Functions transferring metrics to metrics

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Abstract

We study the properties of real functions f for which the compositions $f \circ d$ is a metric for every metric space (X, d). The explicit form is found for the invertible elements of the semigroup $\mathcal{F}$ of all such functions. The increasing functions $f \in \mathcal{F}$ are characterized by the subadditivity condition and a maximal inverse subsemigroup in the set of all these functions is explicitly described. The upper envelope of the set of functions $f \in \mathcal{F}$ with $f(1) = 1$ is found and it leads to the exact constants in Harnack's inequality for functions $f \in \mathcal{F}$.

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1 Introduction

Write $\mathbb{R}^{\oplus}$ for the set of all nonnegative real numbers. For each metric space (X, d) denote by $\mathcal{F}(X) = \mathcal{F}(X, d)$ the set of all functions $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ for which the map

$$X \times X \xrightarrow{d} \mathbb{R}^{\oplus} \xrightarrow{f} \mathbb{R}^{\oplus} \tag{1.1}$$

is a metric on X and, moreover, define the set $\mathcal{F}$ of functions f by the rule

$$(f \in \mathcal{F}) \Leftrightarrow (f \in \mathcal{F}(X) \text{ for every metric space } X). \tag{1.2}$$

The set $\mathcal{F}$ is a special case among the sets of functions $\Phi : \mathbb{R}^{\oplus} \times \cdots \times \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ which generate the metrics d_Φ on products of metric spaces (X_i, d_i), $i = 1, \ldots, n$ by the rule

$$d_\Phi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \Phi(d_1(x_1, y_1), \ldots, d_n(x_n, y_n)).$$

These functions were studied in an interesting paper of A. Bernig, T. Foertsch and V. Schroeder [2]. In that paper the authors write: "The function Φ has to satisfy certain natural conditions ... in order that d_Φ is metric. These conditions still allow strange metrics on the product (even the trivial product when $n = 1$). In particular, Φ does not have to be continuous". In the present paper we consider general $f \in \mathcal{F}$ which, of course, can be discontinuous. Note that [2] deals with the products $X_1 \times \cdots \times X_n$ for arbitrary $n \in \mathbb{N}$ but, in fact, only when Φ is induced by a norm.

There is a simple characterization of functions belonging to $\mathcal{F}$, see Lemma 1 in [2] for a similar "multidimensional" result.

1.1 Theorem. *A function $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ belongs to $\mathcal{F}$ if and only if the following conditions hold:*

(i) *$f(0) = 0$ and $f(t) > 0$ for all $t > 0$;*

(ii) *the inequality*

$$2(f(a) \vee f(b) \vee f(c)) \leq f(a) + f(b) + f(c) \tag{1.3}$$

holds for $a, b, c \in \mathbb{R}^{\oplus}$ whenever

$$2(a \vee b \vee c) \leq a + b + c. \tag{1.4}$$

For the proof it is sufficient to observe that (1.4) holds if and only if we have the following three inequalities

$$a \leq b + c, \quad b \leq a + c \quad \text{and} \quad c \leq a + b.$$

1.2 Remark. Every function $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ belongs to $\mathcal{F}(X)$ if $X = \emptyset$. In this case superposition (1.1) is empty and it is the unique distance function on $\emptyset$. A function $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ belongs to $\mathcal{F}(X)$ for each one-point metric space X if and only if $f(0) = 0$ and, moreover, $f \in \mathcal{F}(X)$ for all two-point metric spaces X if and only if condition (i) of Theorem 1.1 holds.

The set $\mathcal{F}$ coincides with the set of all functions $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ transfering metrics to metrics on three-point metric spaces, i.e.,

$$\mathcal{F} = \cap\{\mathcal{F}(X) : X \text{ are metric spaces with } \operatorname{card} X = 3\}. \tag{1.5}$$

The last statement implies

1.3 Corollary. *Let $\mathbb{R}^2$ be the set of all complex numbers with the usual metric $d_{\mathbb{R}^2}(z_1, z_2) = |z_1 - z_2|$. Then the equality*

$$\mathcal{F} = \mathcal{F}(\mathbb{R}^2) \tag{1.6}$$

holds.

It is a consequence of the commutativity of the next diagram for each $f \in \mathcal{F}(\mathbb{R}^2)$.

$$\begin{tikzcd}
X \times X \arrow[d, "em_X \otimes em_X"'] \arrow[dr, "d_X"] & & \\
\mathbb{R}^2 \times \mathbb{R}^2 \arrow[r, "d_{\mathbb{R}^2}"'] & \mathbb{R}^{\oplus} \arrow[r, "f"'] & \mathbb{R}^{\oplus}
\end{tikzcd}$$

Here d_X and $em_X \otimes em_X$ are the distance functions and, respectively, the direct product of the isometric embeddings em_X of the three-point metric spaces X.

We may replace $\mathbb{R}^2$ in (1.6) by an arbitrary metric space Y which has the property that every three-point metric space X can be isometrically embeded in Y.

The next proposition follows from the definition of the set $\mathcal{F}$.

1.4 Proposition. *Let $f, g \in \mathcal{F}$. The following statements hold.*

(i) *The envelope $f \vee g$,*

$$(f \vee g)(t) := f(t) \vee g(t), \quad t \in \mathbb{R}^{\oplus},$$

belongs to $\mathcal{F}$.

(ii) *The superposition $g \circ f$,*

$$(g \circ f)(t) = g(f(t)), \quad t \in \mathbb{R}^{\oplus},$$

belongs to $\mathcal{F}$.

(iii) *If $\alpha, \beta \in \mathbb{R}^{\oplus}$ and $\alpha \vee \beta > 0$, then we have $\alpha f + \beta g \in \mathcal{F}$.*

Statements (i) and (ii) of the previous proposition show, in particular, that $\mathcal{F}$ is a semigroup with respect to the superposition $\circ$ of functions $f \in \mathcal{F}$ and that the union

$$\hat{\mathcal{F}} := \mathcal{F} \cup \{0\}, \tag{1.7}$$

where 0 is the identically zero function on $\mathbb{R}^{\oplus}$, is a convex cone in the linear space of all real-valued functions on $\mathbb{R}^{\oplus}$. We shall prove some properties of the semigroup $(\mathcal{F}, \circ)$ and of the cone $\hat{\mathcal{F}}$ in the next section of the paper.

