Some inference results on random pure exchange economies

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Abstract

In this paper we apply the theory of large deviations to a random exchange economy. Two types of the observations, namely partial observation and full observation, and their consequences on our a posteriori knowledge about the equilibrium are discussed. A random exchange economy with economic sectors is used as an example.

Keywords: large deviations, random exchange economy, observation, equilibrium, economic sectors.

1 Introduction

In this paper we study a random exchange economy in which the preferences and endowments of the agents are "random" or "subject to exogenous shocks", namely the economy is stage contingent on the economic environment ω. As the general human behavior is, an agent's preferences and endowments may be state contingent. As argued in the seminal works of [Hil71] and [BM73], random characters of the agents might be a better way to interpret human behavior in economic activities than deterministic characters. An interesting problem to pose for a large random exchange economy is to know the a priori relation of random equilibrium prices with their model-based a priori expected values. In particular, since the equilibrium prices of such an economy are random, we want to analyze the existence and convergence properties of the random equilibrium prices, i.e., whether there exist a law of large numbers and a central limit theorem. In contrast with other literature dealing with this topic, we advocate the relevance of the theory of large deviations for the equilibrium results of random exchange economies. By utilizing the theory of large deviations, Nummelin proved the existence and convergence of the equilibrium prices in a general dependent case. The results in [Num00b] and [Num00a] do not depend on some special structures of the agents such as independent and identically distributed ([Hil71] and [BM73]), exchangeability and strong mixing ([BM73]), lattice ([Fol74]) or dependency neighborhoods ([MR00] and [MR02]).

We notice that the existence and convergence results mentioned above heavily rely on the knowledge of the (a priori) probability distribution of the economic environment, $P(d\omega)$. For example, in [MR00],[MR02],[Num00b] and [Num00a], such probability law is required in calculating the expectation of the total excess demand function. Eventually the random equilibrium prices converge to some non-random prices in the convergence set which contains prices making such expectation of the total excess demand vanish. Our choice of the (a priori) probability law can be subjective or empirical, but on the other hand, due to the complexity of the economies, it is difficult to know the ”true” probability law governing the economic environments exactly a priori. Eventually, the (a posteriori) observed equilibrium may deviate significantly from its model-based (a priori) expected value. In fact, it can be argued that the observed equilibrium may represent a large deviation, namely fall outside the region of the validity of the central limit theorem. Thus it will be a natural interest to address the problem of the inference from the a posteriori observation of the equilibrium (possibly representing a large deviation):

- What (a posteriori) knowledge do we know about the equilibrium conditionally on an observation of the realized equilibrium?

In this paper, two types of the observations, namely partial observation and full observation, and their consequences on our a posteriori knowledge about the equilibrium are discussed. By a partially observed equilibrium, we could predict the full equilibrium even we do not know the exact "true" probability distribution of the economic environments. If we have a full observation of the equilibrium price, we are able to calibrate the a priori probability law to a "true" probability law such that our observation could be consistent with the expected equilibrium prices under the "true" probability law.

2 The description of the model

We consider an exchange economy where $l+1$ commodities are traded by n random agents. We assume that we are dealing with a "large economy"; namely in the exact theorems we let $n \to \infty$. In economic literature, an agent can be described in many ways such as by its preference and endowment or by its demand function and endowment. In this paper, a random agent is characterized by its excess demand function $\zeta(\omega, p)$. $\zeta(\cdot)$ is a random vector defined on an underlying probability space $(\Omega, \mathscr{F}, P)$, i.e.,

$$\zeta(\omega, p) \doteq (\zeta^1(\omega, p), \ldots, \zeta^{l+1}(\omega, p)), \omega \in \Omega.$$

Here Ω denotes the set of all possible scenarios in the economy and $p = (p^1, \ldots, p^{l+1})$ is the given market price. The superscripts $1, \ldots, l+1$ denote the $l+1$ commodities.

We make the following two standard assumptions:

1. Homogeneity of degree zero:

$$\zeta^j(ap) = a\zeta^j(p), j = 1, \ldots, l+1 \text{ for every constant } a > 0;$$

2. Walras' law:

$$\sum_{j=1}^{l+1} p^j \zeta^j(p) = 0.$$

Due to the homogeneity of degree zero, the prices can be normed to belong to the price simplex

$$S^l \doteq \{p \in \mathbb{R}^{l+1}_{++} : \|p\|_1 = 1\}.$$

Then it suffices to put the prices under consideration in the set $\Delta \subset \{p \in \mathbb{R}^l_{++} : \|p\|_1 < 1\}$.

Assumption 1 *Throughout this paper we choose Δ to be compact and convex, and to have non-empty interior.*

Walras' law implies that it suffices to consider the excess demand for the first l commodities. Then, without loss of generality, we can just consider an excess demand functions $\zeta : \Omega \times \Delta \to \mathbb{R}^l$ from now on.

