COUNTING MEASURE AND FORKING IN FINITE MODELS

Tapani Hyttinen∗

Abstract

Based on the resent work by E. Hrushovski, using the counting measure, we construct an independence notion to nice classes of finite structure and study when the independence notion is the same notion that we get from the non-forking in the ultraproduct of a representative collection of models from the class.

In the paper [Va], Jouko Väänänen argues that pseudo-finite models (i.e. ultraproducts of finite models) are a good framework for studying first-order logic on finite models and he demonstrates this several ways, e.g. he describes how to use pseudo-finite models to prove various non-definability results for finite models. In the spirit of [Va], in this paper we look at one way of defining an independence notion to a class of finite structure.

A lot of work has been done in which stability theory for finite structures is developed, used or which study questions that can be looked naturally also from this point of view. E.g. the work [Hi] by C. Hill on effective algorithms inverting L^2-invariants, many papers by V. Koponen, see e.g. [Ko], my own contributions, see e.g. [Hy], the study of asymptotic classes of finite structure (and smoothly approximated structures), see e.g. [El], and the recent work [Hr] by E. Hrushovski from which we will use several ideas.

Keywords: Forking, counting measure, finite structures
2010 MSC: primary 03C45, secondary 03C13
∗ Partially supported by the Academy of Finland, grant 1123110.

So let $\mathcal{K}$ be a collection of finite models and for simplicity let Δ be the set of quantifier free formulas. If the class has the joint embedding property, we can find a subset $\{\mathcal{A}_i | \; i<\omega\}$ of $\mathcal{K}$ so that for all $i<\omega$, $\mathcal{A}_i$ is a substructure of $\mathcal{A}_{i+1}$ and every member of $\mathcal{K}$ embeds to some $\mathcal{A}_i$, $i<\omega$, and one can think the sequence $(\mathcal{A}_i)_{i<\omega}$ as a monster model for the class, especially when one can choose the sequence $(\mathcal{A}_i)_{i<\omega}$ well, see [Hy]. Now one can ask, how to find an independence notion to $(\mathcal{A}_i)_{i<\omega}$, something like $\phi(x,a)$ is free over b for Δ-formulas $\phi(x,y)$.

If U is a non-principle ultrafilter on ω, then the non-forking in

$$\mathbf{M} = \Pi_{i<\omega}\mathcal{A}_i/U$$

gives one such notion when we think each $\mathcal{A}_i$ as a substructure of $\mathbf{M}$ in the natural way. But what does this notion tell about the structures $\mathcal{A}_i$, $i<\omega$?

Using ideas from [Hr], from the counting measure on the models $\mathcal{A}_i$, $i<\omega$, one gets another such notion. In this paper we study the properties of this independence notion and in particular the question when the two notions are the same.

Everything we do is very close to what is done in the theory of asymptotic classes. The main difference in the starting points is that we do not require the existence of the dimensions, instead we work directly with the counting measures inside all (Δ-)definable subsets of $\mathcal{A}_i^n$, not just those of the form $\mathcal{A}_i^n$, see Example 2.3 below. Also we ask a bit different questions, in particular we are a lot less ambitious.

There is also a close connection to the theory of NIP theories and to Keisler measure. In particular, Proposition 4.7 in [HP] is close to what we prove here.

1 Counting measure and forking

We let L be a countable vocabulary and we fix a strictly increasing sequence $(\mathcal{A}_i)_{i<\omega}$ of finite L-structures (i.e. L-structures with finite universe) such that for all $i<\omega$, $\mathcal{A}_i$ is a submodel of $\mathcal{A}_{i+1}$. Δ is a collection of first-order L-formulas such that it contains all atomic formulas and is closed under boolean combinations and replacing variables by other variables (and adding dummy variables). The main examples of Δ we have in mind are the set of all quantifier free formulas and the set of all first-order formulas. In some cases we could let Δ be also something other than a collection of first-order formulas, see the end of this section.

By a,b etc. we mean finite sequences of elements from some model and by x,y etc. we mean finite sequences of variables. $a\in\mathcal{A}_i$ means that $a\in(\mathcal{A}_i)^n$ for suitable n. By $t_\Delta(a/A;\mathcal{B})$ we mean the complete Δ-type of a over A in a model $\mathcal{B}$ i.e. $\{\phi(x,b) | \; \phi\in\Delta,\ b\in A,\ \mathcal{B}\models\phi(a,b)\}$. By $t(a/A;\mathcal{B})$ we mean the complete first-order type. We let $\mathcal{A}=\cup_{i<\omega}\mathcal{A}_i$. By formulas, definability, elementary etc. mean mean the usual first-order concepts (not with respect to Δ). So e.g. we say that $X\subseteq\mathcal{A}_i^n$ is definable if for some first-order formula $\phi(x,y)$ and $b\in\mathcal{A}_i$, $X=\phi(\mathcal{A}_i,b)=\{a\in\mathcal{A}_i | \; \mathcal{A}_i\models\phi(a,b)\}$. The number n is called the arity of X

and in this case we also say that X is definable over b. We add Δ to the notion (e.g. Δ-definable) when we use only formulas from Δ (i.e. Δ-formulas).

We fix a non-principal ultrafilter U on ω.

We write $\mathcal{A} \models_U \phi(a)$ if $\{i < \omega | \ a \in \mathcal{A}_i, \ \mathcal{A}_i \models \phi(a)\} \in U$ and by $\phi^U(\mathcal{A}, a)$ we mean the set $\{b \in \mathcal{A} | \ \mathcal{A} \models_U \phi(b, a)\}$ and $\phi^U(\mathcal{A}_i, a) = \phi^U(\mathcal{A}, a) \cap \mathcal{A}_i$. Also we say that $(\mathcal{A}_i)_{i<\omega}$ is a Δ-sequence if for all $i < \omega$, Δ-formula ϕ and $a \in \mathcal{A}_i$, $\mathcal{A}_i \models \phi(a)$ iff $\mathcal{A}_{i+1} \models \phi(a)$ (i.e. $\mathcal{A}_i$ is a Δ-elementary submodel of $\mathcal{A}_{i+1}$). Then if $(\mathcal{A}_i)_{i<\omega}$ is a Δ-sequence, $\phi(x, y)$ a Δ-formula and $a, b \in \mathcal{A}_i$, $\phi^U(\mathcal{A}_i, b) = \phi(\mathcal{A}_i, b)$ and $\mathcal{A}_i \models \phi(a, b)$ iff $\mathcal{A} \models_U \phi(a, b)$.

Abusing the notation, we say that $X = (X_i)_{i<\omega}$ is (Δ-) definable if there are a (Δ-) formula $\phi(x, y)$ and $a_i \in \mathcal{A}_i$ such that for all $i < \omega$, $X_i = \phi(\mathcal{A}_i, a_i)$. If for all $i < \omega$, we can choose $a_i = a \in \mathcal{A}$ (X_i is e.g. $\emptyset$ if $a \notin \mathcal{A}_i$) we say that X is definable over a and the arity is $length(x)$. We write also $\mathcal{A}$ for the sequence $(\mathcal{A}_i)_{i<\omega}$ of sets and $\mathcal{A}^n$ for $(\mathcal{A}_i^n)_{i<\omega}$. It will be clear from the context, weather we mean $(\cup_{i<\omega} \mathcal{A}_i)^n$ or $(\mathcal{A}_i^n)_{i<\omega}$.

