $\mathcal{F}_{t-} = \vee_{s<t}\mathcal{F}_s$ and $\mathcal{F}_{t+} = \cap_{u>t}\mathcal{F}_u$.

Also, in order for the calculations above to make sense we need the to assume that $\mathbf{P}_t^{T,\theta} \ll \mathbf{P}_t$ for all $t < T$. Or, since the both measures are Gaussian, we may as well assume that:

(A1) $\mathbf{P}_t \sim \mathbf{P}_t^{T,\theta}$ for all $t < T$.

From equation (6) we see that assumption (A1) says that $\langle \hat{X}_{T|\cdot} \rangle_t < \langle \hat{X}_{T|\cdot} \rangle_T$ for all $t < T$. So, another way of stating assumption (A1) is that the value of X_T cannot be predicted for certain by using the information $\mathcal{F}_t^X$ only. Indeed,

$$\langle \hat{X}_{T|\cdot} \rangle_{T,t} = \mathbf{Var}\left(\hat{X}_{T|t}\right)$$

is the prediction error of $\hat{X}_{T|t}$. Let us note that in general the measures $\mathbf{P}_T$ and $\mathbf{P}_T^{T,\theta}$ are of course singular, since $X^{T,\theta}$ is degenerate at T.

In what follows $\beta_{T,\theta}$ is a non-anticipative functional acting on Gaussian (prediction) martingales m:

$$\beta_{T,\theta}(m)_t := \frac{\theta - m_t}{\langle m \rangle_{T,t}}.$$

The following proposition is the key tool in finding a non-anticipative representation.

Proposition 6. *Let X be a Gaussian process on $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the assumptions (A0) and (A1). Then the bridge measure $\mathbf{P}^{T,\theta}$ on $(\Omega, \mathcal{F})$ may be represented as*

$$\mathrm{d}\mathbf{P}_t^{T,\theta} \;=\; L_t^{T,\theta}\,\mathrm{d}\mathbf{P}_t,$$

where

$$L_t^{T,\theta} \;=\; \exp\left(\int_0^t \beta_{T,\theta}\big(\hat{X}_{T|\,\cdot}\big)_s\,\mathrm{d}\hat{X}_{T|s} - \frac{1}{2}\int_0^t \beta_{T,\theta}\big(\hat{X}_{T|\,\cdot}\big)_s^2\,\mathrm{d}\langle \hat{X}_{T|\,\cdot}\rangle_s\right).$$

Proof. The claim follows from equation (6). Indeed, just use Itô's formula with the martingale $\hat{X}_{T|\,\cdot}$ to the function

$$g(t,x) \;:=\; -\frac{1}{2}\frac{(\theta - x)^2}{\langle \hat{X}_{T|\,\cdot}\rangle_{T,t}},$$

and there you have it. □

2.4 Non-anticipative Representation

In order to come back from the "prediction martingale level" to the actual process we still need one assumption.

(A2) The non-anticipative linear mapping F_T sending the path of the Gaussian process X to the path of its prediction martingale $\hat{X}_{T|\,\cdot}$ is injective.

The assumption (A2) says simply that the process X may be recovered from $\hat{X}_{T|\,\cdot}$ by $X = F_T^{-1}\big(\hat{X}_{T|\,\cdot}\big)$. Also, note that the assumption (A2) implies that the prediction filtration and the original filtration are the same: $\mathbf{F}^X = \mathbf{F}^{\hat{X}_{T|\,\cdot}}$.

Let m be a Gaussian martingale. We denote by $S_{T,\theta}(m)$ the unique solution of the differential equation

$$\mathrm{d}m_t^{T,\theta} \;=\; \mathrm{d}m_t + \beta_{T,\theta}\big(m^{T,\theta}\big)_t\,\mathrm{d}\langle m^{T,\theta}\rangle_t \tag{7}$$

with initial condition $m_0^{T,\theta} = \zeta$, i.e.

$$\begin{aligned} m_t^{T,\theta} &= S_{T,\theta}(m)_t \\ &= \zeta + (\theta - \zeta)\frac{\langle m^{T,\theta}\rangle_t}{\langle m^{T,\theta}\rangle_T} + \langle m^{T,\theta}\rangle_{T,t}\int_0^t \frac{\mathrm{d}m_s}{\langle m^{T,\theta}\rangle_{T,s}}. \end{aligned}$$

In order to see that $S_{T,\theta}(m)$ is indeed the solution to (7) one just uses the integration by parts. It is also worth noticing that classical theory of differentiation applies here: The differential equation (7) may be understood pathwise. Finally, note that by the Girsanov's theorem the brackets of the Gaussian martingales m and $m^{T,\theta}$ coincide: $\langle m\rangle = \langle m^{T,\theta}\rangle$.

Let us now abuse the notation slightly and set

$$S_T(m) \;\; := \;\; S_{T,0}(m).$$

Then we have the decomposition

$$S_{T,\theta}(m) \;\; = \;\; \theta K_T(m) + S_T(m),$$

where

$$K_T(m) \;\; := \;\; \frac{\langle m \rangle}{\langle m \rangle_T}$$

and S_T is independent of θ.