The total excess demand is the sum of all individual excess demands in the economy, defined as

$$\begin{aligned} Z_n(\omega, p) &\doteq (Z_n^1(\omega, p), \dots, Z_n^l(\omega, p)) \\ &= (\sum_{i=1}^{n} \zeta_i^1(\omega, p), \dots, \sum_{i=1}^{n} \zeta_i^l(\omega, p)). \end{aligned}$$

The random equilibrium prices (r.e.p.) are defined as those price vectors $p_n^* = \{p_n^*(\omega), \omega \in \Omega\}$ at which the (random) total excess demand vanishes:

$$Z_n(\omega, p_n^*(\omega)) = 0.$$

Formally, the random equilibrium prices comprise a random set:

$$\pi_n^* = \{\pi_n^*(\omega); \omega \in \Omega\}$$

where

$$\pi_n^*(\omega) = \{p_n^* \in \Delta : Z_n(\omega, p_n^*) = 0\}$$

denotes the set of equilibrium prices at the realized economic environment. Thus $p_n^*(\omega)$ denotes an arbitrary element of $\pi_n^*(\omega)$. Let $EZ_n(p) \doteq \int Z_n(\omega, p)P(d\omega)$ denote the (non-random) expected total excess demand function.

From now on, we will omit the parameter ω where there is no misunderstanding. We define a price depending total characteristic of the economy, $X_n(p)$. The equilibrium behavior of $X_n(p)$ is described by the graph $\pi_n^*(X)$ of the random map $n^{-1}X_n(\cdot)$ with the random equilibrium set π_n^* as its support, i.e.,

$$\pi_n^*(X) \doteq \{(p_n^*, n^{-1}X_n(p_n^*)); Z_n(p_n^*) = 0\}.$$

Assumption 2 *We assume the limit $\mu(p) \doteq \lim_{n\to\infty} n^{-1}EX_n(p)$ exists.*

3 Main results

Let $\Lambda_n(\alpha; p)$ denote the Laplace transform of the total excess demand $Z_n(p)$ under an a priori chosen probability law $P(d\omega)$ governing the economic environments:

$$\Lambda_n(\alpha; p) \doteq Ee^{\alpha \cdot Z_n(p)} = \int e^{\alpha \cdot Z_n(\omega,p)} P(d\omega), \alpha \in \mathbb{R}^l$$

and let

$$\Lambda_{X;n}(\alpha,\beta; p) \doteq Ee^{\alpha \cdot Z_n(p)+\beta \cdot X_n(p)}, \alpha \in \mathbb{R}^l, \beta \in \mathbb{R}^d$$

denote the Laplace transform of the pair $(Z_n(p), X_n(p))$. $\Lambda_n(p)$ is defined as the minimum of $\Lambda_n(\alpha; p)$:

$$\Lambda_n(p) \doteq \min_{\alpha \in \mathbb{R}^l} \Lambda_n(\alpha; p).$$

Let $c_n(\alpha; p)$ and $c_(X; n)$ be the logarithm of the Laplace transforms, i.e.,

$$c_n(\alpha; p) \doteq \log \Lambda_n(\alpha; p) = \log Ee^{\alpha \cdot Z_n(p)}$$

and

$$c_{X;n}(\alpha,\beta; p) \doteq \log \Lambda_{X;n}(\alpha,\beta; p) = \log Ee^{\alpha \cdot Z_n(p)+\beta \cdot X_n(p)}.$$

The logarithm of the Laplace transform of a random variable is known to be a convex function (see e.g. [Bil95], p.148).

Assumption 3 *We assume that the limits $c(\alpha; p) = \lim_{n\to\infty} n^{-1} c_n(\alpha; p)$ and $c_X(\alpha,\beta; p) = \lim_{n\to\infty} n^{-1} c_{X;n}(\alpha,\beta; p)$ exist and they converge uniformly over α and β.*

The economic entropy functions $I : \Delta \to \mathbb{R}_+$ and the economic entropy function with respect to the X, $I_X : \Delta \times \mathbb{R}^d \to \mathbb{R}_+$ are defined as

$$I(p) \doteq -\inf_{\alpha \in \mathbb{R}^l} c(\alpha; p) = \hat{c}(0; p)$$

and

$$I_X(p, w) \doteq \sup_{(\alpha,\beta)\in\mathbb{R}^{l+d}} (\beta \cdot w - c_X(\alpha,\beta; p)) = \hat{c}_X(0,\beta; p)$$

respectively, where

$$\hat{c}(v, ; p) \doteq \sup_{\alpha \in \mathbb{R}^l} (\alpha \cdot v - c(\alpha; p))$$

and

$$\hat{c}_X(v, w; p) \doteq \sup_{(\alpha,\beta)\in\mathbb{R}^{l+d}} (\alpha \cdot v + \beta \cdot w - c_X(\alpha,\beta; p)).$$

Assumption 4 *We assume that there exists a parameter* $\alpha = \alpha(p) \in \mathbb{R}^l$ *such that* $\partial c/\partial\alpha(\alpha(p); p) = 0$.

It follows

$$I(p) = -c(\alpha(p); p).$$

We make some assumptions on the mean total excess demand function, $n^{-1}Z_n$ and the mean of the price-depending economic variable, $n^{-1}X_n$. We notice that the first part of Assumption 7 on $n^{-1}Z_n$ can be derived from Assumption 6.