We let μ_i be the counting measure on $\mathcal{A}_i$ i.e. for $A, B \subseteq \mathcal{A}_i$, $\mu_i(A/B) = |A \cap B|/|B|$ (say $\mu_i(A/B) = 1$ if $B = \emptyset$) and for all definable sequences X and Y, we write $\mu_i(X/Y) = |X_i \cap Y_i|/|Y_i|$. By $\mu(X/Y)$ we mean $lim_{i \to \infty} \mu_i(X/Y)$ if this limit exists. We define $\mu^*(X/Y)$ to be the infimum of those rationals q such that $\{i < \omega | \ \mu_i(X/Y) \leq q\} \in U$. Notice that for all Y the map $X \mapsto \mu^*(X/Y)$ is some kind of measure on the set of all definable sequences. Notice also that $\mu(X/Y) = 0$ implies $\mu^*(X/Y) = 0$ or more generally $\mu(X/Y) = r$ implies $\mu^*(X/Y) = r$. So, at least in many cases, $\mu^*(X/Y)$ can be calculated without knowing U, see the examples in Section 2.

1.1 Definition.

(i) We say that $(\mu_i)_{i<\omega}$ *is* U*-continuous if for every* Δ*-formula* $\phi(x, y)$, $a \in \mathcal{A}$ *and* $\epsilon > 0$, *there is a first-order formula* $\psi(y)$ *such that* $\mathcal{A} \models_U \psi(a)$ *and for all* $b \in \mathcal{A}$, *if* $\mathcal{A} \models_U \psi(b)$, *then*

$$\{i < \omega | \ |1 - (|\phi(\mathcal{A}_i, a)|/|\phi(\mathcal{A}_i, b)|)| \leq \epsilon\} \in U,$$

where $0/0 = 1$ *and* $n/0 = 2$ *for* $n > 0$.

(ii) We say that $(\mu_i)_{i<\omega}$ *is uniformly* U*-continuous if for every* Δ*-formula* $\phi(x, y)$ *and* $\epsilon > 0$, *there are* $n < \omega$ *and first-order formulas* $\psi_j(y)$, $j \leq n$, *such that* $\mathcal{A} \models_U \forall y \bigvee_{j \leq n} \psi_j(y)$ *and for all* $j \leq n$

$$\{i < \omega | \ \text{for all } a, b \in \psi_j(\mathcal{A}_i), \ |1 - (|\phi(\mathcal{A}_i, a)|/|\phi(\mathcal{A}_i, b)|)| \leq \epsilon\} \in U,$$

where again $0/0 = 1$ *and* $n/0 = 2$ *for* $n > 0$.

(iii) We say that $(\mu_i)_{i<\omega}$ *is* U, Δ*-continuous if (i) holds with the additional requirement that the formula* ψ *can always be chosen from* Δ. *Uniformly* U, Δ*-continuous is defined similarly.*

1.2 Lemma.

(i) Uniform U-continuity implies U-continuity.

(ii) Suppose $(\mu_i)_{i<\omega}$ is U-continuous, $\epsilon > 0$ and that $X = (\phi(\mathcal{A}_i, a))_{i<\omega}$ and $Y = (\psi(\mathcal{A}_i, a))_{i<\omega}$ have the same arity and ϕ and ψ are Δ-formulas. Then there is a first-order formula $\theta(y)$ such that $\mathcal{A} \models_U \theta(a)$ and for all b, if $\mathcal{A} \models_U \theta(b)$, then, letting $X' = (\phi(\mathcal{A}_i, b))_{i<\omega}$ and $Y' = (\psi(\mathcal{A}_i, b))_{i<\omega}$,

$$|\mu^*(X/Y) - \mu^*(X'/Y')| \leq \epsilon.$$

(iii) Suppose $(\mu_i)_{i<\omega}$ is uniformly U-continuous, $\epsilon > 0$ and ϕ and ψ are Δ-formulas. Then there are first-order formulas $\theta_i(y)$, $i \leq n$, such that $\mathcal{A} \models_U \forall y \bigvee_{i\leq n} \theta_i(y)$ and for all $j \leq n$ and $a_i, b_i \in \mathcal{A}_i$, $i < \omega$, if $\{i < \omega | \ \mathcal{A}_i \models \theta_j(a_i) \wedge \theta_j(b_i)\} \in U$, then letting $X = (\phi(\mathcal{A}_i, a_i))_{i<\omega}$, $Y = (\psi(\mathcal{A}_i, a_i))_{i<\omega}$, $X' = (\phi(\mathcal{A}_i, b_i))_{i<\omega}$ and $Y' = (\psi(\mathcal{A}_i, b_i))_{i<\omega}$,

$$\{i < \omega | \ |\mu_i(X_i/Y_i) - \mu_i(X'_i/Y'_i)| \leq \epsilon\} \in U.$$

Proof. We prove (iii), (ii) is similar and (i) is clear. Let $\epsilon > 0$ and $\delta = min\{\epsilon, 1/2\}/4$. By the assumption, it is enough to show that if $|1 - |X_i|/|X'_i|| \leq \delta$ and $|1 - |Y_i|/|Y'_i|| \leq \delta$, then $||X_i|/|Y_i| - |X'_i|/|Y'_i|| \leq \epsilon$ (assuming that $X_i \subseteq Y_i$ and $X'_i \subseteq Y'_i$ i.e. $|X_i|/|Y_i| \leq 1$ and $|X'_i|/|Y'_i| \leq 1$). Now

$$(|X_i|/|X'_i|)/(|Y_i|/|Y'_i|) \leq (1+\delta)/(1-\delta)$$

and

$$(|Y_i|/|Y'_i|)/(|X_i|/|X'_i|) \leq (1+\delta)/(1-\delta)$$

and so

$$||X_i|/|Y_i| - |X'_i|/|Y'_i|| \leq |1 - (1+\delta)/(1-\delta)| \leq 2\delta/(1-\delta) \leq 4\delta \leq \epsilon.$$

□

Before futher studies of this measure from the point of view of independence, we notice that there is a connection between the continuity properties defined above and the elimination of quantifiers.

If $(\mathcal{A}_i)_{i<\omega}$ is a Δ-sequence, then we say that $\mathcal{A}$ is Δ-atomic if for all $a \in \mathcal{A}$ there is $\phi(x) \in t_\Delta(a/\emptyset; \mathcal{A})$ such that for all $b \in \mathcal{A}$, $\mathcal{A} \models \phi(b)$ implies $t_\Delta(b/\emptyset; \mathcal{A}) = t_\Delta(a/\emptyset; \mathcal{A})$. We say that $\mathcal{A}$ is strongly Δ-atomic if for all $a \in \Pi_{i<\omega}\mathcal{A}_i/U$ there is $\phi(x) \in t_\Delta(a/\emptyset; \Pi_{i<\omega}\mathcal{A}_i/U)$ such that for all $b \in \Pi_{i<\omega}\mathcal{A}_i/U$, $\Pi_{i<\omega}\mathcal{A}_i/U \models \phi(b)$ implies $t_\Delta(b/\emptyset; \Pi_{i<\omega}\mathcal{A}_i/U) = t_\Delta(a/\emptyset; \Pi_{i<\omega}\mathcal{A}_i/U)$.