The following theorem is the analogy of the non-anticipative representation (3).

Theorem 7. *Let X be a Gaussian process with mean μ and covariance R satisfying (A0), (A1) and (A2). Then the bridge $X^{T,\theta}$ from $(0, \mu(0))$ to (T, θ) admits the non-anticipative representation*

$$X_t^{T,\theta} = \left(F_T^{-1} S_{T,\theta} F_T\right)(X)_t \tag{8}$$

$$= \theta \left(F_T^{-1} K_T F_T\right)(X)_t + \left(F_T^{-1} S_T F_T\right)(X)_t \tag{9}$$

$$= \theta \frac{R(T,t)}{R(T,T)} + X_t^{T,0}. \tag{10}$$

Moreover, the original process X may be recovered from the bridge $X^{T,\theta}$ by

$$X_t \;\; = \;\; \left(F_T^{-1} S_{T,\theta}^{-1} F_T\right) \left(X^{T,\theta}\right)_t . \tag{11}$$

Proof. Let us first prove the equations (8)–(10). By the equation (3) we already know the contribution coming from θ. Indeed, we must have

$$\left(F_T^{-1} K_T F_T\right)(X)_t \;\; = \;\; \theta \frac{R(T,t)}{R(T,T)}.$$

So, we may assume that $\theta = 0$ and consider the corresponding bridge $X^{(T,0)}$. Now, we map X to its prediction martingale $F_T(X)$. Then $(S_T F_T)(X)$ is the solution of the stochastic differential equation (7) with $m = F_T(X)$ and the initial condition $\zeta = \mu(T)$. Consequently, the Girsanov's theorem and Proposition 6 tells us that

$$\mathrm{Law}\Big(\left(S_T F_T\right)(X) \, ; \, \mathbf{P} \Big) \;\; = \;\; \mathrm{Law}\big(F_T(X) \, ; \, \mathbf{P}^{T,\theta} \big) . \tag{12}$$

So, the claim (8) follows simply by recovering the process X by using the map F_T^{-1} on both sides of the equation (12).

The equation (11) is now obvious, since $S_{T,\theta}$ is invertible. Indeed,

$$\begin{aligned} S_{T,\theta}^{-1}\left(F_T\left(X^{T,\theta}\right)\right)_t &= S_{T,\theta}^{-1}\left(\hat{X}_{T|\,\cdot}^{T,\theta}\right)_t \\ &= \hat{X}_{T|\,t}^{T,\theta} + \int_0^t \beta\left(\hat{X}_{T|\,\cdot}^{T,\theta}\right)_s \mathrm{d}\langle \hat{X}_{T|\,\cdot}^{T,\theta}\rangle_s \end{aligned}$$

This finishes the proof. □

Remark 8. For the differential equation (2) have the following formal analogy

$$X_t^{T,\theta} \quad = \quad X_t + F_T^{-1}\left(\int_0^s \frac{\theta - F_T(X^{T,\theta})_u}{\langle F_T(X)\rangle_{T,u}}\,\mathrm{d}\langle F_T(X)\rangle_u\,;\, s \le t\right)_t .$$

In the following sections we consider some special Gaussian bridges and give the somewhat abstract Theorem 7, and highly abstract Remark 8, more concrete forms. In particular, we consider cases where the operators F_T and F_T^{-1} may be represented as Wiener integrals.

3 Bridges of Gaussian Martingales

The case of Gaussian martingales is extremely simple. Indeed, the analogy to the Brownian case is complete.

Proposition 9. *Let M be a continuous Gaussian martingale with strictly increasing bracket $\langle M\rangle$ and $M_0 = \xi$. Then the M-bridge $M^{T,\theta}$ admits the representations*

$$\mathrm{d}M_t^{T,\theta} = \mathrm{d}M_t + \frac{\theta - M_t^{T,\theta}}{\langle M\rangle_{T,t}}\,\mathrm{d}\langle M\rangle_t, \qquad M_0^{T,\theta} = \xi, \tag{1}$$

$$M_t^{T,\theta} = \xi + (\theta - \xi)\frac{\langle M\rangle_t}{\langle M\rangle_T} + \langle M\rangle_{T,t}\int_0^t \frac{\mathrm{d}M_s}{\langle M\rangle_{T,s}}, \tag{2}$$

$$M_t^{T,\theta} = \theta\frac{\langle M\rangle_t}{\langle M\rangle_T} + \left(M_t - \frac{\langle M\rangle_t}{\langle M\rangle_T}M_T\right). \tag{3}$$

Moreover, we have

$$\begin{aligned} \mathbf{E}M_t^{T,\theta} &= \xi + (\theta - \xi)\frac{\langle M\rangle_t}{\langle M\rangle_T}, \\ \mathbf{Cov}(M_t^{T,\theta}, M_s^{T,\theta}) &= \langle M\rangle_{t\wedge s} - \frac{\langle M\rangle_t\langle M\rangle_s}{\langle M\rangle_T}. \end{aligned}$$