2 Examples and properties of functions in $\mathcal{F}$

2.1 Example. Let a and b be positive real numbers. The following functions belong to $\mathcal{F}$ for every $a, b > 0$:

$$g_b(x) = bx, \quad x \in \mathbb{R}^{\oplus}, \qquad f_a(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x > 0 \end{cases}, \tag{2.1}$$

$$\varphi(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \text{ is positive and irrational} \\ 1 + \frac{1}{m(x)} & \text{if } x \text{ is positive and rational} \end{cases} \tag{2.2}$$

where $m(x)$ is the smallest positive integer m such that $x = \frac{n}{m}$ and $n \in \mathbb{N}$.

The function φ is a particular case of the following construction.

2.2 Example. Let $a > 0$. If the function $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ satisfies the double inequality

$$a \leq f(x) \leq 2a$$

for all $x > 0$ and if $f(0) = 0$, then $f \in \mathcal{F}$.

Some other examples of functions $f \in \mathcal{F}$ will be given in Section 4 of the paper.

2.3 Remark. The function φ, see (2.2), is a variant of Thomae's function, also known as the Riemann function [1, Example 5.1.6(h)]. The set of the discontinuities of φ is the set of all nonnegative rational numbers. All possible sets of discontinuities of functions $f \in \mathcal{F}$ will be completely described in the end of Section 3.

Consider now the pointwise limits of functions from $\mathcal{F}$.

2.4 Theorem. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a pointwise convergent sequence of functions $f_n \in \mathcal{F}$ and let*

$$f(t) := \lim_{n\to\infty} f_n(t) \tag{2.3}$$

for each $t \in \mathbb{R}^{\oplus}$. Then either f belongs to $\mathcal{F}$ or $f(t) = 0$ for all $t \in \mathbb{R}^{\oplus}$.

2.5 Corollary. *The convex cone $\hat{\mathcal{F}}$, see* (1.7), *is the closure of the set $\mathcal{F}$ with respect to the pointwise convergence topology in the linear space of all real-valued functions on $\mathbb{R}^{\oplus}$.*

The following lemma is a particular case of the corresponding multidimensional result, see Remark ii), [2, p. 502].

2.6 Lemma. *Let $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ be a function satisfying condition* (ii) *of Theorem 1.1. Then the inequality*

$$f(t) \leq 2f(x) \tag{2.4}$$

holds whenever $0 \leq t \leq 2x$.

Proof of Theorem 2.4. Since $f_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, we have the inequality

$$2(f_n(a) \vee f_n(b) \vee f_n(c)) \leq f_n(a) + f_n(b) + f_n(c)$$

whenever $2(a \vee b \vee c) \leq a + b + c$ and $a, b, c \in \mathbb{R}^{\oplus}$. Letting $n \to \infty$ we obtain that condition (ii) of Theorem 1.1 holds for the limit function f. Moreover it is clear that $f(0) = 0$ and that $f(x) \geq 0$ for all $x \in \mathbb{R}^{\oplus}$. Consequently if $f \notin \mathcal{F}$, then there is $x_0 > 0$ such that $f(x_0) = 0$. Applying Lemma 2.6 we see that $f(x) = 0$ for all $x \in [0, 2x_0]$. The second application of this lemma gives the same equality on $[0, 4x_0]$, the third one gives it on $[0, 8x_0]$ and so on. Thus $f(x) = 0$ for all $x \in \mathbb{R}^{\oplus}$. $\square$

The following corollary is another form of Theorem 1.1.

2.7 Corollary. *Let $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ be a function such that: $f(0) = 0$ and there is $x_0 \in \mathbb{R}^{\oplus}$ with $f(x_0) \neq 0$ and condition* (ii) *of Theorem 1.1 holds for this f. Then f belongs to $\mathcal{F}$.*

Proof. It follows from Theorem 1.1 that $f \in \mathcal{F}$ if and only if $f(x) > 0$ for all $x > 0$. If $f(x_1) = 0$ for some $x_1 > 0$, then using (2.4) we can prove the equality $f(x) = 0$ for all $x \in \mathbb{R}^{\oplus}$. The last equality contradicts $f(x_0) \neq 0$. □

Topological properties of $(X, f \circ d)$ are simple consequence of Lemma 2.6. Recall that two metrics d and ρ on a set X are called equivalent if d and ρ induce exactly the same open sets, i.e., the same topology on X. The next lemma is valid.

2.8 Lemma. *Let (X, d) be a metric space and let $f \in \mathcal{F}(X)$. The metrics d and $f \circ d$ induce the same topology on X if and only if for every sequence $\{x_n\}_{n \in \mathbb{N}}$, $x_n \in X$ and every point $a \in X$, we have either*

$$\lim_{n\to\infty} d(x_n, a) = 0 \quad \textit{and} \quad \lim_{n\to\infty} f(d(x_n, a)) = 0 \tag{2.5}$$

or

$$\lim_{n\to\infty} \sup d(x_n, a) > 0 \quad \textit{and} \quad \lim_{n\to\infty} \sup f(d(x_n, a)) > 0. \tag{2.6}$$

2.9 Theorem. *Let $f \in \mathcal{F}$. The metrics d and $f \circ d$ are equivalent for every metric space (X, d) if and only if*

$$\lim_{t\to 0} f(t) = 0. \tag{2.7}$$

Proof. Suppose that d and $f \circ d$ are equivalent for every metric space (X, d). Let $\{t_n\}_{n \in \mathbb{N}}$ be an arbitrary sequence of nonnegative real numbers such that $\lim_{n\to\infty} t_n = 0$. Using Lemma 2.8 with $X = \mathbb{R}^{\oplus}$, $d(x, y) = |x - y|$ and $a = 0$ we obtain from (2.5) that $\lim_{n\to\infty} f(t_n) = 0$. Hence f is continuous at the point zero, so we have (2.7).