Notice that if L is finite and relational and Δ is the set of all quantifier free formulas, then for all $n < \omega$, the number of $t_\Delta(a/\emptyset; \mathcal{A})$, $a \in \mathcal{A}^n$ is finite and also it is easy to see that the number of these types is finite iff $\mathcal{A}$ is strongly Δ-atomic (cf. Ryll-Nardzewski). However, we will not use the direction from right to left of this fact in the proof of the next lemma, since in some cases in which these measures may work (e.g. continuous logic), the direction is not true.

1.3 Lemma. *Assume that $(\mathcal{A}_i)_{i<\omega}$ is a Δ-sequence and that Δ is closed under subformulas.*

(i) Every $\mathcal{A}_i$ is a Δ-elementary submodel of $\mathcal{A}$ and $\mathcal{A}$ is a Δ-elementary submodel of $\Pi_{i<\omega}\mathcal{A}_i/U$.

(ii) Assume that $(\mu_i)_{i<\omega}$ is U,Δ-continuous and $\mathcal{A}$ is Δ-atomic. Then $\mathcal{A}$ is Δ-homogeneous i.e. if a and b have the same Δ-type then there is an automorphism f of $\mathcal{A}$ such that $f(a)=b$.

(iii) If $(\mu_i)_{i<\omega}$ is uniformly U,Δ-continuous and $\mathcal{A}$ is strongly Δ-atomic, then $\mathcal{A}$ is an elementary submodel of $\Pi_{i<\omega}\mathcal{A}_i/U$ and $\Pi_{i<\omega}\mathcal{A}_i/U$ has Δ-elimination of quantifiers, i.e, in $\Pi_{i<\omega}\mathcal{A}_i/U$, every first-order formula is equivalent with a Δ-formula.

Proof. (i) is immediate by the assumption on Δ and (ii) can be proved as the first claim in (iii). So we prove (iii): For the first claim of (iii), it is clearly enough to show that the second player wins the Ehrenfeuch-Fraıssé-game of length ω played between $(\mathcal{A},e)$ and $(\Pi_{i<\omega}\mathcal{A}_i/U,e)$, where $e\in\mathcal{A}$ (so we prove more than what we claimed). This is easy: She plays so that the sequences of the elements played so far have the same Δ-type over e. We show how she makes her second move in the case $e=\emptyset$ and on the first move $a\in\mathcal{A}$ and $b\in\Pi_{i<\omega}\mathcal{A}_i/U$ are chosen and on the second round the first player has chosen c from $\mathcal{A}$, the other cases are similar or easier. Let $\phi(x,y)$ be the Δ-formula that isolates $t_\Delta(ca/\emptyset;\mathcal{A})$. Let $\psi_j(y)$ be as in the definition of uniform U,Δ-continuity for $\phi(x,a)$ and $\epsilon=1/2$ chosen so that $\mathcal{A}\models\psi_j(a)$. Then $\Pi_{i<\omega}\mathcal{A}_i/U\models\psi_j(b)$ and let $X\in U$ be such that for all $i\in X$, $\mathcal{A}_i\models\psi_j(b_i)$ ($b=(b_i)_{i<\omega}/U$). By the choice of ψ_j, there is $Y\in U$ such that $Y\subseteq X$ and for all $i\in Y$, $|1-|\phi(\mathcal{A}_i,a)|/|\phi(\mathcal{A}_i,b_i)||\leq 1/2$. Clearly we may assume that Y is chosen so that $a,c\in\mathcal{A}_i$ for all $i\in Y$. But then for $i\in Y$, $\phi(\mathcal{A}_i,b_i)$ can not be empty and thus the second player can choose $d=(d_i)_{i<\omega}/U\in\Pi_{i<\omega}\mathcal{A}_i/U$ such that for all $i\in Y$, $\mathcal{A}_i\models\phi(d_i,b_i)$.

For the second claim of (iii), let $\phi(x)$ be an arbitrary first-order formula. By (ii) there are Δ-formulas ψ_{kj}, $k\in I$ and $j\in J_k$, such that

$$\mathcal{A}\models\forall x(\phi(x)\leftrightarrow\bigvee_{k\in I}\bigwedge_{j\in J_k}\psi_{kj}).$$

By the proof above,

$$\Pi_{i<\omega}\mathcal{A}_i/U\models\forall x(\phi(x)\leftrightarrow\bigvee_{k\in I}\bigwedge_{j\in J_k}\psi_{kj}).$$

Since $\Pi_{i<\omega}\mathcal{A}_i/U$ is ω_1-saturated,

$$Th(\Pi_{i<\omega}\mathcal{A}_i/U)\models\forall x(\phi(x)\leftrightarrow\bigvee_{k\in I}\bigwedge_{j\in J_k}\psi_{kj}).$$

From this the claim follows by compactness. □

We extend the vocabulary L to a vocabulary L^* by adding new predicates $I^{\phi,\psi}$ for all Δ-formulas $\phi = \phi(x,y)$ and $\psi = \psi(x,y)$. We interpret these in $\mathcal{A}$ and in each $\mathcal{A}_i$ so that $(a,b) \in I^{\phi,\psi}$ if $\mu^*((\phi(\mathcal{A}_i,a))_{i<\omega}/(\psi(\mathcal{A}_i,b))_{i<\omega}) = 0$. We write $\mathcal{A}_i^*$ for what we get from $\mathcal{A}_i$ after adding the interpretations for the predicates $I^{\phi,\psi}$ and $\mathcal{A}^*$ is defined similarly. We let $\mathbf{M}^*$ be $\Pi_{i<\omega}\mathcal{A}_i^*/U$ and the embedding $a \mapsto (a_i)_{i<\omega}/U$, $a \in \mathcal{A}$, is considered as the identity map, where $a_i = a$ for all $i < \omega$.

We let $\mathbf{M}$ be the reduct of $\mathbf{M}^*$ to the vocabulary L and we recall that by Lemma 1.3 (iii), if $(\mathcal{A}_i)_{i<\omega}$ is a Δ-sequence, $\mathcal{A}$ is strongly Δ-atomic and $(\mu_i)_{i<\omega}$ is uniformly U,Δ-continuous, then $\mathcal{A}$ is an elementary submodel of $\mathbf{M}$.

1.4 Remark. *For all $a,b \in \mathcal{A}$, $\mathcal{A}^* \models I^{\phi,\psi}(a,b)$ iff $\mathbf{M}^* \models I^{\phi,\psi}(a,b)$.*

Proof. Immediate by the definitions. □

For the rest of this paper, we make the assumption below. This is done because we are interested in the properties of the models $\mathcal{A}_i$ that are first-order in the original similarity type (at least first-order in the sense of continuous logic). However, many of the results below hold if instead of $\mathbf{M}$ we work in $\mathbf{M}^*$ and modify definitions and assumptions suitably, see [Hr].

1.5 Assumption. *From now on, we assume that $(\mu_i)_{i<\omega}$ is U-continuous.*

1.6 Lemma.

(i) For all $a,b,c,d \in \mathcal{A}$, if $t(ab/\emptyset;\mathbf{M}) = t(cd/\emptyset;\mathbf{M})$, then $(a,b) \in I^{\phi,\psi}$ iff $(c,d) \in I^{\phi,\psi}$. If in addition $(\mu_i)_{i<\omega}$ is uniformly U-continuous, this holds for all $a,b,c,d \in \mathbf{M}$.