Proof. Since M is continuous and $\langle M\rangle$ is strictly increasing the assumption (A0) and (A1) are satisfied. The assumption (A2) is trivial in this case. Now, the solution of (1) is (2) and this is just the equation (8) where F_T is the identity operator. Representation (3) as well as the mean and covariance functions come from the representation (3). Indeed, for Gaussian martingales we have $R(t,s) = \langle M\rangle_{t\wedge s}$. □

Remark 10. Actually, one can deduce the result of Proposition 9 without using the "Bayes–Itô–Girsanov machinery" introduced in Section 2. Indeed, the result follows quite easily from equations (2)–(4) and the representation of the Gaussian martingale M as the time-changed Brownian motion $W_{\langle M\rangle}$.

4 Bridges of Wiener Predictable Processes

Let us first consider abstract Wiener-integration with respect to Gaussian processes. The *linear space* $\mathcal{H}_t$ of a Gaussian process X is the closed Gaussian subspace of $L^2(\Omega,\mathcal{F},\mathbf{P})$ generated by the random variables X_s, $s \le t$. For the prediction martingale of X it is well known that $\hat{X}_{T|t} \in \mathcal{H}_t$. Let $\mathcal{E}_t$ denote the space of elementary functions over $[0,t]$ equipped with the inner product generated by the covariance of X:

$$\left\langle\!\left\langle \mathbf{1}_{[0,s)}, \mathbf{1}_{[0,u)} \right\rangle\!\right\rangle \;:=\; R(s,u),$$

Let Λ_t be the completion of $\mathcal{E}_t$ in the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. Now the mapping

$$\mathcal{I}_t : \mathbf{1}_{[0,s)} \mapsto X_s$$

extends to an isometry between Λ_t and $\mathcal{H}_t$. We call this extension the *abstract Wiener integral.*

Alas, the space Λ_t is not in general a space of functions (or more precisely a space of equivalence classes of functions). However, we can find a subspace of it whose elements may be identified as (equivalence classes) of functions. Viz. the space $\tilde{\Lambda}_t$ which consists of such function f that

$$\sup_{\pi} \sum_{s_i,s_j\in\pi} f(s_{i-1})f(s_{j-1}) \left\langle \mathbf{1}_{[s_{i-1},s_i)}, \mathbf{1}_{[s_{j-1},s_j)} \right\rangle \;<\; \infty.$$

Here the supremum is taken over all partitions π of the interval $[0,t]$. The reason to take a supremum instead of letting the mesh of the partition go to zero is that the $\langle\!\langle\cdot,\cdot\rangle\!\rangle$-norm of a function may increase when multiplied by an indicator function. For details of this phenomenon in the case of fractional Brownian motion see Bender and Elliot [1].

If $f \in \tilde{\Lambda}_t$ then we write

$$\int_0^t f(s)\,\mathrm{d}X_s \;:=\; \mathcal{I}_t[f]. \tag{1}$$

So, the Wiener integral (1) of a function $f \in \tilde{\Lambda}_t$ is defined as a $\langle\!\langle \cdot, \cdot \rangle\!\rangle$-limit of simple functions. Note that if $t \leq T$ then $\tilde{\Lambda}_t \subset \tilde{\Lambda}_T$ and $\mathcal{I}_t[f] = \mathcal{I}_T\left[f \mathbf{1}_{[0,t)}\right]$ for $f \in \tilde{\Lambda}_T$.

Since the operator F_T is linear and non-anticipative we have

$$\hat{X}_{T|t} \;=\; \mathcal{I}_t\left[p_{T,t}\right]$$

for some $p_{T,t} \in \Lambda_t$. We assume now that this prediction kernel $p_{T,t}$ is actually a function in $\tilde{\Lambda}_t$:

(A3) There exists a Volterra kernel p_T such that $p_T(t,\cdot) \in \tilde{\Lambda}_t$ for all t and m may be represented as the Wiener integral

$$\hat{X}_{T|t} \;=\; \int_0^t p_T(t,s)\,\mathrm{d}X_s. \tag{2}$$

Representation (2) suggests that, if we are lucky enough, the inverse operator F_T^{-1} may be represented as a Wiener integral with respect to $\hat{X}_{T|\cdot}$. This is the meaning of the next assumption we make.

(A4) There exists a Volterra kernel p_T^* such that the original Gaussian process X may be reconstructed from the prediction martingale m as a Wiener integral

$$X_t \;=\; \int_0^t p_T^*(t,s)\,\mathrm{d}\hat{X}_{T|s}. \tag{3}$$

Remark 11. The Wiener integral in (A4) may understood as an abstract Wiener integral or, as well, as the stochastic integral with respect to the martingale m. Indeed, in this case Λ_t is the function space $L^2([0,t],\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle)$. Also, assumption (A4) gives us an alternative way of defining the Wiener integral (2). Indeed, let the operator P_T^* be the linear extension of the map $\mathbf{1}_{[0,t)} \mapsto p_T^*(t,\cdot)$. Then the assumption (A3) may be restated as:

> The operator P_T^* has the indicator functions $\mathbf{1}_{[0,t)}$, $t \in (0,T]$, in its image.