Assume now that (2.7) holds but there exists a metric space (X, d) such that the metrics d and $f \circ d$ are not equivalent. By Lemma 2.8 we can find a sequence $\{x_n\}_{n \in \mathbb{N}}$ and a point a in the space (X, d) such that either

$$\lim_{n\to\infty} d(x_n, a) = 0 \quad \text{but} \quad \limsup_{n\to\infty} f(d(x_n, a)) > 0 \tag{2.8}$$

or

$$\lim_{n\to\infty} \sup d(x_n, a) > 0 \quad \text{but} \quad \lim_{n\to\infty} f(d(x_n, a)) = 0. \tag{2.9}$$

Relations (2.8) contradict (2.7) so we have (2.9) which means that there are $t_0 > 0$ and a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\{x_n\}_{n \in \mathbb{N}}$ such that

$d(x_{n_k}, a) \geq t_0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} f(d(x_{n_k}, a)) = 0$. Inequality (2.4) implies

$$f(t_0) \leq 2f(d(x_{n_k}, a))$$

for each $k \in \mathbb{N}$. Letting $k \to \infty$, we obtain the inequality $f(t_0) \leq 0$ with $t_0 > 0$, contrary to the condition $f \in \mathcal{F}$. □

Recall that a metric space (X, d) is discrete if each subset of X is open.

2.10 Example. The functions φ and f_a, see (2.1) and (2.2) are examples of functions such that the space $(X, \varphi \circ d)$ and $(X, f_a \circ d)$ are discrete for every metric space (X, d). Certainly limit relation (2.7) does not hold with these functions.

The following theorem shows that we describe a typical situation in Example 2.10.

2.11 Theorem. *Let $f \in \mathcal{F}$. The following statements are equivalent.*

(i) *The function f is discontinuous at zero.*

(ii) *There is $a > 0$ such that $f(x) \geq a$ for all $x > 0$.*

(iii) *The space $(X, f \circ d)$ is discrete for every metric space (X, d).*

Proof. The implication (ii)⇒(iii) is trivial and (iii)⇒(i) follows from Theorem 2.9. To prove (i)⇒(ii) suppose that there is a sequence of strictly positive numbers x_n, $n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} f(x_n) = 0.$$

Let $\varepsilon > 0$ and let n be a natural number such that $f(x_n) \leq \varepsilon$. Lemma 2.6 implies

$$|f(t) - f(0)| = f(t) \leq 2f(x_n) \leq 2\varepsilon$$

for all $t \in [0, x_n]$. Hence f is continuous at 0. The implications (i)⇒(ii) follows. □

2.12 Corollary. *The following statements hold.*

(i) *If (X, d) is a discrete metric space, then d and $f \circ d$ are equivalent for each $f \in \mathcal{F}$.*

(ii) *If (X, d) is a nondiscrete metric space, $f \in \mathcal{F}$, and if d and $f \circ d$ are equivalent, then for every metric space (Y, ρ) the metrics ρ and $f \circ \rho$ are equivalent.*

It was noted in Proposition 1.4 that $f \vee g \in \mathcal{F}$ for each two $f, g \in \mathcal{F}$. Using Lemma 2.6 we can generalize this fact for the upper envelops of nonvoid sets $\mathcal{G} \subseteq \mathcal{F}$ with no restrictions to card$(\mathcal{G})$.

Let $\mathcal{G}$ be a nonvoid subset of $\mathcal{F}$. The upper envelop of $\mathcal{G}$ is the smallest function $g : \mathbb{R}^{\oplus} \to [0, \infty]$ such that $f(t) \leq g(t)$ for all $t \in \mathbb{R}^{\oplus}$. We shall write $f_{\mathcal{G}}$ or $\bigvee\limits_{f \in \mathcal{G}} f$ for the upper envelope of $\mathcal{G}$.

2.13 Proposition. *Let $\mathcal{G}$ be a nonvoid subset of $\mathcal{F}$. The upper envelope $f_{\mathcal{G}}$ belongs to $\mathcal{F}$ if and only if there is $t_0 > 0$ such that $f_{\mathcal{G}}(t_0) < \infty$.*

Proof. Let $a, b, c \in \mathbb{R}^{\oplus}$ be numbers for which (1.4) holds. Then, for every $g \in \mathcal{G}$, we have the inequality

$$2(g(a) \vee g(b) \vee g(c)) \leq f_{\mathcal{G}}(a) + f_{\mathcal{G}}(b) + f_{\mathcal{G}}(c),$$

so that

$$2(f_{\mathcal{G}}(a) \vee f_{\mathcal{G}}(b) \vee f_{\mathcal{G}}(c)) \leq f_{\mathcal{G}}(a) + f_{\mathcal{G}}(b) + f_{\mathcal{G}}(c).$$

Moreover since $\mathcal{G} \neq \emptyset$, we have $f_{\mathcal{G}}(0) = 0$ and $0 < f_{\mathcal{G}}(t) \leq \infty$ for each $t > 0$.

Suppose that there is $t_0 > 0$ such that $f_{\mathcal{G}}(t_0) < \infty$. Using Lemma 2.6 we obtain the double inequality

$$g(t) \leq 2g(t_0) \leq 2f_{\mathcal{G}}(t_0)$$

for all $t \in [0, 2t_0]$. Consequently

$$f_{\mathcal{G}}(t) = \bigvee_{g \in \mathcal{G}} g(t) \leq 2f_{\mathcal{G}}(t_0) < \infty$$

folds for all $t \in [0, 2t_0]$. Repeating this procedure we see that $f_{\mathcal{G}}$ is finite on the set $\bigcup\limits_{n=1}^{\infty} [0, 2^n t_0] = \mathbb{R}^{\oplus}$. Hence if $f_{\mathcal{G}}$ is finite in a point $t_0 \in (0, \infty)$, then $f_{\mathcal{G}} \in \mathcal{F}$. The converse statement is evidently true. $\square$

2.14 Corollary. *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions belonging to $\mathcal{F}$ and let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers. If there is $t_0 > 0$ such that the series $\sum_{n=0}^{\infty} \alpha_n f_n(t_0)$ is convergent, then the series $\sum_{n=0}^{\infty} \alpha_n f_n(t)$ converges uniformly on the bounded subsets of $\mathbb{R}^{\oplus}$ and the sum belongs to $\mathcal{F}$.*

2.15 Remark. The corollary generalizes statement (iii) of Proposition 1.4. The uniform convergence on a bounded set $B \subseteq \mathbb{R}^{\oplus}$ means that

$$\lim_{m \to \infty} \left(\sup_{t \in B} \left(\sum_{n=m}^{\infty} \alpha_n f_n(t) \right) \right) = 0,$$

and this is a consequence of Lemma 2.6 applied to the sums $\sum_{n=m}^{\infty} \alpha_n f_n(t)$, $m = 0, 1, 2, \ldots$.