(ii) If $\phi(x,y)$ and $\phi'(x,y)$ are equivalent in $\mathbf{M}$, i.e. $\{i<\omega \mid \mathcal{A}_i \models \forall x \forall y(\phi \leftrightarrow \phi')\} \in U$, and similarly for $\psi(x,y)$ and $\psi'(x,y)$, then the interpretations of $I^{\phi,\psi}$ and $I^{\phi',\psi'}$ are the same (e.g. in $\mathcal{A}^*$).

Proof. (i): Immediate by Lemma 1.2.

(ii): Immediate by the definitions. □

Suppose Y is Δ-definable sequence over b. We write $a \in I_b^{\phi,Y}$ if $(a,b) \in I^{\phi,\psi}$, where ψ is any formula such that $(\psi(\mathcal{A}_i,b))_{i<\omega} = Y$.

Let X and Y be definable subsets of $\mathbf{M}$, say $X = \phi(\mathbf{M},a)$ and $Y = \psi(\mathbf{M},b)$, where $a = (a_i)_{i<\omega}/U$ and $b = (b_i)_{i<\omega}/U$. We write $\mu^*(X/Y)$ for the infimum of the rationals q such that

$$\{i<\omega \mid \mu_i(\phi(\mathcal{A}_i,a_i)/\psi(\mathcal{A}_i,b_i)) \leq q\} \in U.$$

We also write X_i for $\phi(\mathcal{A}_i,a_i)$ (and so $\mu^*(X/Y) = \mu^*((X_i)_{i<\omega}/(Y_i)_{i<\omega})$). Notice that X_i may depend on the choice of ϕ and a but not too much, only in a small set. In fact:

1.7 Lemma.

(i) The definition of μ^* does not depend on the choice of $\phi(x,a)$ and $\psi(x,b)$ nor on the choice of the representatives $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$.

(ii) For all X and Y Δ-definable over $a \in \mathcal{A}$ by $\phi_0(x,a)$ and $\phi_1(x,a)$, repectively, and $\epsilon > 0$, there is a first-order formula $\psi(y)$ such that $\mathcal{A} \models_U \psi(a)$ and for all $b \in \mathcal{A}$ if $\mathcal{A} \models_U \psi(b)$ and $X' = \phi_0(\mathbf{M}, b)$ and $Y' = \phi_1(\mathbf{M}, b)$, then $|\mu^*(X/Y) - \mu^*(X'/Y')| \le \epsilon$.

(iii) If $(\mu_i)_{i<\omega}$ is uniformly U-continuous, then for all X and Y Δ-definable over $a \in \mathbf{M}$ by $\phi_0(x,a)$ and $\phi_1(x,a)$, repectively, and $\epsilon > 0$, there is a first-order formula $\psi(y)$ such that $\mathbf{M} \models \psi(a)$ and for all $b \in \mathbf{M}$ if $\mathbf{M} \models \psi(b)$ and $X' = \phi_0(\mathbf{M}, b)$ and $Y' = \phi_1(\mathbf{M}, b)$, then $|\mu^*(X/Y) - \mu^*(X'/Y')| \le \epsilon$.

Proof. (i): If $\mathbf{M} \models \forall x(\phi(x,a) \leftrightarrow \phi'(x,a'))$, then

$$W = \{i < \omega |\ \mathcal{A}_i \models \forall x(\phi(x,a_i) \leftrightarrow \phi'(x,a'_i))\} \in U.$$

Thus for all $i \in X$ and $c \in \mathcal{A}_i$, $c \in \phi(\mathcal{A}_i, a_i)$ iff $c \in \phi(\mathcal{A}_i, a'_i)$ i.e. for all $i \in W$, $|\phi(\mathcal{A}_i, a_i)| = |\phi(\mathcal{A}_i, a'_i)|$. And so the claim is immediate by the definitions.

(ii) and (iii) are immediate by Lemma 1.2 and the definitions. □

Notice that for all Δ-definable subsets Y of $\mathbf{M}^n$, $X \mapsto \mu^*(X/Y)$ is kind of a measure on the Δ-definable subsets of $\mathbf{M}^n$ (the domain of μ^* is not a σ-algebra) and letting I_Y be the set of all Δ-definable subsets X of $\mathbf{M}$ of measure 0 (i.e. $\mu^*(X/Y) = 0$), then I_Y is an ideal. Notice also that if Y is definable over $b \in \mathcal{A}$, then for all $a \in \mathcal{A}$, $\phi(\mathbf{M}, a) \in I_Y$ iff $a \in I_b^{\phi, Y'}$ where $Y' = (Y_i)_{i<\omega}$.

When we speak about order indiscernible sequences, we mean that they are order indiscernible in $\mathbf{M}$ (or in some elementary extension of $\mathbf{M}$) and in the sence of the first-order logic.

1.8 Definition.

(i) We say that I_Y is an S1-ideal if for all Δ-formulas $\phi(x,y)$, for all b such that Y is Δ-definable over b and order-indiscernible $(a_i)_{i<\omega}$ over b, if for some $n < \omega$, $\bigcap_{i \le n} \phi(\mathbf{M}, a_i) \in I_Y$, then $\phi(\mathbf{M}, a_0) \in I_Y$.

(ii) We say that I_Y is an S1-ideal over $(b, \mathcal{A})$ if Y is Δ-definable over $b \in \mathcal{A}$ and the following holds: for all Δ-formulas $\phi(x,y)$ and order-indiscernible $(a_i)_{i<\omega}$ over b, if for all $i < \omega$, $a_i \in \mathcal{A}$ and $\bigcap_{i<2} \phi(\mathbf{M}, a_i) \in I_Y$, then $\phi(\mathbf{M}, a_0) \in I_Y$.

Notice that assuming uniform U-continuity, the requirement in Definition 1.8 (i) above does not depend on the choice of b: By Erdös-Rado, for any c and order-indiscernible $(a_i)_{i<\omega}$ over b, there is order-indiscernible $(b_i)_{i<\omega}$ over bc such that $t((b_i)_{i \le n}/b; \mathbf{M}) = t((a_i)_{i \le n}/b; \mathbf{M})$ for any $n < \omega$.

The following Lemma and its corollary are based on ideas from [Hr].

1.9 Lemma.

(i) For all $Y \subseteq \mathbf{M}$ Δ-definable over $b \in \mathcal{A}$, I_Y is an S1-ideal over $(b, \mathcal{A})$.

(ii) If $(\mu_i)_{i<\omega}$ is uniformly U-continuous, then for all Δ-definable Y, I_Y is an S1-ideal.

Proof. (i) Suppose not. Let $(a_i)_{i<\omega}$ and $\phi(x,y)$ witness that the claim is not true. Let $\epsilon = \mu^*(\phi(\mathbf{M}, a_0)/Y) > 0$, $m \ge 4/\epsilon$ be a natural number and

$\delta = \epsilon/m^3$. By U-continuity (i.e. Lemma 1.7 (ii)) and the definition of μ^*, we can find $i < \omega$ such that for all $j < m$, $\mu_i(\phi(\mathcal{A}_i, a_j)/Y_i) > 3\epsilon/4$ and for all $j < k < m$, $\mu_i((\phi(\mathcal{A}_i, a_j) \cap \phi(\mathcal{A}_i, a_k))/Y_i) < \delta$. But then we have a contradiction since $\mu_i(\bigcup_{k<m} \phi(\mathcal{A}_i, a_k)/Y_i) > 1$:

Let $X = \bigcup_{i<m}(\{i\} \times |\phi(\mathcal{A}_i, a_j) \cap Y_i|)$ and let $f : X \to \mathbf{M}^{length(x)}$ be such that for all $i < m$, $f \restriction (\{i\} \times |\phi(\mathcal{A}_i, a_j) \cap Y_i|)$ is onto $\phi(\mathcal{A}_i, a_j) \cap Y_i$ and one to one. Let Z be the set of those $c \in X$ such that for some $d \in X - \{c\}$, $f(d) = f(c)$. By the choice of i and ϵ, $|X| \geq 3|Y_i|$ and by the choice of Z, $|X - Z| \leq |\bigcup_{k<m}(\phi(\mathcal{A}_i, a_k) \cap Y_i)|$, since $f \restriction (X - Z)$ is one to one. Thus it suffices to show that $|Z| \leq |Y_i|$. But

$$|Z| \leq \Sigma_{k<j<m} m|\phi(\mathcal{A}_i, a_k) \cap \phi(\mathcal{A}_i, a_j) \cap Y_i|$$

and thus $|Z| \leq m^2(m(\delta|Y_i|)) = \epsilon|Y_i| \leq |Y_i|$.