In this case we may define Wiener integrals with respect to X as

$$\int_0^t f(s)\,\mathrm{d}X_s \;:=\; \int_0^t P_T^*[f](s)\,\mathrm{d}\hat{X}_{T|s}$$

for such f that $P_T^*[f] \in L^2([0,t],\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle)$. Moreover, in this case

$$p_T(t,\cdot) \;=\; (P_T^*)^{-1}\left[\mathbf{1}_{[0,t)}\right].$$

Indeed, this is the approach taken in the next section.

Remark 12. Obviously (A4) implies (A2). Also, we have implicitly assumed that X is centred with $X_0 = 0$. However, adding a mean function to X causes no difficulties. Indeed, let $\tilde{m}$ be the prediction martingale of the centred process $X - \mu$ and let $\tilde{p}_T$ and $\tilde{p}^*_T$ be the kernels associated to this centred process. Then

$$\hat{\tilde{X}}_{T|\,t} = \hat{X}_{T|\,t} - \mu(T),$$

$$X_t = \int_0^t \tilde{p}^*_T(t,s)\,\mathrm{d}\hat{X}_{T|\,s} + \mu(t),$$

$$\hat{X}_{T|\,t} = \int_0^t \tilde{p}_T(t,s)\,\mathrm{d}(X_s - \mu(s)) + \mu(T).$$

Remark 13. The relation (3) says that the covariance R of X may be written as

$$R(t,s) \;=\; \int_0^{t\wedge s} p^*_T(t,u)p^*_T(s,u)\;\mathrm{d}\langle \hat{X}_{T|\,\cdot}\rangle_u. \tag{4}$$

So, p^*_T is a "square root" of R. Note, however, that in general a decomposition like (4) is by no means unique, even if the measure is given. This means that from an equation like (4) we cannot deduce the kernel p^*_T even if we knew the measure $\mathrm{d}\langle \hat{X}_{T|\,\cdot}\rangle$ induced by the bracket $\langle \hat{X}_{T|\,\cdot}\rangle$.

We have the following analogue of representations (2) and (3).

Proposition 14. *Let X be a Gaussian process with covariance R satisfying (A0), (A1), (A3) and (A4). Then the bridge $X^{T,\theta}$ satisfies the integral equation*

$$X^{T,\theta}_t \;=\; X_t + \int_0^t \left\{\theta - \int_0^s p_T(s,u)\,\mathrm{d}X^{T,\theta}_u\right\} \frac{p^*_T(t,s)}{\langle \hat{X}_{T|\,\cdot}\rangle_{T,s}}\,\mathrm{d}\langle \hat{X}_{T|\,\cdot}\rangle_s. \tag{5}$$

Moreover $X^{T,\theta}$ admits the non-anticipative representation

$$X^{T,\theta}_t \;=\; \theta\frac{R(T,t)}{R(T,T)} + X_t - \int_0^t \phi_T(t,s)\,\mathrm{d}X_s, \tag{6}$$

where

$$\phi_T(t,s) \;=\; \int_s^t \left\{\int_s^u \frac{p_T(v,s)}{\langle \hat{X}_{T|\,\cdot}\rangle^2_{T,v}}\,\mathrm{d}\langle \hat{X}_{T|\,\cdot}\rangle_v - \frac{p_T(u,s)}{\langle \hat{X}_{T|\,\cdot}\rangle_{T,u}}\right\} p^*_T(t,u)\,\mathrm{d}\langle \hat{X}_{T|\,\cdot}\rangle_u.$$

Remark 15. Note that unlike the equations (2) and (1) the equation (5) is not of differential form. Indeed, it is clear by now that the differential connection is characteristic to the martingale case.

Proof. [Proposition 14] Consider the prediction martingale $\hat{X}_{T|\cdot}$. Using the relation (7) i.e.

$$\mathrm{d}\hat{X}^{T,\theta}_{T|t} \;=\; \mathrm{d}\hat{X}_{T|t} + \frac{\theta - \hat{X}^{T,\theta}_{T|t}}{\langle \hat{X}_{T|\cdot}\rangle_{T,t}}\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_t$$

with (3) yields

$$X^{T,\theta}_t \;=\; X_t + \int_0^t \left\{\theta - \hat{X}^{T,\theta}_{T|s}\right\} \frac{p^*_T(t,s)}{\langle \hat{X}_{T|\cdot}\rangle_{T,s}}\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_s. \tag{7}$$

The integral equation (5) follows now from (7) and (2).