Let $\mathcal{G}_1$ be a family of all functions $f \in \mathcal{F}$ such that $f(1) = 1$. To find the explicit form of the upper envelop $f_{\mathcal{G}_1}$ we introduce the following function μ. For every $x \in \mathbb{R}^{\oplus}$ define $\mu(x)$ as the smallest $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that $x \leq n$. It easily follows that

$$\mu(x) = \begin{cases} x & \text{if } x \in \mathbb{N} \\ [x] + 1 & \text{if } x \in \mathbb{R}^{\oplus} \setminus \mathbb{N} \end{cases} \tag{2.10}$$

where $[x]$ is the integral part of x. The function μ is increasing and, for each $x \in \mathbb{R}$, satisfies

$$x \leq \mu(x) \quad \text{and} \quad \mu(\mu(x)) = \mu(x) \tag{2.11}$$

and

$$\mu(x + y) \leq 1 + \mu(x) \quad \text{and} \quad \mu(m + y) = \mu(m + 1) \tag{2.12}$$

if $0 < y \leq 1$ and $m \in \mathbb{N}$.

2.16 Theorem. *The function $f_{\mathcal{G}_1}$ has the representation*

$$f_{\mathcal{G}_1}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2 & \text{if } 0 < x < 1 \\ \mu(x) & \text{if } 1 \leq x < \infty. \end{cases} \tag{2.13}$$

To prove this theorem we need the following two lemmas.

2.17 Lemma. *Each function $f \in \mathcal{F}$ is subadditive, that is*

$$f(x + y) \leq f(x) + f(y) \tag{2.14}$$

holds for all $x, y \in \mathbb{R}^{\oplus}$.

Proof. It follows from Remark i) in [2, p. 502]. □

2.18 Lemma. *Let $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ be a function such that $f(0) = 0$ and $f(x) > 0$ for all $x > 0$. If the inequality*

$$f(z) \leq f(x) + f(y) \tag{2.15}$$

holds whenever

$$0 \leq z \leq x + y \quad \text{and} \quad x \wedge y > 0, \tag{2.16}$$

then $f \in \mathcal{F}$.

Proof. To verify $f \in \mathcal{F}$ it is sufficient to show that condition (ii) of Theorem 1.1 is true. Suppose that (1.4) holds. It implies the inequalities

$$a \leq b + c, \quad b \leq c + a \quad \text{and} \quad c \leq a + b.$$

If $a \wedge b \wedge c = 0$, then we have $a = b$ or $b = c$ or $c = a$ so (1.3) is trivially true. The implication (2.16)⇒(2.15) yields (1.3) if $a \wedge b \wedge c > 0$. Hence condition (ii) of Theorem 1.1 holds if (2.16) implies (2.15). □

Proof of Theorem 2.16. For convenience we introduce the function

$$\nu(x) := \begin{cases} 2 & \text{if } x \in (0,1) \\ \mu(x) & \text{if } x \in \mathbb{R}^{\oplus} \setminus (0,1). \end{cases} \tag{2.17}$$

We must prove the equality

$$\nu = f_{\mathcal{G}_1}. \tag{2.18}$$

We shall first prove that

$$\nu(t) \geq f_{\mathcal{G}_1}(t) \tag{2.19}$$

for each $t \in \mathbb{R}^{\oplus}$.

Let $f \in \mathcal{G}_1$. First note that

$$f(0) = \nu(0) = 0 \quad \text{and} \quad f(1) = \nu(1) = 1.$$

Let $t \in (0,1) \cup (1,2]$. Inequality (2.4) implies

$$f(t) \leq 2f(1) = 2$$

and moreover, by (2.10) and (2.17), we have $\nu(t) = 2$. Consequently the inequality

$$f(t) \leq \nu(t) \tag{2.20}$$

holds for all $t \in [0,2]$. To extend this inequality to intervals $[0,n]$ with natural $n > 2$ we use the induction on n. Suppose that (2.20) holds for $t \in [0,m]$ where $m \geq 2$. Let $x \in (0,1]$. Using Lemma 2.17 we obtain

$$f(m+x) \leq f(m+x-1) + f(1). \tag{2.21}$$

The inductive hypothesis implies

$$f(m+x-1) \leq \nu(m+x-1). \tag{2.22}$$

Since $m+x-1 \in [1,\infty)$ and $\nu|_{[1,\infty)} = \mu|_{[1,\infty)}$ and since μ is increasing, we have

$$\nu(m+x-1) \leq \nu(m) = m.$$

It follows from this, (2.22), (2.21) and (2.12) that

$$\begin{aligned} f(m+x) \leq \nu(m+x-1) + f(1) = \nu(m) + 1& \\ = m+1 = \nu(m+1) = \nu(m+x).& \end{aligned}$$

Hence if (2.20) holds on $[0,m]$ it also holds on $[0,m+1]$. Thus (2.20) holds for all $t \in \mathbb{R}^{\oplus}$ and all $f \in \mathcal{G}_1$. It implies (2.19).

Next we prove the converse inequality

$$\nu(t) \leq f_{\mathcal{G}_1}(t) \tag{2.23}$$

for all $t \in \mathbb{R}^{\oplus}$. To this end, we establish the membership relation $\nu \in \mathcal{G}_1$ which evidently implies (2.23).