Assumption 5 *For each* $p \in \Delta$ *the first derivative of the mean total excess demand,* $n^{-1}Z_n'$ *exists and is uniformly non-singular almost surely; that means, there is a constant* $A_{-1}(p) < \infty$ *such that*

$$|(n^{-1}Z_n'(p))^{-1}| = |nZ_n'(p)^{-1}| \leq A_{-1}(p) \textit{ for all } n \in N, \textit{a.s.}.$$

Assumption 6 *The second derivative of the mean total excess demand,* $n^{-1}Z_n''$ *is bounded on some closed neighborhood* $\bar{U}$ *of the given price p; namely, there is a constant* $A_2(p) < \infty, \varepsilon_2(p) > 0$ *and we define* $\bar{U} \doteq \bar{U}(p, \varepsilon_2(p))$ *such that*

$$|n^{-1}Z_n''(q)| \leq A_2(p) \textit{ a.s. for all } q \in \bar{U}.$$

Assumption 7 *The first derivatives of the mean of total excess demand,* $n^{-1}Z_n'$ *and the mean of the macroeconomic variable,* $n^{-1}X_n$ *are bounded on some closed neighborhood* $\bar{U}$ *of the given price p; namely, there is a constant* $A_1(p) < \infty, \varepsilon_1(p) > 0$ *and we define* $\bar{U} \doteq \bar{U}(p, \varepsilon_1(p))$ *such that*

$$|n^{-1}Z_n'(q)| \leq A_1 \textit{ and } |n^{-1}X_n'(q)| \leq A_1 \textit{ a.s. for all } q \in \bar{U}.$$

For the purpose of inference, we distinguish two types of a posteriori observations of the equilibrium: the partial observation and the full observation.

3.1 Partial observation

The idea of the expectation of the total excess demand $EZ_n(p)$ is essential in getting the equilibrium prices of a random exchange economy. In a large random exchange economy, the equilibrium prices should be in the equilibrium set containing prices p such that $\lim_{n\to\infty} EZ_n(p) = 0$. Unfortunately, due to the complexity of the economy, it is difficult to know the exact "true" probability law governing the economic environments a priori. Consequently this makes it difficult to construct the exact equilibrium prices from the model. With the help of a partial observation in which the equilibrium price or some price-depending economic variable are observed only partially, results in this section provide us one way to calculate not only the full equilibrium price but also the equilibrium behavior of the price-depending economic variable.

A mathematical definition of partial observation is that the prevailing equilibrium graph $(p_n^*, n^{-1}X_n(p_n^*))$ is in some convex observation set $C \subset \mathbb{R}^{l+d}$ which is consistent with the partial observation, i.e.,

$$\pi_n^*(X) \cap C \neq \emptyset.$$

The next theorem shows that our best inferences on the equilibrium price and the equilibrium value of the price-depending mean total characteristic of the economy, $n^{-1}X_n$, conditionally on a partial observation of the equilibrium graph $\pi_n^*(X)$ in the observation set C, are the entropy minimizing prices p_C and the entropy minimizing values X_C. In other words, the equilibrium graph $\pi_n^*(X)$ converges to the graph $\pi_C(X)$ which minimizes the entropy I_X among the observation set C. Furthermore, one may argue that for "almost all" convex sets $C \in \mathbb{R}^{l+d}$ the entropy minimizing set $\pi_C^*(X)$ is one point set (see [Num00a], Remark 3.1). In such cases, we have a more precise result saying that the equilibrium price p_n^* converges to the unique entropy minimizing price p_C and the mean of the equilibrium macroeconomic variable $n^{-1}X_n(p_n^*)$ converges to the unique entropy minimizing result X_C.

Theorem 1 *Suppose that* ***Assumptions 1 - 7*** *are satisfied and* C *is a convex set having non-empty interior such that* $I_X(C) < \delta_1$ *with some constant* $\delta_1 > 0$. *Then for all* $\varepsilon > 0$ *we have*

$$\lim_{n\to\infty} P(\pi_n^*(X) \subset U(\pi_C(X), \varepsilon) | \pi_n^*(X) \cap C \neq \emptyset) = 1.$$

Proof The mathematical result which leads to this theorem first appeared in [Num00a] Corollary 4.4, but its proof contains some errors. We provide a correct proof in this paper.

We first prove an upper bound estimate of the random graph $\pi_n^*(X)$:

$$\limsup_{n\to\infty} n^{-1} \log P(\pi_n^*(X) \cap \{I_X \geq \delta\} \neq \emptyset) \leq -\delta$$

for all $0 \le \delta \le \delta_1$ where $\delta_1 > 0$ is the constant mentioned in the Theorem such that the level sets $\bar{L}_X(\delta) \doteq \{I_X \le \delta\}$ are compact.

Let $b < \infty$ be an arbitrary fixed constant so that

$$K(b) \doteq \{(p, w) \mid \|w - \mu(p)\|_\infty \le b\}$$

becomes a compact set. It is proved in [Num00a] Theorem 5 that the level set $\{I_X \le \delta\}$ is also compact, so we have some $b < \infty$ such that

$$\{I_X \le \delta\} \subset K(b).$$

Let $F \doteq \{I_X \ge \delta\} \cap K(b)$, then F is compact and

$$\begin{aligned}
&P\{\pi_n^*(X) \cap \{I_X \ge \delta\} \neq \emptyset\} \\
&\le P\{\pi_n^*(X) \cap F \neq \emptyset\} + P\{K^c(b)\} \\
&\le P\{\pi_n^*(X) \cap F \neq \emptyset\} + P\{\|\frac{1}{n}X_n(p) - \mu(p)\|_\infty \ge b \text{ for some } p \in D\}.
\end{aligned}$$