(ii) As (i). Notice that the witness for the counter assumption that I_Y is not an S1-ideal can always be chosen so that $n = 2$. □

Notice that in the proof above we showed a bit more than what we claim: If Y is Δ-definable over $b \in \mathcal{A}$, $\phi(x, a)$, $a \in \mathcal{A}$, is a Δ-formula and $\mu^*(\phi(\mathbf{M}, a)/Y) = \epsilon > 0$, then for any natural number $m > 4/\epsilon$, there are no $a_i \in \mathcal{A}$, $i < m$, such that $a_0 = a$, $\mu^*(\phi(\mathbf{M}, a_0) \wedge \phi(\mathbf{M}, a_1))/Y) = 0$ and for all $k < j < m$, $t(a_k a_j/b; \mathbf{M}) = t(a_0 a_1/b; \mathbf{M})$.

Recall that $\phi(x, a)$ divides over (countable) B if there is a sequence $(a_i)_{i<\omega}$ and $n < \omega$ such that $a_0 = a$, $(a_i)_{i<\omega}$ is order-indiscernible over B and

$$\mathbf{M} \models \neg\exists x \bigwedge_{i \leq n} \phi(x, a_i).$$

We say that $\phi(x, a)$ forks over B if there are formulas $\phi_i(x, a_i)$, $i < n$, such that each ϕ_i divides over B and

$$\mathbf{M} \models \forall x(\phi \to \vee_{i<n} \phi_i).$$

We write $a \downarrow_C B$, if no $\phi(x, b) \in t(a/CB; \mathbf{M})$ forks over C.

We say that $\phi(x, a)$, $a \in \mathcal{A}$, splits strongly over $b \in \mathcal{A}$ inside $\mathcal{A}$ if there is a sequence $(a_i)_{i<\omega} \subseteq \mathcal{A}$ order-indiscernible over b such that $a_0 = a$ and $\phi(x, a_0) \wedge \phi(x, a_1)$ is not realized in $\mathbf{M}$.

1.10 Corollary.

(i) If $a \in \mathcal{A}$, Δ-formula $\phi(x, a)$ splits strongly over $b \in \mathcal{A}$ inside $\mathcal{A}$ and $Y \subseteq \mathbf{M}^{length(x)}$ is a non-empty set Δ-definable over b, then $\phi(\mathbf{M}, a) \in I_Y$.

(ii) Suppose $(\mu_i)_{i<\omega}$ is uniformly U-continuous. If Δ-formula $\phi(x, a)$, $a \in \mathbf{M}$, divides over $b \in \mathcal{A}$ and $Y \subseteq \mathbf{M}^{length(x)}$ is a non-empty set Δ-definable over b, then $\phi(\mathbf{M}, a) \in I_Y$.

(iii) Suppose $(\mu_i)_{i<\omega}$ is uniformly U-continuous. If Δ-formulas $\psi_i(x, a_i)$, $a_i \in \mathbf{M}$ and $i \leq n$, divide over $b \in \mathcal{A}$ and $Y \subseteq \mathbf{M}^{length(x)}$ is a non-empty set Δ-definable over b, then $\vee_{i \leq n} \psi_i(\mathbf{M}, a_i) \in I_Y$.

Proof. The proof of (i) is similar to the proof of (ii). For (ii), let a_i, $i<\omega$, and $n<\omega$ witness that $\phi(x,a)$ divides over b. By ω_1-saturation of $\mathbf{M}$, we may assume that $a_0=a$. Then $X=\cap_{i\le n}\phi(\mathbf{M},a_i)=\emptyset$ and thus $X\in I_Y$. By Lemma 1.9, $\phi(\mathbf{M},a)=\phi(\mathbf{M},a_0)\in I_Y$.

(iii): So suppose each $\psi_i(x,a_i)$ divides over b. Then by (ii),

$$\mu^*(\vee_{i\le n}\psi_i(\mathbf{M},a_i)/Y)\le\Sigma_{i\le n}\mu^*(\psi_i(\mathbf{M},a_i)/Y)=0.$$

□

1.11 Definition.

(i) We say that Δ*-formula* $\phi(x,a)$ μ^**-forks over* b *if for all non-empty sets* Y *of arity* $length(x)$ Δ*-definable over* b *in* $\mathbf{M}$, $\phi(\mathbf{M},a)\in I_Y$ *(i.e.* $\mu^*(\phi(\mathbf{M},a)/Y)=0$*).*

(ii) Let p *be a* Δ*-type (i.e. for some* x, p *is a collection of* Δ*-formulas* $\phi(x,a)$, $a\in\mathbf{M}$*). We say that* p μ^**-forks over* b *if there are* $n\in\omega$ *and formulas* $\phi_i(x,a_i)\in p$, $i\le n$, *such that* $\wedge_{i\le n}\phi_i(x,a_i)$ μ^**-forks over* a.

(iii) For countable $C\subseteq\mathbf{M}$ *and finite* $B\subseteq\mathbf{M}$, *We write* $a\downarrow^*_B C$ *if the type* $t_\Delta(a/BC;\mathbf{M})$ *does not* μ^**-fork over* B.

(iv) For countable $A,C\subseteq\mathbf{M}$ *and finite* $B\subseteq\mathbf{M}$, *we write* $A\downarrow^*_B C$ *if for all (finite sequences)* $a\in A$, $t_\Delta(a/BC;\mathbf{M})$ *does not* μ^**-fork over* B.

Above there is a slight missuse of notation: If $A=\{a_0,...,a_n\}$ and $a=(a_0,...,a_n)$, then it may happen that $a\downarrow_B C$ but $A\not\downarrow_B C$, see Example 2.4.

1.12 Lemma. *Let* $\phi(x,y)$ *be a* Δ*-formula and* p *a countable* Δ*-type over* $\mathbf{M}$.

(i) If $\phi(x,a)$, $a\in\mathbf{M}$, *does not* μ^**-fork over* A *and* $A\subseteq B$ *(both finite), then* $\phi(x,a)$ *does not* μ^**-fork over* B.

(ii) If $a,b,c,d\in\mathcal{A}$ *and* $t(ab/\emptyset;\mathbf{M})=t(cd/\emptyset;\mathbf{M})$, *then* $\phi(x,a)$ μ^**-forks over* b *iff* $\phi(x,c)$ μ^**-forks over* d. *If* $(\mu_i)_{i<\omega}$ *is uniformly* U*-continuous, then this holds for all* $a,b,c,d\in\mathbf{M}$.