Since $\nu(1) = 1$ and $\nu(0) = 0$ and $\nu(t) > 0$ for $t > 0$ it is sufficient, by Lemma 2.18, to show that

$$\nu(z) \leq \nu(z) + \nu(y) \tag{2.24}$$

whenever

$$z \leq x + y \quad \text{and} \quad x \wedge y > 0. \tag{2.25}$$

If $z \in [0, 2]$, then $\nu(z) \leq 2$ so (2.24) holds because $\nu(x) \geq 1$ for all $x > 0$. If $z \in (2, \infty)$ and $0 < x \wedge y \leq 1$, then (2.25) implies the inequality $x \vee y \geq z - 1$ and, in addition, we have $\mu(z) = \nu(z)$ and $\mu(z-1) = \nu(z-1)$, and $\mu(x \vee y) = \nu(x \vee y)$. Consequently using (2.13) we obtain

$$\begin{aligned}\nu(x) + \nu(y) = \nu(x \wedge y) + \nu(x \vee y) \geq 1 + \nu(x \vee y) &= 1 + \mu(x \vee y) \\ &\geq 1 + \mu(z-1) \geq \mu(z) = \nu(z).\end{aligned}$$

Consider now the case where $z \in (2, \infty)$ and $x \wedge y \geq 1$. In this case $\mu(z) = \nu(z)$, $\mu(x) = \nu(x)$ and $\mu(y) = \nu(y)$. Suppose that (2.25) holds. Now the additivity of μ on the set $\mathbb{N}$ and (2.11) yield

$$\begin{aligned}\nu(x) + \nu(y) &= \mu(x) + \mu(y) = \mu(\mu(x)) + \mu(\mu(y)) \\ &= \mu(\mu(x) + \mu(y)) \geq \mu(x + y) \geq \mu(z) = \nu(z).\end{aligned} \tag{2.26}$$

Consequently (2.24) holds in all cases. Hence $\nu \in \mathcal{G}_1$ and (2.23) follows. Inequalities (2.23) and (2.19) imply the desired equality (2.18). □

Using Theorem 2.16 we can find the exact constant in Harnack's inequality for functions from $\mathcal{F}$.

2.19 Corollary. *Let A be a nonvoid compact subset of $(0, \infty)$ and let*

$$m := \min\{x : x \in A\}, \quad M := \max\{x : x \in A\}.$$

Then, for every function $f \in \mathcal{F}$, the inequality

$$\sup\{f(x) : x \in A\} \leq \mu\Big(\frac{M}{m}\Big) \inf\{f(x) : x \in A\} \tag{2.27}$$

holds with μ defined by (2.10)*. This inequality transforms into the equality for the function $f = \mu \circ g_{\frac{1}{m}}$, see* (2.1)*.*

Proof. Write

$$s(f) = \sup\{f(x) : x \in A\} \quad \text{and} \quad i(f) := \inf\{f(x) : x \in A\}.$$

If $M = m$ or if $s(f) = i(f)$, then (2.27) holds because

$$\mu\Big(\frac{M}{m}\Big) \geq \mu(1) = 1.$$

Consequently, without loss of generality we assume that $s(f) > i(f)$ and that $M > m$. In this case for all sufficiently small $\varepsilon > 0$ there is $a = a(\varepsilon) \in A$ such that

$$i(f) \leq f(a) \leq i(f) + \varepsilon < s(f).$$

Write $i_\varepsilon = f(a_\varepsilon)$ and define

$$\psi_\varepsilon(x) = \begin{cases} 0 & \text{if } x = 0 \\ i_\varepsilon & \text{if } 0 < f(x) \leq i_\varepsilon \\ f(x) & \text{if } f(x) > i_\varepsilon. \end{cases}$$

It is clear that $\psi_\varepsilon = f \vee f_{i_\varepsilon} \in \mathcal{F}$, see Example 2.1. The superposition $g_{\frac{1}{f(a)}} \circ \psi_\varepsilon \circ g_a$ belongs to the set $\mathcal{G}_1$. Thus, by Theorem 2.16 we have

$$\sup\{g_{\frac{1}{f(a)}} \circ \psi_\varepsilon \circ g_a(t) : t \in g_a^{-1}(A)\} \leq \sup\{f_{\mathcal{G}_1}(t) : t \in g_a^{-1}(A)\} \tag{2.28}$$

where

$$g_a^{-1}(A) := \{t \in \mathbb{R}^{\oplus} : g_a(t) \in A\} = \frac{1}{a}A = \Big\{\frac{1}{a}x : x \in A\Big\}.$$

We claim that

$$\sup\Big\{f_{\mathcal{G}_1}(t) : t \in \frac{1}{a}A\Big\} \leq \mu\Big(\frac{M}{m}\Big). \tag{2.29}$$

Indeed, it is clear if $\frac{1}{a}A \subseteq (0, 1]$, because

$$f_{\mathcal{G}_1}(x) \leq 2 \quad \text{for all } a \in (0, 1]$$

and because the inequality $M > m$ implies $\frac{M}{m} > 1$ so $\mu(\frac{M}{m}) \geq 2$. If there is $t > 1$ such that $t \in \frac{1}{a}A$, then (2.13) implies

$$\sup\Big\{f_{\mathcal{G}_1}(t) : t \in \frac{1}{a}A\Big\} = f_{\mathcal{G}_1}\Big(\frac{M}{a}\Big) = \mu\Big(\frac{M}{a}\Big) \leq \mu\Big(\frac{M}{m}\Big).$$

The left part in (2.28) can be rewritten as

$$\sup\{g_{\frac{1}{f(a)}} \circ \psi_\varepsilon \circ g_a(t) : t \in g_a^{-1}(A)\} = \frac{1}{f(a)} \sup\{\psi_\varepsilon(t) : t \in A\} = \frac{1}{f(a)} s(f).$$

The last equality (2.28) and (2.29) imply

$$\frac{1}{f(a)} s(f) \leq \mu\Big(\frac{M}{m}\Big)$$

or, in the equivalent form,

$$s(f) \le f(a_\varepsilon)\mu\Big(\frac{M}{\varepsilon}\Big).$$

Letting $\varepsilon \to 0$ we obtain (2.27).

The rest part of Corollary 2.19 can be proved by simple computation. □

2.20 Remark. The function μ is subadditive, in fact it was proved in (2.26). Hence, by Theorem 4.1, μ belongs to $\mathcal{F}$. Consequently the constant $\mu(\frac{M}{m})$ is the best possible for inequality (2.27).