We first consider $P\{\pi_n^*(X) \cap F \neq \emptyset\}$. For a fixed $\varepsilon_0 > 0$, since F is compact, we choose $\{(p_i, w_i)\}$ to be a finite subset of F such that for each $(p, n^{-1}X_n(p)) \in F$ there is some (p_i, w_i) such that

$$|p - p_i| \le \varepsilon_0,$$

$$|\frac{1}{n}X_n(p) - w_i| \le \varepsilon_0.$$

Due to Assumption 7, we have

$$|\frac{1}{n}Z_n(p) - \frac{1}{n}Z_n(p_i)| \le A_1\varepsilon_0 \doteq \varepsilon_1,$$

$$\begin{aligned}
|\frac{1}{n}X_n(p_i) - w_i| &\le |\frac{1}{n}X_n(p_i) - \frac{1}{n}X_n(p)| + |\frac{1}{n}X_n(p) - w_i| \\
&\le (A_1 + 1)\varepsilon_0 \doteq \varepsilon_2.
\end{aligned}$$

Thus

$$\begin{aligned}
&P\{\pi_n^*(X) \cap F \neq \emptyset\} \\
&= P\{\exists p \in D, \text{ such that } Z_n(p) = 0 \text{ and } (p, \frac{1}{n}X_n(p)) \in F\} \\
&\le P\{|\frac{1}{n}Z_n(p_i)| \le \varepsilon_1 \text{ and } |\frac{1}{n}X_n(p_i) - w_i| \le \varepsilon_2\}.
\end{aligned}$$

By the standard large deviation upper bound estimate, we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log P\{\pi_n^*(X) \cap F \neq \emptyset\} \\
&\leq \max_i \limsup_{n\to\infty} \frac{1}{n} \log P\{|\frac{1}{n} Z_n(p_i)| \leq \varepsilon_1 \text{ and } \frac{1}{n} X_n(p_i) \in \bar{U}(w_i, \varepsilon_2)\} \qquad (1) \\
&\leq \max_i (- \inf_{|v|\leq\varepsilon_1, w\in\bar{U}(w_i,\varepsilon_2)} \hat{c}_X(v, w; p_i)).
\end{aligned}
$$

Since $\hat{c}_X$ is a C^2-map, the inf can be attained on the domain $\{|v| \leq \varepsilon_1, w \in \bar{U}(w_i, \varepsilon_2)\}$. Let us expand the inf with the Lagrange remainder at $(0, w_i)$:

$$
\begin{aligned}
&\hat{c}_X(v, w; p_i) \\
&= \hat{c}_X(0, w_i; p_i) + \varepsilon_1 \frac{\partial \hat{c}_X}{\partial v}(\theta\varepsilon_1, w + \theta(\varepsilon_2 + \varepsilon_0)) + (\varepsilon_2 + \varepsilon_0) \frac{\partial \hat{c}_X}{\partial w}(\theta\varepsilon_1, w_i + \theta\varepsilon_2) \\
&= I_X(p_i, w_i) + M_1\varepsilon_1 + M_2\varepsilon_2
\end{aligned}
$$

with $M_1, M_2 < \infty$ since $\hat{c}_X$ is C^2. Thus we have

$$
\begin{aligned}
(1) &\leq \max_i (-(I_X(p_i, w_i) + M_1\varepsilon_1 + M_2\varepsilon_2)) \\
&= -\min_i (I_X(p_i, w_i) + M_1\varepsilon_1 + M_2\varepsilon_2) \\
&\leq -\delta - \min_i (M_1\varepsilon_1 + M_2\varepsilon_2).
\end{aligned}
$$

Since $(p_i, w_i) \in F$ hence $I_X(p_i, w_i) \geq \delta$. We also have ε_0 arbitrary fixed and $\varepsilon_1 \to 0, \varepsilon_2 \to 0$ when $\varepsilon_0 \to 0$. So we get the result:

$$
\limsup_{n\to\infty} \frac{1}{n} \log P\{\pi_n^*(X) \cap F \neq \emptyset\} \leq -\delta.
$$

Now let us consider $P\{\|\frac{1}{n} X_n(p) - \mu(p)\|_\infty \geq b$ for some $p \in D\}$. Since D is compact, we have for every $p \in D$ there is some p_i such that

$$
|\frac{1}{n} X_n(p) - \frac{1}{n} X_n(p_i)| \leq A_1\varepsilon_0 \doteq \varepsilon_3,
$$

$$
|\mu(p) - \mu(p_i)| = \lim_{n\to\infty} |E(\frac{1}{n} X_n(p) - \frac{1}{n} X_n(p_i))| \leq \varepsilon_3,
$$

where $|p - p_i| \leq \varepsilon_0$. Then we can choose ε_0 small enough such that

$$
\begin{aligned}
&\|\frac{1}{n} X_n(p_i) - \mu(p_i)\|_\infty \\
&= \|(\frac{1}{n} X_n(p_i) - \frac{1}{n} X_n(p)) + (\frac{1}{n} X_n(p) - \mu(p)) + (\mu(p) - \mu(p_i))\|_\infty \\
&\geq b - 2\varepsilon_3 \geq \frac{b}{2}.
\end{aligned}
$$

By the standard large deviation upper bound estimate:

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log P\{\|\frac{1}{n}X_n(p) - \mu(p)\|_\infty \geq b \text{ for some } p \in D\} \\
&\leq \max_i \limsup_{n\to\infty} \frac{1}{n} \log P\{\|\frac{1}{n}X_n(p_i) - \mu(p_i)\|_\infty \geq \frac{b}{2}\} \qquad (2)\\
&\leq \max_i (-\inf_{\|w-\mu(p_i)\|_\infty \geq \frac{b}{2}} \hat{c}_X(w;p_i)).
\end{aligned}
$$

Here $\hat{c}_X(w;p)$ denotes the convex conjugate function of the map $c_X(0,\beta;p) \doteq \lim_{n\to\infty} \frac{1}{n} \log Ee^{\beta\cdot X_n(p)}$. By [Num00a] Lemma 4.2, we have

$$
(2) \leq -\frac{b}{2} + \frac{M}{2}.
$$

Here $M \doteq \sup_{|\beta|\leq 1, p\in D} |\partial^2 c_X(0,\beta;p)| < \infty$. Since b is arbitrary large, we take $b \geq M + 2\delta$, then the order of $P\{\pi_n^*(X) \cap F \neq \emptyset\}(= e^{-n\delta}$ according to (1)) dominates the order of $P\{\| n^{-1}X_n(p) - \mu(p) \|_\infty \geq b$ for some $p \in D\}(= e^{-n(\frac{b}{2}-\frac{M}{2})}$ according to (2)). So we have proved the final assertion:

$$
\limsup_{n\to\infty} \frac{1}{n} \log P\{\pi_n^*(X) \cap \{I_X \geq \delta\} \neq \emptyset\} \leq -\delta.
$$

By [Num00a] Theorem 10, a lower bound estimate can be found:

$$
\liminf_{n\to\infty} n^{-1} \log P(\pi_n^*(X) \cap H \neq \emptyset) \geq -I_X(H).
$$

and as a corollary of the upper bound and lower bound estimates, it is proved in [Num00a] Corollary 4.1 that for any convex set C

$$
\lim_{n\to\infty} n^{-1} \log P(\pi_n^*(X) \cap C \neq \emptyset) = -I_X(C).
$$

By defining $C_\varepsilon \doteq \bar{C} \cap U(\pi_C^*(X), \varepsilon)^c$, it can be shown that

$$
P\{\pi_n^*(X) \cap C_\varepsilon \neq \emptyset | \pi_n^*(X) \cap C \neq \emptyset\} \leq e^{n(-I(C_\varepsilon)+I(C))}.
$$

By the continuity of $I(\cdot)$ and the fact $C_\varepsilon \subset \bar{C} \setminus \pi_C^*(X)$, we have $I(C_\varepsilon) > I(\bar{C}) = I(C)$. Therefore

$$
\lim_{n\to\infty} P(\pi_n^*(X) \subset U(\pi_C^*(X), \varepsilon) | \pi_n^*(X) \cap C \neq \emptyset) = 1
$$

and the assertion follows. □

3.2 Full observation

The law of large numbers of random exchange economies ([Num00b],[MR00],[MR02]) argues that the observed equilibrium price converges to some price in the convergence set consisting of prices such that the expectation of the total excess demand vanishes. The probability law governing the economic environment, which is required in calculating the expectation of the total excess demand, can be chosen subjectively or empirically. But in most cases it is difficult to know this probability law exactly a priori. We are interested in the problem that how a a posteriori fully observed equilibrium price could help us to calibrate the a priori probability law and get a "true" probability law consistent with our observation.

Due to the law of large numbers of random exchange economies, when the number of agents in the economy is large, our observed equilibrium price should be in a close neighborhood of some model-based expected equilibrium price under the "true" probability law governing the economic environment. In a precise mathematical form, a full observation is the occurrence of the event

$$\pi_n^* \cap U(p,\delta) \neq \emptyset.$$

Denote the a priori chosen probability law by $P(d\omega)$ and the a posteriori (after calibration) "true" probability law by $P(d\omega|\pi_n^* \cap U(p,\delta) \neq \emptyset)$ where p is the a posteriori fully observed equilibrium price.

We define the probability law under the observation p as

$$P(d\omega|p) \doteq \Lambda_n(p)^{-1} e^{\alpha(p)\cdot Z_n(\omega,p)} P(d\omega),$$

and denote the expectation under the probability law by $E(\cdot|p)$.

Theorem 2 *Suppose that **Assumptions 1 - 7** are satisfied. Then for any fixed* $\varepsilon > 0$*, all sufficiently small* $\delta > 0$*:*

$$\lim_{n\to\infty} P(n^{-1}|X_n^* - E(X_n|p)| < \varepsilon|\pi_n^* \cap U(p,\delta) \neq \emptyset) = 1.$$

Proof With a full observation p the observation set becomes

$$C \doteq U(p,\delta) \times \mathbb{R}^d.$$

The contraction principle ([Num00a] Theorem 4) says that for all $p \in \Delta$,

$$\begin{aligned} I(p) &= \inf_{w \in \mathbb{R}^d} I_X(p, w) \\ &= I_X(p, w(p)) \end{aligned}$$

where

$$w(p) \doteq \frac{\partial c_X}{\partial \beta}(\alpha(p), 0; p).$$

We then have

$$\begin{aligned} w(p) &= \frac{\partial c_X}{\partial \beta}(\alpha(p), 0; p) \\ &= \lim_{n \to \infty} n^{-1} \frac{E(X_n(p, \omega) e^{\alpha(p) \cdot Z_n(p,\omega)})}{E e^{\alpha(p) \cdot Z_n(p,\omega)}}. \end{aligned}$$