(iii) If $a,b\in\mathcal{A}$ *and* $\phi(x,a)$ *does not* μ^**-fork over* b, *then there is* $\psi(y,z)\in t(ab/\emptyset;\mathbf{M})$ *such that if* $c,d\in\mathcal{A}$ *and* $\mathbf{M}\models\psi(c,d)$, *then* $\phi(x,c)$ *does not* μ^**-fork over* d. *If* $(\mu_i)_{i<\omega}$ *is uniformly* U*-continuous, then this holds for* $\mathbf{M}$ *in place of* $\mathcal{A}$.

(iv) If $p\vdash\phi(x,b)$ *and* $\phi(x,b)$ *is a* Δ*-formula such that it* μ^**-forks over* a, *then* p μ^**-forks over* a.

(v) If $t_\Delta(a/b;\mathbf{M})$ *is algebraic, then* $a\downarrow^*_b c$ *for all* c.

(vi) If $t_\Delta(a/b;\mathbf{M})$ *is not algebraic but* $t_\Delta(a/bc;\mathbf{M})$ *is, then* $a\not\downarrow^*_b c$.

Proof. (i) follows from the fact that if Y witnesses that $\phi(x,a)$ does not μ^*-fork over A, then it is Δ-definable over A and thus also over B and thus it witnesses that $\phi(x,a)$ does not μ^*-fork over B.

(v) is clear since if $\phi(\mathbf{M},b)$ contains n elements, then for all Δ-definable X which contain a, $\mu^*(X/\phi(M,b))\ge 1/n>0$.

(vi): Clearly we can find algebraic $\phi(x,bc) \in t_\Delta(a/bc:\mathbf{M})$ such that

$$\phi(x,bc) \vdash t_\Delta(a/bd;\mathbf{M}).$$

But then if Y is Δ-definable over b and $\phi(\mathbf{M},bc) \cap Y \neq \emptyset$, Y is not algebraic. Thus $\mu^*(\phi(\mathbf{M},bc)/Y) = 0$.

The rest are immediate by Lemma 1.7 and definitions. □

We say that $\downarrow^*$ is symmetric if for all $a,b,c \in \mathbf{M}$, $a \downarrow^*_b c$ implies $c \downarrow^*_b a$.

The proof of Lemma 1.13 (iii) below uses ideas from [Hr].

1.13 Lemma. *Let $a,b,c,d \in \mathbf{M}$, $\phi(x,y)$ be a Δ-formula and p a countable Δ-type over $\mathbf{M}$.*

(i) If $\phi(\mathbf{M},a) \neq \emptyset$, then $\phi(x,a)$ does not μ^*-fork over a.

(ii) $a \downarrow^*_b cd$ implies $a \downarrow^*_{bc} d$ and $a \downarrow^*_b c$. In particular, if $\downarrow^*$ is symmetric, $a \downarrow^*_b c$ and d is a subsequence of a, then $d \downarrow^*_b c$.

(iii) If p does not μ^*-fork over b, then for all countable $A \subseteq \mathbf{M}$ there is $a \in \mathbf{M}$ such that $a \models p$ and $a \downarrow^*_b A$.

Proof. For (i), just let Y in the definition of μ^*-forking be $\phi(\mathbf{M},a)$ and (ii) is immediate by Lemma 1.12 (i). So we prove (iii): By compactness and the fact that $\mathbf{M}$ is ω_1-saturated, it is enough to show that

$$\theta = \wedge_{i\leq n}\phi_i(x,b^i) \wedge \wedge_{i\leq n}\neg\psi_i(x,c^i)$$

is consistent, where $\phi_i(x,b^i) \in p$ and $\psi_i(x,c^i)$ are such that they μ^*-fork over b and $c^i \in A$. Let Y witness that $\wedge_{i\leq n}\phi_i(x,b^i)$ does not μ^*-fork over b i.e. Y is Δ-definable over b and $\mu^*(\wedge_{i\leq n}\phi_i(\mathbf{M},b^i)/Y) \geq \epsilon > 0$. Now by the definition of μ^*-forking, one can see that $X \in U$, when X is the set of all $j < \omega$ such that for all $i \leq n$, $\mu_j(\psi_i(\mathcal{A}_j,c^i_j)/Y_j) < \epsilon/(2n+2)$ and $\mu_j(\wedge_{k\leq n}\phi_k(\mathcal{A}_j,b^i_j)/Y_j) > \epsilon/2$. Then we can find $a = a/U \in \mathbf{M}$ such that for all $j \in X$, $a_j \in \cap_{i\leq n}\phi_i(\mathcal{A}_i,b^i_j) - \cup_{j\leq n}\psi_i(\mathcal{A}_j,c^i_j)$. Clearly a realizes θ. □

1.14 Corollary. *Assume that $\downarrow^*$ is symmetric. Suppose $B,C \subseteq \mathbf{M}$, B finite and C countable and $A \subseteq \mathbf{M}$ is countable. If $A \downarrow^*_B C$, then for all countable $D \subseteq \mathbf{M}$, there is A' such that*

$$t_\Delta(A'/BC;\mathbf{M}) = t_\Delta(A/BC;\mathbf{M})$$

and $A \downarrow^*_B D$.

Proof. Write $A = (a_i)_{i<\omega}$ and by x, x_k etc. we mean finite sequences of variables from the set $\{v_i|\ i < \omega\}$. Let $\phi(x,b) \in t_\Delta((a_i)_{i<\omega}/BC;\mathbf{M})$ and $\psi_k(x_k,d_k)$, $k < n$ and $d_k \in D$, be such that each $\psi_k(x_k,d_k)$ μ^*-forks over B. Again it is enough to show that $\theta = \phi(x,b) \wedge \wedge_{k<n}\neg\psi(x_k,a_k)$ is realised in $\mathbf{M}$. Let $m < \omega$ be such that if v_i appears in x or in some x_k, then $i < m$. By Lemma 1.13 (iii), there is $(a'_i)_{i<m} \in \mathbf{M}$ such that $t_\Delta((a'_i)_{i<m}/BC;\mathbf{M}) = t_\Delta((a_i)_{i<m}/BC)$ and $(a'_i)_{i<m} \downarrow^*_B D$. By Lemma 1.13 (ii), $(a'_i)_{i<m}$ realizes θ. □

Let $\mathbf{M}^{**}$ be some large very saturated elementary extension of $\mathbf{M}$.

1.15 Definition. *Suppose* $(\mu_i)_{i<\omega}$ *is uniformly* U*-continuous.*

(i) For $a, b \in \mathbf{M}$*, we say that* $t(a/b;\mathbf{M})$ *is good if for all countable* $C \subseteq \mathbf{M}$ *there is* a' *such that* $t(a'/b;\mathbf{M}) = t(a/b;\mathbf{M})$, $a' \downarrow^*_b C$ *and* $a' \notin C$.

(ii) For $a, b, c \in \mathbf{M}^{**}$*, we write* $a \downarrow^*_b c$ *if for all (some),* $a', b', c' \in \mathbf{M}$ *with* $t(a'b'c'/\emptyset;\mathbf{M}) = t(abc/\emptyset;\mathbf{M}^{**})$, $a \downarrow^*_b c$

Notice also that if $t(a/b;\mathbf{M})$ is not algebraic and $t(a/b;\mathbf{M}) \not\vdash \vee_{i\leq n}\psi_i(x,b_i)$ for any Δ-formulas $\psi_i(x,b_i)$ which μ^*-fork over b, then $t(a/b;\mathbf{M})$ is good.