2.21 Remark. By Lemma 2.17 all functions $f \in \mathcal{F}$ are subadditive. This fact implies, of course, the existence of analogs to some results of the present paper in the general theory of subadditive functions. Cf., for example, Proposition 2.13 with Theorem 7.2.2 in [6] or Theorem 2.4 with Theorems 7.2.3 and 7.3.3 in [6].

3 The smallest ideal and the largest subgroup of the semigroup $(\mathcal{F}, \circ)$

The sets

$$\mathcal{A} := \{f_a : a \in \mathbb{R}^{\oplus} \setminus \{0\}\} \quad \text{and} \quad \mathcal{B} := \{g_b : b \in \mathbb{R}^{\oplus} \setminus \{0\}\}, \tag{3.1}$$

where

$$g_b(x) = bx,\ x \in \mathbb{R}^{\oplus} \quad \text{and} \quad f_a(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x > 0, \end{cases}$$

see Example 2.1, have interesting, purely algebraic characterizations in the semigroup $(\mathcal{F}, \circ)$.

3.1 Definition. *Let S be a semigroup. An element $i \in S$ is a left zero of S if $ix = i$ for all $x \in S$. A nonvoid set $\Gamma \subseteq S$ is a bilateral ideal of S if $(xb)y \in \Gamma$ for all $x, y \in \Gamma$ and every $b \in S$.*

3.2 Proposition. *The set $\mathcal{A}$ is the smallest bilateral ideal of the semigroup $(\mathcal{F}, \circ)$.*

Proof. It is known and easy to prove, that the set of all left zeros of a semigroup S if the smallest bilateral ideal of S if this set is nonvoid. See [4, §1.1, Exercise 1.6] for the dual statement.

For all $g \in \mathcal{F}$ and each $f_a \in \mathcal{A}$, we have the equalities

$$g \circ f_a = f_{g(a)} \quad \text{and} \quad f_a \circ g = f_a. \tag{3.2}$$

Equalities (3.2) imply that $\mathcal{A}$ is a bilateral ideal. Write $\hat{\mathcal{A}}$ for the smallest bilateral ideal in $\mathcal{F}$. Evidently we have inclusion $\hat{\mathcal{A}} \subseteq \mathcal{A}$. The second equality in (3.2) means that each f_a is a left zero of $\mathcal{F}$. Consequently we obtain the converse inclusion $\mathcal{A} \subseteq \hat{\mathcal{A}}$. □

3.3 Corollary. *If e is a left zero of $\mathcal{F}$, then there is $a \in \mathbb{R}^{\oplus} \setminus \{0\}$ such that $e = f_a$. There are no right zeros in $\mathcal{F}$.*

Proof. The first statement follows from Proposition 3.2. The nonuniqueness of the left zeros implies the second one. For details see [4, §1.1]. $\square$

3.4 Definition. *Let S be a semigroup with the unit e, i.e., $ex = xe = x$ for each $x \in S$. An element $i \in S$ is invertible in S if there is $b \in S$ such that*

$$ib = e = bi. \tag{3.3}$$

This b will be called the *inverse* element for i. The following result is well known. See, for example, [4, §1.7, Theorem 1.10].

3.5 Lemma. *Let S be a semigroup with the unit e. The following statements hold.*

(i) *The set U of all invertible in S elements is a subgroup in S.*

(ii) *If B is a subgroup in S and $e \in B$, then $B \subseteq U$.*

(iii) *If i is invertible in S, then the inverse element for i is unique. Moreover if $bi = e$ or $ib = e$ for the invertible i, then b is the inverse element for i.*

It is clear that g_1 is the unit of the semigroup $(\mathcal{F}, \circ)$. Write $\mathcal{U}$ for the set of all invertible in $(\mathcal{F}, \circ)$ elements.

3.6 Theorem. *We have the equality*

$$\mathcal{U} = \mathcal{B},$$

where $\mathcal{B}$ was defined in (3.1), that is a function $f \in \mathcal{F}$ invertible in $(\mathcal{F}, \circ)$ if and only if there is $b > 0$ such that $f = g_b$.

This theorem and statement (ii) of Lemma 3.5 imply the next characterization of $\mathcal{B}$.

3.7 Corollary. *The set $\mathcal{B}$ is the largest subgroup of $(\mathcal{F}, \circ)$ containing the unit of $(\mathcal{F}, \circ)$.*

We shall prove the following form of Theorem 3.6.

3.8 Theorem. *Let i be an invertible element of $(\mathcal{F}, \circ)$. If $i(1) = 1$, then $i = g_1$, that is*

$$i(x) = x \tag{3.4}$$

for all $x \in \mathbb{R}^{\oplus}$.

Note that Theorem 3.8 implies Theorem 3.6 because $\mathcal{B}$ is a subgroup of $(\mathcal{F}, \circ)$ and $\mathcal{B} \subseteq \mathcal{U}$ and $\mathcal{B} \ni g_1$ and the value of the superposition $f_{i^{-1}(1)} \circ i$ at the point 1 is 1 where $i^{-1}(1) = \frac{1}{i(1)}$.

We separate the proof of Theorem 3.8 into several lemmas.

3.9 Lemma. *Let $f \in \mathcal{F}$. The inequality*

$$|f(x+y) - f(x)| \leq f(y) \tag{3.5}$$

holds for each $x, y \in \mathbb{R}^{\oplus}$.

Proof. Inequality (3.5) can be rewritten as

$$-f(y) \leq f(x+y) - f(x) \leq f(y). \tag{3.6}$$

The right inequality in (3.6) follows from Lemma 2.17. To prove the left one note that (1.4) holds with $a = x + y,\ b = x$ and $c = x$ for all $x, y \in \mathbb{R}^{\oplus}$. Hence, using Theorem 1.1, we obtain

$$2f(x) \leq 2(f(x) \vee f(y) \vee f(x+y)) \leq f(x+y) + f(x) + f(y)$$

which implies the double inequality $-f(y) \leq f(x+y) - f(x)$. □

3.10 Lemma. *Let i be invertible in $(\mathcal{F}, \circ)$. Then i is a continuous bijection of the set $\mathbb{R}^{\oplus} = [0, \infty)$.*

Proof. The first equality in (3.3) implies that i is a surjection, the second one shows that i is an injection. Consequently i is bijective.