The second equality follows the uniform convergence assumption of c_X. Therefore we can see $w(p)$ as the limit of mean expectation of X_n under the probability law

$$P(d\omega|p) \doteq \frac{e^{\alpha(p) \cdot Z_n(p,\omega)}}{E e^{\alpha(p) \cdot Z_n(p,\omega)}} P(d\omega),$$

i.e.,

$$w(p) = \lim_{n \to \infty} n^{-1} E(X_n|p).$$

Then $\pi_C(X)$ has the form:

$$\pi_C(X) = \{(p_C, \lim_{n \to \infty} n^{-1} E(X_n|p_C)); p_C \in U(p, \delta)\}.$$

Assumption 7 indicates that $Z_n(\cdot)$ and $X_n(\cdot)$ are continuous on some closed neighborhood of the given price p. It follows that

$$|E(X_n|p) - E(X_n|p_C)| < \varepsilon_n(\delta) \text{ when } |p - p_C| < \delta$$

where $\varepsilon_n(\delta) \to 0$ when $\delta \to 0$. Thus

$$U(\pi_C(X), \varepsilon(n, \delta)) \subset U((p, n^{-1} E(X_n|p)), \delta)$$

where $\varepsilon(n, \delta) \to 0$ when $\delta \to 0$. By Theorem 1 we can conclude that

$$\lim_{n \to \infty} P(\pi_n^*(X) \subset U(\pi_C(X), \varepsilon) | \pi_n^*(X) \cap C \neq \emptyset) = 1.$$

It follows that for all sufficiently small $\delta > 0$

$$\lim_{n\to\infty} P(\pi_n^*(X) \subset U((p, n^{-1}E(X_n|p)), \delta)|\pi_n^* \cap U(p,\delta) \neq \emptyset) = 1.$$

Hence

$$\lim_{n\to\infty} P(n^{-1}|X_n^* - E(X_n|p)| < \varepsilon|\pi_n^* \cap U(p,\delta) \neq \emptyset) = 1.$$

□

This theorem asserts that the probability law $P(d\omega|p)$ is a good approximation of the a posteriori probability law conditionally on the observation of the r.e.p. at p. For a check, we see that due to the law of large numbers, the expectation of total excess demand under the a posteriori probability law must vanish at our observed equilibrium prices p, i.e.,

$$\lim_{n\to\infty} E(Z_n(p)|\pi_n^* \cap U(p,\delta) \neq \emptyset) \doteq \lim_{n\to\infty} \int_\Omega Z_n(\omega, p) P(d\omega|\pi_n^* \cap U(p,\delta) \neq \emptyset) = 0.$$

If we take $P(d\omega|\pi_n^* \cap U(p,\delta) \neq \emptyset)$ to be $P(d\omega|p)$, then it is easy to verify

$$\begin{aligned}\lim_{n\to\infty} E(Z_n(p)|p) &\doteq \lim_{n\to\infty} \int_\Omega Z_n(\omega, p) \frac{e^{\alpha(p)\cdot Z_n(p,\omega)}}{Ee^{\alpha(p)\cdot Z_n(p,\omega)}} P(d\omega) \\ &= \lim_{n\to\infty} \frac{1}{Ee^{\alpha(p)\cdot Z_n(p,\omega)}} \frac{\partial}{\partial\alpha} Ee^{\alpha(p)\cdot Z_n(p,\omega)} \\ &= \lim_{n\to\infty} \frac{\partial c_n}{\partial\alpha}(\alpha(p); p) \\ &= 0.\end{aligned}$$

The last equality follows the definition of $\alpha(p)$.

The probability law $P(d\omega|p)$ we used in this theorem is an analogy to the canonical probability law in thermodynamics. Therefore this result has another interpretation from the thermodynamical point of view. On can compare it with [Num09], Theorem 5 where a different approach based on a "δ-neighborhood observation" is used.

4 An application: A random exchange economy with sectors

In this section we apply our main results to a random exchange economy with economic sectors. Our interests in such a sectorial economy are inspired by [LN01].

In [LN01], a disequilibrium status of the sectorial economy is discussed, while in this section we are interested in the properties when the sectorial economy is in equilibrium.

The economic sectors can represent different groups of agents in an economy or different countries or organizations in a global trade model. We choose the collection of all sectorial excess demands to be the price depending total characteristic of the economy, $X_n(p)$. Because of the close connection between the total excess demand and the sectorial excess demands, the entropy functions possess some special properties. We can also prove an important result saying that when the whole economy is in equilibrium, each economic sector is also in equilibrium under the same equilibrium price.

We consider a random exchange economy consisting of K sectors each having n_k random agents and its sectorial excess demand function $Z^{(k)}_{n_k}(p,\omega), k = 1,\dots,K$. Let

$$\underline{Z}_n(p) \doteq (Z^{(1)}_{n_1}(p),\dots,Z^{(K)}_{n_K}(p)) \in \mathbb{R}^{Kl}$$

denote the collection of all sectorial excess demands where $n = \sum_{k=1}^K n_k$ is the total number of agents in the economy. Thus the total excess demand of the economy becomes $Z_n(p) = \sum_{k=1}^K Z^{(k)}_{n_k}$. In the economic equilibrium, we have $Z_n(p) = 0$.