Let us now derive the non-anticipative representation (6). Inserting the solution $\hat{X}^{T,\theta}_{T|\cdot} = S_{T,\theta}\big(\hat{X}_{T|\cdot}\big)$ to the equation (7) we obtain

$$\begin{aligned}
X^{T,\theta}_t &= X_t + \int_0^t \frac{p^*_T(t,s)}{\langle \hat{X}_{T|\cdot}\rangle_{T,s}} \left\{\theta \frac{\langle \hat{X}_{T|\cdot}\rangle_{T,s}}{\langle \hat{X}_{T|\cdot}\rangle_{T}} + \langle \hat{X}_{T|\cdot}\rangle_{T,s} \int_0^s \frac{\mathrm{d}\hat{X}_{T|u}}{\langle \hat{X}_{T|\cdot}\rangle_{T,u}}\right\} \mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_s \\
&= X_t + \frac{\theta}{\langle \hat{X}_{T|\cdot}\rangle_{T}} \int_0^t p^*_T(t,s)\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_s \\
&\quad + \int_0^t \int_0^s \frac{\mathrm{d}\hat{X}_{T|u}}{\langle \hat{X}_{T|\cdot}\rangle_{T,u}} p^*_T(t,s)\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_s \\
&=: X_t + \theta f_T(t) + \Phi_T\big(\hat{X}_{T|\cdot}\big)_t.
\end{aligned}$$

Note now that

$$X_T \;=\; \hat{X}_{T|T} \;=\; \int_0^T p^*_T(T,s)\,\mathrm{d}\hat{X}_{T|s},$$

which implies that $p^*_T(T,s) = \mathbf{1}_{[0,T)}(s)$. Consequently, by (4)

$$\int_0^t p^*_T(t,s)\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_s \;=\; \int_0^{T\wedge t} p^*_T(T,s) p^*_T(t,s)\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_s \;=\; R(T,t),$$

and, since $\langle \hat{X}_{T|\cdot}\rangle_T = R(T,T)$, we have

$$f_T(t) \;=\; \frac{R(T,t)}{R(T,T)}$$

(a fact that we actually knew already by (4)). Now we want to express $\Phi_T\big(\hat{X}_{T|\cdot}\big)$ in terms of X. We proceed by integrating by parts:

$$\int_0^s \frac{\mathrm{d}\hat{X}_{T|u}}{\langle \hat{X}_{T|\cdot}\rangle_{T,u}} \;=\; \frac{\hat{X}_{T|s}}{\langle \hat{X}_{T|\cdot}\rangle_{T,s}} - \int_0^s \frac{\hat{X}_{T|u}}{\langle \hat{X}_{T|\cdot}\rangle^2_{T,u}}\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_u. \tag{8}$$

Using the assumption (A4) to (8) and changing the order of integration we obtain

$$\begin{aligned}\int_0^s \frac{\mathrm{d}\hat{X}_{T|u}}{\langle \hat{X}_{T|\cdot}\rangle_{T,u}} &= \frac{1}{\langle \hat{X}_{T|\cdot}\rangle_{T,s}} \int_0^s p_T(s,u)\,\mathrm{d}X_u \\ &\quad - \int_0^s \frac{1}{\langle \hat{X}_{T|\cdot}\rangle_{T,u}^2} \int_0^u p_T(u,v)\,\mathrm{d}X_v\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_u \\ &= \int_0^s \left\{ \frac{p_T(s,u)}{\langle \hat{X}_{T|\cdot}\rangle_{T,s}} - \int_u^s \frac{p_T(v,u)}{\langle \hat{X}_{T|\cdot}\rangle_{T,v}^2}\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_v \right\} \mathrm{d}X_u.\end{aligned}$$

Thus,

$$\begin{aligned}&\Phi_T\big(\hat{X}_{T|\cdot}\big)_t \\ &= \int_0^t \int_0^s \left\{ \frac{p_T(s,u)}{\langle \hat{X}_{T|\cdot}\rangle_{T,s}} - \int_u^s \frac{p_T(v,u)}{\langle \hat{X}_{T|\cdot}\rangle_{T,v}^2}\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_v \right\} \mathrm{d}X_u p_T^*(t,s)\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_s \\ &= \int_0^t \int_s^t \left\{ \frac{p_T(u,s)}{\langle \hat{X}_{T|\cdot}\rangle_{T,u}} - \int_s^u \frac{p_T(v,s)}{\langle \hat{X}_{T|\cdot}\rangle_{T,v}^2}\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_v \right\} p_T^*(t,u)\,\mathrm{d}\langle \hat{X}_{T|\cdot}\rangle_u\,\mathrm{d}X_s \\ &= -\int_0^t \phi_T(t,s)\,\mathrm{d}X_s.\end{aligned}$$

This proves the decomposition (6). □

5 Bridges of Volterra Processes

The result of the previous section is still rather implicit. Indeed, we have no explicit relation between the covariance R of X and the bracket $\langle \hat{X}_{T|\cdot}\rangle$ of the prediction martingale m. Moreover, in general there is no simple way of finding, or even insuring the existence, of the kernels p_T^* and p_T. In this section we consider a model where these connections are clear, although the formulas turn out to be rather complicated.