1.16 Lemma. *Suppose* $(\mu_i)_{i<\omega}$ *is uniformly* U*-continuous and* $\mathbf{M}$ *is stable. Then for all* $a, b, c \in \mathbf{M}$*, the following holds: if* $t(c/b;\mathbf{M})$ *is good and* $a \downarrow^*_b c$*, then either* $a \not\downarrow_b c$ *or* $c \downarrow^*_b a$*. In particular, if* $\mathbf{M}$ *has* Δ*-elimination of quantifiers,* $\downarrow^*$ *is symmetric.*

Proof. Suppose not i.e. $a \downarrow_b c$ and $c \not\downarrow^*_b a$. Since $t(c/b;\mathbf{M})$ is good and $\mathbf{M}$ is ω_1-saturated by compactness as in the proof of Lemma 1.3 (iii), we can find $a_i, c_i \in \mathbf{M}^{**}$, $i < \kappa = \beth_{(2^\omega)^+}$ so that for all $i < \kappa$, $t(a_ic_i/b;\mathbf{M}^{**}) = t(ac/b;\mathbf{M})$ and $a_i \downarrow_{bc_i} \cup_{j<i}a_jc_j$ (and so $a_i \downarrow_b \cup_{j<i}a_jc_j$), $c_i \notin \cup_{j<i}ba_jc_j$ and $c_i \downarrow^*_b d$ for all $d \in \cup_{j<i}a_jb_j$. By Erdös-Rado, we may assume that $(a_ic_i)_{i<\omega}$ is an (infinite) indiscernible sequence over b and by ω_1-saturation of $\mathbf{M}$ that $a_ic_i \in \mathbf{M}$ for all $i < \omega$ and that $a_0 = a$ and $c_0 = c$. By the choise of c_i, $c_i \downarrow^*_b a_j$ if $i > j$. If $i < j$, then $stp(c_i/b;\mathbf{M}) = stp(c/b;\mathbf{M})$ and $stp(a_j/b;\mathbf{M}) = stp(a/b;\mathbf{M})$ and since both $a \downarrow_b c$ and $a_j \downarrow_b c_i$, $t(ac/b;\mathbf{M}) = t(a_jc_i/b;\mathbf{M})$ and so $c_i \not\downarrow^*_b a_j$. By Lemma 1.12 (ii), this is a contradiction with the stability of $\mathbf{M}$.

For the in particular part we assume that $\mathbf{M}$ has Δ-elimination of quantifiers and that $a \downarrow^*_b c$ and show that $c \downarrow^*_b a$. We observe first that by Lemma 1.13 (i) and (iii) and Lemma 1.12 (v), if $t(c/b;\mathbf{M})$ is not algebraic, then it is good. But if $t(c/b;\mathbf{M})$ is algebraic, then by Lemma 1.12 (v), $c \downarrow^*_b a$. So we may assume that $t(c/b;\mathbf{M})$ is good. By Δ-elimination of quantifiers, Corollary 1.10 (iii) and Lemma 1.12 (iv), $a \downarrow_b c$ and so by what we showed above $c \downarrow^*_b a$. □

1.17 Theorem. *Suppose* $(\mu_i)_{i<\omega}$ *is uniformly* U*-continuous,* $\mathbf{M}$ *is stable and has* Δ*-elimination of quantifiers. Then for all* $a, b, c \in \mathbf{M}$*,* $a \downarrow_b c$ *iff* $a \downarrow^*_b c$.

Proof. By Δ-elimination of quantifiers, we have already shown the implication from right to left in Lemma 1.12 (iv) and Corollary 1.10 (iii). So we prove the other direction. By Corollary 1.14, Lemma 1.16 and Δ-elimination of quantifiers, we can find a countable elementary submodel $\mathcal{B}$ of $\mathbf{M}$ such that $b \in \mathcal{B}$ and $\mathcal{B} \downarrow^*_b ac$. Then $\mathcal{B} \downarrow_b ac$ and so $a \downarrow_b \mathcal{B}c$, in particular $a \downarrow_{\mathcal{B}} c$. Also by monotonicity and symmetry (i.e. Lemma 1.16), $a \downarrow^*_b \mathcal{B}$. So by Corollary 1.14 and Δ-elimination of quantifiers, there is a' such that $t(a'/\mathcal{B};\mathbf{M}) = t(a/\mathcal{B};\mathbf{M})$ and $a' \downarrow^*_b \mathcal{B}c$. Since the implication from right to left holds $a' \downarrow_{\mathcal{B}} c$. Thus $t(a/bc;\mathbf{M}) = t(a'/bc;\mathbf{M})$. Since $a' \downarrow^*_b c$, Lemma 1.12 (ii) implies that $a \downarrow^*_b c$. □

Before the examples we want to observe, that although we have assumed that Δ consists of first-order formulas, there are cases in which we may loosen this assumption: Suppose that the vocabulary L is finite and relational and let $k \in \omega$

be greater than the arity of any relation in L. By L^k we mean the set of all $L_{\omega_1\omega}$-sentences such that at most k variables appear in them. We let Δ be the closure of L^k under (finite) boolean combinations (so in Δ-formulas, any finite number of variables may appear). Finally suppose that $(\mathcal{A}_i)_{i<\omega}$ is such that $\mathcal{A}_i$ is L^k-elementary submodel of $\mathcal{A}_{i+1}$. Then each $\mathcal{A}_i$ is Δ-elementary submodel of $\mathbf{M} = \Pi_{i<\omega}\mathcal{A}_i/U$ and we may think the formulas of L^k as atomic formulas. When we do this, $(\mathcal{A}_i)_{i<\omega}$ is a Δ-sequence and $\mathcal{A}$ is strongly Δ-atomic, in fact, for each n the number of $t_\Delta(a/\emptyset;\mathcal{A})$, $a \in \mathcal{A}^n$, is finite. Now it is easy to see that everything we have done above goes through also in this situation.

Also one expects that what we have done above can also be done for continuous logic ($\sim$ fussy logic).

1.18 Open question. *If in Theorem 1.17 uniform U-continuity is replaced by just U-continuity, can one still prove that $a \downarrow_b c$ iff $a \downarrow_b^* c$, for all $a, b, c \in \mathcal{A}$.*

2 Examples

In this section we give examples. The first three fit to our context and the last two are counterexamples.

2.1 Example. *Let F be a finite field and for all $i < \omega$, let $\mathcal{A}_i$ be an $i+1$-dimensional vector space over F chosen so that $\mathcal{A}_i$ is a subspace of $\mathcal{A}_{i+1}$. We let Δ be the set of all quantifier free formulas. Now $(\mathcal{A}_i)_{i<\omega}$ is a Δ-sequence, the number of $t_\Delta(a/\emptyset;\mathcal{A})$, $a \in \mathcal{A}^n$, is finite for all $n < \omega$ and $(\mu_i)_{i<\omega}$ is uniformly U,Δ-continuous, in fact, for Δ-formulas $\phi(x,y)$ and $\psi(x,y)$ and $a \in \mathcal{A}$ such that $\mathcal{A} \models \forall x(\phi(x,a) \rightarrow \psi(x,a))$, letting deg be the Morley degree,*

$$\mu(\phi(\mathcal{A},a)/\psi(\mathcal{A},a)) = \mu^*(\phi(\mathcal{A},a)/\psi(\mathcal{A},a)) = deg(\phi(x,a))/deg(\psi(x,a))$$

if the generic element in $\phi(\mathbf{M},a)$ and the generic element in $\psi(\mathbf{M},a)$ have the same dimension i.e. the sets have the same Morley rank and otherwise

$$\mu(\phi(\mathcal{A},a)/\psi(\mathcal{A},a)) = \mu^*(\phi(\mathcal{A},a)/\psi(\mathcal{A},a)) = 0.$$

Also this value is clearly definable (and not only approximately) with a Δ-formula and since the number of possible Δ-types is finite, this can be done uniformly. Of course, $\mathbf{M}$ is stable and has Δ-elimination of quantifiers.