Remark 1 *For example, we could assume that agents in the same sector are independent and identical, and agent i_k (the subscription k means the agent is in sector k) has a Cobb-Douglas utility function so that the utility of the consumption bundle $x = (x^1,\dots,x^l)$ is of the form*

$$u_{i_k}(x) = \prod_{j=1}^{l} (x^j)^{\alpha^j_{i_k}}$$

where $\alpha^1_{i_k},\dots,\alpha^l_{i_k}$ are nonnegative share parameters satisfying $\sum_{j=1}^l \alpha^j_{i_k} = 1$. Denote agent i_k's initial endowment by $e_{i_k} \doteq (e^1_{i_k},\dots,e^l_{i_k})$. In a random model, the share parameters α_{i_k} as well as the initial endowments $e_{i_k}, i_k = 1,\dots,n_k, k = 1,\dots,K$ are random variables. For more details on a random Cobb-Douglas exchange economy one could see [Num09], p.21.

We then define

$$c(\alpha;p) \doteq \lim_{n\to\infty} n^{-1} \log E e^{\alpha\cdot Z_n(p)}$$

and

$$c_{\underline{Z}}(\alpha,\beta;p) \doteq \lim_{n\to\infty} n^{-1} \log E e^{\alpha\cdot Z_n(p)+\beta\cdot \underline{Z}_n(p)}$$

These two limits are assumed to exist and converge uniformly over α and β. The entropy function is then defined by

$$I(p) \doteq -\inf_{\alpha\in\mathbb{R}^l} c(\alpha;p)$$

and

$$I_{\underline{Z}}(p,w) \doteq \sup_{(\alpha,\beta)\in\mathbb{R}^{l+Kl}} (\beta\cdot w - c_{\underline{Z}}(\alpha,\beta;p))$$

Notice $Z_n(p) = \sum_{k=1}^K Z_{n_k}^{(k)}$ and let $\alpha\underline{1} = (\underbrace{\alpha,\ldots,\alpha}_{K})$, thus

$$\begin{aligned} c_{\underline{Z}}(\alpha,\beta;p) &= \lim_{n\to\infty} n^{-1} \log E e^{\alpha\cdot Z_n(p)+\beta\cdot \underline{Z}_n(p)} \\ &= \lim_{n\to\infty} n^{-1} \log E e^{(\alpha\underline{1}+\beta)\cdot \underline{Z}_n(p)} \\ &= \lim_{n\to\infty} n^{-1} \log E e^{\underline{\alpha}\cdot \underline{Z}_n(p)} \\ &\doteq c^*_{\underline{Z}}(\underline{\alpha};p) \end{aligned}$$

and the economic entropy function for a sectorial economy with respect to $\underline{Z}_n$ can be written as

$$\begin{aligned} I_{\underline{Z}}(p,w) &= \sup_{(\alpha,\beta)\in\mathbb{R}^{l+Kl}} (\beta\cdot w - c_{\underline{Z}}(\alpha,\beta;p)) \\ &= \sup_{(\alpha,\underline{\alpha})\in\mathbb{R}^{l+Kl}} ((\underline{\alpha}-\alpha\underline{1})\cdot w - c^*_{\underline{Z}}(\underline{\alpha};p)) \\ &= \sup_{\underline{\alpha}\in\mathbb{R}^{Kl}} (\underline{\alpha}\cdot w - c^*_{\underline{Z}}(\underline{\alpha};p)) - \inf_{\alpha\in\mathbb{R}^l} \alpha\underline{1}\cdot w. \end{aligned}$$

Proposition 1 *In a random exchange economy with K sectors,*

$$I_{\underline{Z}}(p,w) = \sup_{\underline{\alpha}\in\mathbb{R}^{Kl}} (\underline{\alpha}\cdot w - c^*_{\underline{Z}}(\underline{\alpha};p)) - \inf_{\alpha\in\mathbb{R}^l} \alpha\underline{1}\cdot w.$$

We assume further that $\alpha = \alpha(p) \in \mathbb{R}^l$ is the unique solution of the equation

$$\frac{\partial c}{\partial \alpha} = 0.$$

It follows that

$$c(\alpha;p) = c^*_{\underline{Z}}(\alpha\underline{1};p),$$

$$I(p) = -c^*_{\underline{Z}}(\alpha(p)\underline{1}; p)$$
$$\leq -\inf_{\underline{\alpha}\in\mathbb{R}^{Kl}} c^*_{\underline{Z}}(\underline{\alpha}; p)$$

and

$$I_{\underline{Z}}(p, w) = \sup_{(\alpha,\beta)\in\mathbb{R}^{l+Kl}} (\beta \cdot w - c_{\underline{Z}}(\alpha, \beta; p)) \geq \sup_{(\alpha,\underline{\alpha})\in\mathbb{R}^{l+Kl}} ((\underline{\alpha} - \alpha\underline{1}) \cdot w) + I(p).$$