(A5) There exists a Volterra kernel k and a continuous Gaussian martingale M with strictly increasing bracket $\langle M\rangle$ such that X admits a representation

$$X_t = \int_0^t k(t,s)\,\mathrm{d}M_s. \tag{1}$$

Remark 16. Since M is continuous, $\langle M\rangle$ is also continuous. Also, if $\langle M\rangle$ is not strictly increasing on an interval $[a,b]$, say, then nothing happens on that interval. Consequently, we could just remove it.

Remark 17. The connection between the covariance R and the kernel k is

$$R(t,s) \;=\; \int_0^{t\wedge s} k(t,u)k(s,u)\,\mathrm{d}\langle M\rangle_u. \tag{2}$$

Moreover, if R admits the representation (2) with some measure $\mathrm{d}\langle M\rangle$, then X admits the representation (1).

Now we define the Wiener integral with respect to X by using the way described in Remark 11. Let K extend the relation $\mathrm{K} : \mathbf{1}_{[0,t)} \mapsto k(t,\cdot)$ linearly. So, we have

$$\begin{aligned}\int_0^T f(t)\,\mathrm{d}X_t &= \int_0^T \mathrm{K}[f](t)\,\mathrm{d}M_t, \\ \int_0^T g(t)\,\mathrm{d}M_t &= \int_0^T \mathrm{K}^{-1}[g](t)\,\mathrm{d}X_t\end{aligned} \tag{3}$$

for any $g \in L^2([0,T],\mathrm{d}\langle M\rangle)$ and such functions f that are in the preimage of $L^2([0,T],\mathrm{d}\langle M\rangle)$ under K.

We need to have the inverse K^{-1} defined for a large enough class of functions. Thus, we assume

(A6) For any $t \le T$ the equation

$$\mathrm{K}f \;=\; \mathbf{1}_{[0,t)}$$

has a solution in f.

(A7) For any $t \le T$ the equation

$$\mathrm{K}g \;=\; \mathbf{1}_{[0,t)}k(T,\cdot)$$

has a solution in g.

By the assumption (A6), we have a reverse representation to (1). Indeed,

$$M_t \;=\; \int_0^t k^*(t,s)\,\mathrm{d}X_s, \tag{4}$$

where we have denoted

$$k^*(t,s) \;:=\; \mathrm{K}^{-1}\left[\mathbf{1}_{[0,t)}\right](s).$$

Since M is a martingale we have

$$\mathrm{d}m_t \;=\; k(T,t)\,\mathrm{d}M_t.$$

By assumption (A6) we have $k(t,s) \neq 0$ for $s < t$ $\mathrm{d}\langle M\rangle$-almost everywhere (and as $\langle M\rangle$ is strictly increasing also $\mathrm{d}t$-almost everywhere). So, we may write

$$\mathrm{d}M_t \;=\; \frac{\mathrm{d}\hat{X}_{T|t}}{k(T,t)}.$$

Thus,

$$X_t \;=\; \int_0^t \frac{k(t,s)}{k(T,s)}\,\mathrm{d}\hat{X}_{T|s}$$

and we have the assumption (A4) satisfied with

$$p_T^*(t,s) \;=\; \frac{k(t,s)}{k(T,s)}.$$

Consequently, the assumption (A2) is also satisfied. Also, the assumption (A6) implies the assumption (A1), since

$$\mathrm{d}\langle \hat{X}_{T|\,\cdot}\rangle_t \;=\; k(T,t)^2\,\mathrm{d}\langle M\rangle_t.$$

Indeed, this implies that $\langle \hat{X}_{T|\,\cdot}\rangle$ is strictly increasing.

For the kernel p_T we find the representation by using the assumption (A7) as follows:

$$\begin{aligned}
\hat{X}_{T|t} &= \int_0^t k(T,s)\,\mathrm{d}M_s \\
&= \int_0^t \mathrm{K}\left[\mathbf{1}_{[0,T)}\right](s)\,\mathrm{d}M_s \\
&= \int_0^t \mathrm{K}^{-1}\left[\mathbf{1}_{[0,t)}\mathrm{K}\left[\mathbf{1}_{[0,T)}\right]\right](s)\,\mathrm{d}X_s \\
&= \int_0^t \Big\{\mathrm{K}^{-1}\mathrm{K}\left[\mathbf{1}_{[0,t)}\right](s) + \mathrm{K}^{-1}\left[\mathbf{1}_{[0,t)}\mathrm{K}\left[\mathbf{1}_{[t,T)}\right]\right](s)\Big\}\,\mathrm{d}X_s \\
&= X_t + \int_0^t \Psi_T(t,s)\,\mathrm{d}X_s,
\end{aligned}$$

where we have denoted

$$\Psi_T(t,s) \;:=\; \mathrm{K}^{-1}\left[\mathbf{1}_{[0,t)}\mathrm{K}\left[\mathbf{1}_{[t,T)}\right]\right](s). \tag{5}$$

So, we have found that

$$\begin{aligned}
\mathrm{d}\langle \hat{X}_{T|\,\cdot}\rangle_t &= k(T,t)^2\,\mathrm{d}\langle M\rangle_t, \\
p_T(t,s) &= \mathbf{1}_{[0,t)}(s) + \Psi_T(t,s), \\
p_T^*(t,s) &= \frac{k(t,s)}{k(T,s)}
\end{aligned}$$

and we may rewrite Proposition 14 as follows.