The function i is continuous at the point zero. Indeed, in the opposite case statement (ii) of Theorem 2.11 implies the existence $a > 0$ such that $i(x) \geq a$ for all $x > 0$, contrary to the surjectivity of i.

Suppose now that t is an arbitrary positive number. Let $\delta \in (0, t)$. Using inequality (3.5) for $f = i$ with $x = t,\ y = \delta$ and with $x = t - \delta,\ y = \delta$ we obtain

$$|i(t+\delta) - i(t)| \leq i(\delta) = |i(\delta) - i(0)| \tag{3.7}$$

and

$$|i(t-\delta) - i(t)| \leq |i(\delta) - i(0)|. \tag{3.8}$$

Since i is continuous at 0, the estimates (3.7), (3.8) yield the continuity of i at the point t. □

Let i be an invertible element of the semigroup $(\mathcal{F}, \circ)$. In what follows we denote by i^{-1} the inverse element for i, see Definition 3.4.

3.11 Lemma. *Let i be invertible in $(\mathcal{F}, \circ)$ and let $i(1) = 1$. Then the equality*

$$i(n) = n \tag{3.9}$$

holds for all $n \in \mathbb{N}$.

Proof. The equality $i(1) = 1$ implies that $i \in \mathcal{G}_1 = \{f \in \mathcal{F} : f(1) = 1\}$. Consequently we have $i(x) \le f_{\mathcal{G}_1}(x)$. In particular, using (2.13) and (2.10) we obtain

$$i(n) \le f_{\mathcal{G}_1}(n) = \mu(n) = n \tag{3.10}$$

for all natural numbers n.

The equality $i(1) = 1$ implies also $i^{-1}(1) = 1$ because $1 = i^{-1}(i(1))$. Hence $i^{-1} \in \mathcal{G}_1$ so the inequality

$$i^{-1}(n) \le n \tag{3.11}$$

holds for all $n \in \mathbb{N}$. Suppose now that there is $n_0 \in \mathbb{N}$ such that $i(n_0) < n_0$. Since i^{-1} is invertible in $(\mathcal{F}, \circ)$, the function i^{-1} is strictly increasing. Hence the last inequality implies $n_0 = i^{-1}(i(n_0)) < i^{-1}(n_0)$, contrary to (3.11). $\square$

It can be proved that an increasing function $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ with $f(0) = 0$ and $f(x) > 0$ for $x > 0$ belongs to $\mathcal{F}$ if and only if $f(x + y) \le f(x) + f(y)$. See Theorem 4.1 in the next section.

3.12 Lemma. *Let i be invertible in $(\mathcal{F}, \circ)$ and let $a > 0$. Then the cut-off function*

$$i_a(x) = \begin{cases} i(x) & \text{for } 0 \le x \le a \\ i(a) & \text{for } x > a. \end{cases}$$

belongs to $\mathcal{F}$.

Proof. Lemma 3.10 implies that i_a is increasing. Hence it is sufficient to prove that

$$i_a(x + y) \le i_a(x) + i_a(y) \tag{3.12}$$

for all $x, y \in \mathbb{R}^{\oplus}$. Inequalities (3.12) and $i(x+y) \le i(x)+i(y)$ are equivalent if $x + y \le a$. If $x + y > a$ and $x \vee y \ge a$ then (3.12) follows from the definition of i_a. If $x + y > a$ but $x < a$ and $y < a$, then $i_a(x + y) \le i(x + y) \le i(x) + i(y) = i_a(x) + i_a(y)$. $\square$

Write F_g for the set of all fixed points of a function $g : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$, i.e., $F_g = \{x : g(x) = x\}$.

3.13 Lemma. *Let i be invertible in $(\mathcal{F}, \circ)$ and let $i(1) = 1$. Then the point 0 is a limit point of the set F_i.*

Proof. Suppose that 0 is an isolated point of the set F_i. By Lemma 3.10 the function i is continuous. Hence F_i is closed. Since F_i is closed and $1 \in F_i$, there is $t_0 \in (0, 1]$ such that

$$i(t_0) = t_0 \quad \text{and} \quad i(t) \ne t$$

for each $t \in (0, t_0)$ and, by continuity, either

$$i(t) < t \tag{3.13}$$

for all $t \in (0, t_0)$, or

$$i(t) > t \tag{3.14}$$

for all $t \in (0, t_0)$. Consider firstly the case (3.13). By Lemma 3.12 the cut-off function i_{t_0} belongs to $\mathcal{F}$. For every positive integer n, denote by $i_{t_0}^{(n)}$ the n-th iteration of the function i_{t_0}, that is

$$i_{t_0}^{(1)} = i_{t_0}, \quad i_{t_0}^{(2)} = i_{t_0}^{(1)} \circ i_{t_0}^{(1)}, \quad \dots, \quad i_{t_0}^{(n)} = i_{t_0}^{(1)} \circ i_{t_0}^{(n-1)}$$

and so on. Since i is strictly increasing, inequality (3.13) implies

$$t_0 > t > i_{t_0}^{(1)}(t) > i_{t_0}^{(2)}(t) > \cdots > i_{t_0}^{(n)}(t) > \cdots \geq 0 \tag{3.15}$$

for all $t \in (0, t_0)$. Moreover we have

$$t_0 = i_{t_0}^{(n)}(t) \quad \text{for each } n = 1, 2, \dots \text{ and all } t \geq t_0.$$

Consequently the sequence $\{i_{t_0}^{(n)}\}_{n \in \mathbb{N}}$ is pointwise convergent. Write

$$i_{t_0}^{\infty}(t) = \lim_{n \to \infty} i_{t_0}^{(n)}(t)$$

for every $t \in \mathbb{R}^{\oplus}$. It is clear that $i_{t_0}^{\infty}(t_0) = t_0$. Hence, by Theorem 2.4, $i_{t_0}^{\infty} \in \mathcal{F}$. Theorem 1.1 implies that

$$i_{t_0}^{\infty}(t) \neq 0 \tag{3.16}$$

for all $t > 0$. Let $t_1 \in (0, t_0)$. Since i is a continuous function, from (3.15) we obtain

$$i(i_{t_0}^{\infty}(t_1)) = i(\lim_{n \to \infty} i_{t_0}^{(n)}(t_1)) = \lim_{n \to \infty} i_{t_0}^{(n+1)}(t_1) = i_{t_0}^{\infty}(t_1).$$

Hence $i_{t_0}^{\infty}(t_1)$ is a fixed point of i. Moreover (3.15) and (3.16) show that $i_{t_0}^{\infty}(t_1) \in (0, t_0)$, contrary to inequality (3.13).