By the contraction principle ([Num00a] Theorem 4) we know for all $p \in \Delta$,

$$I(p) = \inf_{w\in\mathbb{R}^{Kl}} I_{\underline{Z}}(p, w) = I_{\underline{Z}}(p, w(p))$$

where

$$w(p) = \frac{\partial c_{\underline{Z}}}{\partial \beta}(\alpha(p), 0; p).$$

Then the following inequality should hold

$$I_{\underline{Z}}(p, w(p)) = I(p) \geq \sup_{(\alpha,\underline{\alpha})\in\mathbb{R}^{l+Kl}} ((\underline{\alpha} - \alpha\underline{1}) \cdot w(p)) + I(p).$$

It is to say

$$\sup_{(\alpha,\underline{\alpha})\in\mathbb{R}^{l+Kl}} ((\underline{\alpha} - \alpha\underline{1}) \cdot w(p)) \leq 0.$$

Since $\underline{\alpha} \in \mathbb{R}^{Kl}$ and $\alpha \in \mathbb{R}^l$ can be chosen arbitrarily, we have

$$w(p) = 0,$$

hence

$$-c^*_{\underline{Z}}(\alpha(p)\underline{1}; p) = I(p) = I_{\underline{Z}}(p, 0) = -\inf_{\underline{\alpha}\in\mathbb{R}^{Kl}} c^*_{\underline{Z}}(\underline{\alpha}; p).$$

Thus we have proved

Proposition 2 *In a random exchange economy with K sectors,*

$$I(p) = I_{\underline{Z}}(p, 0)$$

and

$$\frac{\partial c^*_{\underline{Z}}}{\partial \underline{\alpha}}(\alpha(p)\underline{1}; p) = 0.$$

A partial observation in our example could be that only the prices of some l' commodities, say $q^1, \ldots, q^{l'}$, are observed and prices of the rest of the commodities $l'+1, \ldots, l$ remain unknown or if we observe the price of a given commodity bundle $b = (b^1, \ldots, b^l) \in \mathbb{R}^l$, i.e., $b \cdot p = q$ for some known price q. In these cases, the observation set C is $C = B \times \mathbb{R}^d$ where

$$B \doteq \{p \in \mathbb{R}^l_{++} : p^1 \in U(q^1, \varepsilon), \ldots, p^{l'} \in U(q^{l'}, \varepsilon)\}$$

and

$$B \doteq \{p \in \mathbb{R}^l_{++} : b \cdot p \in U(q, \varepsilon)\}$$

respectively, where $\varepsilon > 0$. The use of ε-neighborhood in constructing the observation sets is due to the fact that we can only have observations on an economy with relatively large but still finite number of agents.

We could also have a partial observation on the price-depending variable X_n. For example we observe the total excess demands of the first k' sectors, $Z^1, \ldots, Z^{k'}$, but have no information about the other sectorial excess demands and the equilibrium prices, then the observation set C becomes

$$C = \mathbb{R}^l \times D \doteq \mathbb{R}^l \times \{X_n \in \mathbb{R}^d : n^{-1}Z^{(1)} \in U(n^{-1}Z^1, \varepsilon), \ldots, n^{-1}Z^{(k')} \in U(n^{-1}Z^{k'}, \varepsilon)\}.$$

Of course, we can have partial observations on both the equilibrium prices and the sectorial excess demands. In that case, the observation set C is of the form $B \times D$.

Having a partial observation, $\pi_n^*(\underline{Z}) \cap C \neq \emptyset$, we can give our inferences on the full equilibrium price and the full collection of the sectorial excess demands, i.e., the equilibrium graph $\pi_n^*(\underline{Z})$ (i.e., $\{(p_n^*, n^{-1}\underline{Z}_n(p_n^*))\}$) will be in the close neighborhood of the entropy minimizing graph $(\pi_C(\underline{Z}))$ (i.e., $\{(p_C, w_C)\}$) with high probability when $n \to \infty$.

Because of the connection between Z_n and $\underline{Z}_n$, the entropy minimizing set $\pi_C(\underline{Z})$ has a special form when we have an observation on the equilibrium price, i.e., our observation set is of the form $B \times \mathbb{R}^{Kl}$. By Proposition 2,

$$\min_{(p,w) \in B \times \mathbb{R}^{Kl}} I_{\underline{Z}}(p, w) = \min_{p \in B} I_{\underline{Z}}(p, 0) = \min_{p \in B} I(p).$$

Thus

$$\begin{aligned} \pi_C(\underline{Z}) &= \{(p, w) = \arg \min_{(p,w) \in B \times \mathbb{R}^{Kl}} I_{\underline{Z}}(p, w)\} \\ &= \{(p_B, 0)\} \end{aligned}$$

where $p_B \in \pi_B \doteq \{p : p = \arg\min_{p\in B} I(p)\}$. According to Theorem 1, we have

Proposition 3 *In a random exchange economy with K sectors, when we have an observation on the equilibrium price, i.e., our observation set is of the form* $B \times \mathbb{R}^{Kl}$*, then for all* $\varepsilon > 0$

$$\pi_n^*(\underline{Z}) \subset U(\{(\pi_B, 0)\}, \varepsilon)$$

almost surely when $n \to \infty$.

This proposition indicates that $n^{-1}\underline{Z}_n(p) = (n^{-1}Z^{(1)}, \ldots, n^{-1}Z^{(k)}) \to 0$. So it has an important interpretation which says that when the whole economy is in equilibrium, each economic sector is also in equilibrium under the same equilibrium price.

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