Proposition 18. *Let X satisfy assumptions (A5), (A6) and (A7). Then the bridge $X^{T,\theta}$ satisfies the integral equation*

$$\begin{aligned} X_t^{T,\theta} &= X_t + \int_0^t \Bigg\{ \theta - X_s^{T,\theta} \\ &\quad - \int_0^s \Psi_T(s,u)\,\mathrm{d}X_u^{T,\theta} \Bigg\} \frac{k(T,s)k(t,s)}{\int_s^T k(T,u)^2\,\mathrm{d}\langle M\rangle_u}\,\mathrm{d}\langle M\rangle_s. \end{aligned} \tag{6}$$

Moreover, the bridge $X^{T,\theta}$ admits the non-anticipative representation

$$X_t^{T,\theta} = \theta\frac{R(T,t)}{R(T,T)} + X_t - \int_0^t \varphi_T(t,s)\,\mathrm{d}X_s, \tag{7}$$

where

$$\begin{aligned} \varphi_T(t,s) = \int_s^t \Bigg\{ \int_s^u \frac{\big(1+\Psi_T(v,s)\big)k(T,v)^2}{\big(\int_v^T k(T,w)^2\,\mathrm{d}\langle M\rangle_w\big)^2}\,\mathrm{d}\langle M\rangle_v \\ - \frac{1+\Psi_T(u,s)}{\int_u^T k(T,v)^2\,\mathrm{d}\langle M\rangle_v} \Bigg\} k(T,u)k(t,u)\,\mathrm{d}\langle M\rangle_u. \end{aligned}$$

6 Fractional Brownian Bridge

The fractional Brownian motion Z is a centred stationary increment Gaussian process with variance $\mathbf{E}\big(Z_t^2\big) = t^{2H}$ for some $H \in (0,1)$. Another way of charaterising the fractional Brownian motion if to say that it is the unique (up to multiplicative constant) centred H-self-similar Gaussian process with stationary increments.

In order to represent the fractional Brownian motion as a Volterra process we first recall some preliminaries of fractional calculus. For details we refer to Samko et al. [8].

Let f be a function over the interval $[0,1]$ and $\alpha > 0$. Then

$$I_\pm^\alpha\,[f]\,(t) := \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{f(s)}{(t-s)_\pm^{1-\alpha}}\,\mathrm{d}s$$

are the *Riemann–Liouville fractional integrals* of order α. For $\alpha \in (0,1)$,

$$D_\pm^\alpha\,[f]\,(t) := \frac{\pm 1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{f(s)}{(t-s)_\pm^\alpha}\,\mathrm{d}s.$$

are the *Riemann–Liouville fractional derivatives* of order α; $I_\pm^0$ and $D_\pm^0$ are identity operators.

If one ignores the troubles concerning divergent integrals and formally changes the order of differentiation and integration one obtains

$$I_{\pm}^{-\alpha} \;=\; D_{\pm}^{\alpha}.$$

We shall take the above as the definition for fractional integral of negative order and use the obvious unified notation.

Now, the fractional Brownian motion is a Volterra process satisfying assumptions (A5) and (A6). Indeed, let K be a weighted fractional integral or differential operator

$$\mathrm{K}\,[f]\,(t) \;:=\; c_H t^{\frac{1}{2}-H} I_{-}^{H-\frac{1}{2}} \left[s^{H-\frac{1}{2}} f(s)\right](t),$$

where

$$c_H \;=\; \sqrt{\frac{2H(H-\frac{1}{2})\Gamma(H-\frac{1}{2})^2}{\mathrm{B}(H-\frac{1}{2}, 2-2H)}}$$

and Γ and B are the gamma and beta functions. Then we have the relation (1) for fractional Brownian motion:

$$Z_t \;=\; \int_0^t \mathrm{K}\left[\mathbf{1}_{[0,t)}\right](s)\,\mathrm{d}W_s, \tag{1}$$

where W is the standard Brownian motion. Thus, the fractional Brownian motion satisfies the assumption (A5).