2.2 Example. *Let p be a prime and $N_i = \Pi_{n\leq i}p_n^i$, where $(p_n)_{n<\omega}$ is the usual enumeration of primes. Let $\mathcal{A}_i$ be a field of size p^{N_i} chosen so that $\mathcal{A}_i$ is a subfield of $\mathcal{A}_{i+1}$ and Δ be the set of quantifier free formulas. Again $\mathbf{M}$ is stable and has Δ-elimination of quantifiers. Also the fact that $(\mu_i)_{i<\omega}$ is uniformly U-continuous can be seen essentially as in Example 2.1 using (the proof of) Main Theorem in [CDM]: We write $MR(\phi(x,a))$ for the Morley rank of $\phi(x,a)$ in $\mathbf{M}$. By [CDM], for every Δ-formula $\phi(x,y)$ there are $n < \omega$ and for $j < n$, a formula*

$\psi_j(y)$ *and a rational number* μ_j, *such that for all* $a \in \mathcal{A}$ *the following holds: For all* $\epsilon > 0$ *and for all* $i < \omega$ *large enough*

$$|q - |\phi(\mathcal{A}_i, a)|/|\mathcal{A}_i^{MR(\phi(x,a))}|| < \epsilon$$

for some

$$q \in \{\mu_j | \ j < n\}$$

and for all $q \in \{\mu_j| \ j < n\}$ *for all* $\epsilon > 0$ *small enough and* $i < \omega$ *large enough,*

$$|q - |\phi(\mathcal{A}_i, a)|/|\mathcal{A}_i^{MR(\phi(x,a))}|| < \epsilon$$

iff $\mathcal{A}_i \models \bigvee\{\psi_j(y)| \ \mu_j = q\}$. *Finally, for all* $j < n$ *and* $i < \omega$, $\mathcal{A}_i \models \psi_j(a) \wedge \psi_j(b)$ *implies* $MR(\phi(x,a)) = MR(\phi(x,b))$. *From this the claim follows immediately. For more on this example and variations, see the theory of pseudo-finite fields.*

2.3 Example. *For all* $i < \omega$, *let* $\mathcal{A}_i = (A_i, E_k)_{k<\omega}$ *be such that* $\mathcal{A}_i$ *consist of functions* $f : \omega \to i+1$ *such that for all* $x > i$, $f(x) = 0$ *and* $f \ E_k \ g$ *if* $f \restriction (k+1) = g \restriction (k+1)$. *Again, let* Δ *be the set of quantifier free formulas. Then* $(\mu_i)_{i<\omega}$ *is uniformly* U, Δ*-continuous,* $\mathbf{M}$ *is stable and has* Δ*-elimination of quantifiers. Notice that this class is not* N*-dimensional asymptotic class for any* $N < \omega$.

2.4 Example. *For all* $i < \omega$, *let* $\mathcal{A}_i = (A_i, E)$ *be such that* $\mathcal{A}_i$ *consists of pairs* $(k, m) \in \omega^2$ *such that* $k \leq i$ *and if* $k = 0$, *then* $m \leq i^2$ *and otherwise* $m \leq i$. $(k,m) \ E \ (k', m')$ *if* $k = k'$. Δ *is the set of quantifier free formulas. Now the class is not* U*-continuous and in* $\mathbf{M}$, $E(x, (0,0))$ *forks over* $\emptyset$ *but it does not* μ^**-fork over* $\emptyset$.

If instead of $k \leq i$ *we require that* $k < 2$, *we get a class which still is not* U*-continuous and in which* $E(x,(1,0))$ *does not fork over* $\emptyset$ *but it* μ^**-forks over* $\emptyset$.

We also want to point out what happens in the latter example with the formula $\phi(x) = \neg E(y,z)$ *in relation with Lemma 1.13. Now letting* $a = (0,0)$ *and* $b = (1.1)$, $t_\Delta((a,b)/\{(1,0)\}; \mathbf{M})$ *does not* μ^**-fork over* $\emptyset$ *(*$\phi(\mathbf{M})$ *witnesses this) while* $t_\Delta((1.0)/\{a,b\}; \mathbf{M})$ *does* μ^**-fork over* $\emptyset$ *and so symmetry fails. Of course also* $t_\Delta(b/\{(1,0)\}; \mathbf{M})$ μ^**-forks over* $\emptyset$ *and thus also the conclusion of the in particular part of Lemma 1.13 (ii) fails.*

2.5 Example. *For all* $i < \omega$, *define* N_i *so that* $N_0 = 3$ *and* $N_{i+1} = 2N_i + 1$. *Let* $\mathcal{A}_i = (A_i, <)$, *where* A_i *is the set of all integers between* $-N_i$ *and* N_i *and* $<$ *is the natural ordering. By replacing* $\mathcal{A}_i$, $i > 0$, *by isomorphic copies, we may assume that the embeddings* $x \mapsto 2x$ *are identical maps. Let* Δ *be the set of quantifier free formulas. Then* $(\mu_i)_{i<\omega}$ *is not* U*-continuous (and* $\mathbf{M}$ *is not stable) and* $-1 < x \wedge x < 1$ $(-1, 1 \in A_0)$ *forks over* $\emptyset$ *but it does not* μ^**-fork over* $\emptyset$. *Notice that here* $\mathbf{M}$ *is elementarily equivalent with* $\omega + \mathbf{Z} \times \mathbf{Q} + \omega^*$.

References

[CDM] Z. Chatzidakis, L. van den Dries and A. Macintyre, Definable sets over finite fields, Journal für die Reine und Angewandte Mathematik, vol. 427, 1992, 107-135.

[El] R. Elwes, Asymptotic classes of finite structures, Journal of Symbolic Logic, vol. 72, 2007, 418-438.

[Hi] C.D. Hill, Efficiently inverting the L^2-invariant through stability theory, extended abstract in Logical Approaches to Barriers in Computing and Complexity, 2010.

[HP] E. Hrushovski and A. Pillay, On NIP and invariant measures, J. Eur. Math. Soc., vol. 13, 2011, 1005-1061.

[Hr] E. Hrushovski, Stable group theory and approximate subgroups, J. Amer. Math. Soc., vol. 25, 2012, 189-243.

[Hy] T. Hyttinen, On stability in finite models, Archive for Mathematical Logic, vol. 39, 2000, 89-102.

[Ko] M. Djordjevic, Finite variable logic, stability and finite models, Journal of Symbolic Logic, vol. 66, 2001, 837-858.

[Va] J. Väänänen, Pseudo-finite model theory, Matematica Contemporanea, vol. 24, 2003, 169-183.

Department of Mathematics and Statistics
P.O.Box 68
00014 University of Helsinki
Finland