The operator K satisfies the assumption (A6). Indeed,

$$\mathrm{K}^{-1}[f](t) \;=\; \frac{1}{c_H} t^{\frac{1}{2}-H} I_{-}^{\frac{1}{2}-H}\left[s^{H-\frac{1}{2}} f(s)\right](t)$$

The kernel Ψ_T has been calculated e.g. in Pipiras and Taqqu [7], Theorem 7.1. Indeed, for any $H \in (0,1)$ we have

$$\Psi_T(t,s) \;=\; \frac{\sin\left(\pi(H+\frac{1}{2})\right)}{\pi} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} \int_t^T \frac{u^{H+\frac{1}{2}}(u-t)^{H+\frac{1}{2}}}{u-s}\,\mathrm{d}u.$$

As for the kernel p_T^* note that for $H \in (0,1)$ we have

$$\begin{aligned} k(t,s) &= \mathrm{K}\left[\mathbf{1}_{[0,t)}\right](s) \\ &= c_H' \left\{ \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H-\tfrac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\,\mathrm{d}u \right\}, \end{aligned}$$

where

$$c_H' \;=\; \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}}.$$

If $H > 1/2$ then we have a slightly simpler expression, viz.

$$k(t,s) \;=\; c'_H \left(H-\tfrac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}\,\mathrm{d}u.$$

For the derivation of these formulas see Norros et al. [6] and Jost [5].

The representations for the fractional Brownian bridge follow now by plugging in our Ψ_T and k to the formulas (6) and (7) in Proposition 18 with $M = W$ and $\mathrm{d}\langle M\rangle_t = \mathrm{d}t$. Unfortunately, it seems that there is really nothing we can do to simplify the resulting formula (except some trivial use of the H-self-similarity), even in the case $H > 1/2$. So, we do not bother to write the equations (6) and (7) again here.

7 Enlargement of Filtration Point of View

Let us denote by $\mathbf{F}^X$ be its natural (continuous) filtration of the Gaussian process X. Setting in our conditioning simply $\theta := X_T$ we may interpret the bridge measure $\mathbf{P}^{(0,X_0)\to(T,X_T)}$ as initial enlargement of the filtration $\mathbf{F}^X$ by the random variable X_T. Let $\mathbf{F}^{X,X_T}$ be this enlarged filtration. We have formally

$$(\Omega,\mathcal{F},\mathbf{F}^X,\mathbf{P}^{(0,X_0)\to(T,X_T)}) \;\simeq\; (\Omega,\mathcal{F},\mathbf{F}^{X,X_T},\mathbf{P}).$$

For the Brownian motion W we have the following: with respect to the measure $\mathbf{P}^\theta$ the Browian motion has the representation

$$W_t = W_t^\theta + \int_0^t \beta_{T,\theta}(W)_s ds = W_s^\theta + \int_0^t \frac{\theta - W_s}{T-s} ds.$$

This means that with respect to the filtration $\mathbf{F}^{W,W_T}$ and measure $\mathbf{P}$ Brownian motion W has the representation

$$W_t = W_t^{\mathbf{F}^{W,W_T}} + \int_0^t \frac{W_T - W_s}{T-s} ds,$$

where Law $(W^{\mathbf{F}^{W,W_T}}|\mathbf{P}) = \mathrm{Law}(W|\mathbf{P})$, $W^{\mathbf{F}^{W,W_T}}$ is a $(\mathbf{P},\mathbf{F}^{W,W_T})$ Brownian motion, but W is a $(\mathbf{P},\mathbf{F}^{W,W_T})$ semimartingale.

Similarly, if we have an arbitrary gaussian process such that the process has a Volterra representation (3)

$$X_t \;=\; \int_0^t p_T^*(t,s)\,\mathrm{d}\hat{X}_{T|s}.$$

we can use the enlargement of filtration results to give a semimartingle representation for the martingale $\hat{X}_{T|\cdot}$ with respect to $(\mathbf{P},\mathbf{F}^{X,X_T})$:

$$\hat{X}_{T|t} \;=\; \hat{X}_{T|t}^{\mathbf{F}^{X,X_T}} + \int_0^t \frac{X_T - \hat{X}_{T|s}}{\langle \hat{X}_{T|\cdot}\rangle_{T,s}} d\langle \hat{X}_{T|\cdot}\rangle_s, \tag{2}$$

where the $(\mathbf{P}, \mathbf{F}^{X,X_T})$-gaussian martingale $\hat{X}_{T|\cdot}^{\mathbf{F}^{X,X_T}}$ has the same law as $\hat{X}_{T|\cdot}$ (see [2, 3] for more details). We can now use (3), (6) and (2) to obtain the following representation for the process X

$$X_t \;=\; X_t^{\mathbf{F}^{X,X_T}} + X_T \frac{R(T,t)}{R(T,T)} - \int_0^t \phi_T(t,s) dX_s, \tag{3}$$

with

$$\phi_T(t,s) \;=\; \int_s^t \left\{ \int_s^u \frac{p_T(v,s)}{\langle \hat{X}_{T|\cdot} \rangle_{T,v}^2} \,\mathrm{d}\langle \hat{X}_{T|\cdot} \rangle_v - \frac{p_T(u,s)}{\langle \hat{X}_{T|\cdot} \rangle_{T,u}} \right\} p_T^*(t,u) \,\mathrm{d}\langle \hat{X}_{T|\cdot} \rangle_u.$$

Acknowledgements

T. Sottinen is grateful for partial support from EU-IHP network DYNSTOCH. We thank M. Lifshits for the example 1.

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