If inequality (3.14) holds for all $t \in (0, t_0)$, then we can reason similarly using i^{-1} instead of i. Indeed, it is easy to see that the fixed points are the same for the functions i and i^{-1}, i.e., $F_i = F_{i^{-1}}$ and that inequality (3.14) implies

$$t = i^{-1}(i(t)) > i^{-1}(t)$$

because, by Lemma 3.10, i^{-1} is strictly increasing.

Thus neither inequality (3.13) nor inequality (3.14) holds. Consequently 0 is a limit point of F_i, as required. □

We recall the definition of the lower right Dini derivative. Let a real-valued function f be defined on a set $A \subseteq \mathbb{R}$ and let $x_0 \in A$. Suppose that

A contains some half-open interval $[x_0, a)$. The lower right Dini derivative D_+ of f at x_0 is defined by

$$D_+ f(x_0) = \liminf_{\substack{x \to x_0 \\ x \in (x_0, a)}} \frac{f(x) - f(x_0)}{x - x_0}. \tag{3.17}$$

Lemma 3.13 implies the following

3.14 Corollary. *Let i be invertible and let $i(1) = 1$. Then the inequality*

$$D_+ i(0) \le 1 \tag{3.18}$$

holds.

We call a map $f : X \to Y$ between metric spaces (X, d) and (Y, ρ) Lipschitz if there is a constant $L \ge 0$ such that

$$\rho(f(x), f(y)) \le L d(x, y)$$

for all $x, y \in X$. The infimum of all real numbers L satisfying the mentioned above inequality is called the Lipschitz constant of f and is denoted by $Lip(f)$.

3.15 Lemma. *Let $f \in \mathcal{F}$. The function f is Lipschitz if and only if*

$$D_+ f(0) < \infty.$$

If f is Lipschitz, then

$$Lip(f) = D_+ f(0). \tag{3.19}$$

Proof. It is sufficient to show that

$$|f(x) - f(y)| \le D_+ f(0) |x - y| \tag{3.20}$$

for all $x, y \in \mathbb{R}^{\oplus}$. If $D_+ f(0) = \infty$, then (3.20) is trivial. Suppose that $D_+ f(0) < \infty$. By Theorem 2.11 this inequality implies the continuity of f at 0. Let $\varepsilon > 0$. Using (3.17) we see that there is a sequence of positive numbers δ_n, $n \in \mathbb{N}$, such that

$$\lim_{n \to \infty} \delta_n = 0 \quad \text{and} \quad f(\delta_n) \le (1 + \varepsilon) \delta_n D_+ f(0). \tag{3.21}$$

Let x and y be points from $\mathbb{R}^{\oplus}$, $x < y$. Define, for every $n \in \mathbb{N}$, the quantity N as

$$N = N(n) = \left[\frac{y - x}{\delta_n} \right] \tag{3.22}$$

where $[\frac{y-x}{\delta_n}]$ is the integral part of $\frac{y-x}{\delta_n}$. Then we can write

$$f(x) - f(y) = \sum_{i=0}^{N-1} (f(x + i\delta_n) - f(x + (i+1)\delta_n)) + f(x + N\delta_n) - f(y).$$

This formula, (3.21) and (3.5) imply

$$
\begin{aligned}
|f(x)-f(y)| &\le \sum_{i=0}^{N-1}(1+\varepsilon)\delta_n D_+ f(0) + f(|(y-x)-N\delta_n|) \\
&= (N-1)(1+\varepsilon)\delta_n D_+ f(0) + f(|(y-x)-N\delta_n|). \quad (3.23)
\end{aligned}
$$

Using (3.22) and the continuity of f at 0 we obtain

$$\lim_{n\to\infty}(N-1)\delta_n = |x-y| \quad \text{and} \quad \lim_{n\to\infty} f(|(y-x)-N\delta_n|) = 0.$$

Consequently (3.23) implies that

$$|f(x)-f(y)| \le (1+\varepsilon)D_+ f(0)|x-y|.$$

Letting $n\to\infty$ we have (3.20). □

We are ready now to finish the

Proof of Theorem 3.8. Let i be an invertible element of $(\mathcal{F},\circ)$ and let $i(1)=1$. Corollary 3.14 and Lemma 3.15 imply that i is a Lipschitz function and $Lip(i)\le 1$. Consequently i is an absolutely continuous function. Hence the derivative $i'(t)$ exists almost every on $[0,\infty)$, and the inequality

$$i'(t) \le 1 \tag{3.24}$$

holds a.e. on $[0,\infty)$, and the equality

$$i(t) = \int_0^t i'(t)\,dt \tag{3.25}$$

holds for each $t\in[0,\infty)$. See, for example, [7, Chapter IX]. Formulas (3.9) and (3.25) imply that

$$n = \int_0^n i'(t)\,dt \tag{3.26}$$

for each $n\in\mathbb{N}$. Inequality (3.24) shows that if (3.26) is true for all $n\in\mathbb{N}$, then $i'(t)=1$ a.e. on $\mathbb{R}^{\oplus}$. Consequently, by (3.25), we have the desired equality

$$i(t) = \int_0^t dt = t$$

for all $t\in\mathbb{R}^{\oplus}$. □

The following continuity properties of functions belonging to $\mathcal{F}$ were, in fact, obtained during the proof of Theorem 3.8.

3.16 Proposition. *Let $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ belong to $\mathcal{F}$.*

(i) *If f is continuous at 0, then f is uniformly continuous on $\mathbb{R}^{\oplus}$.*

(ii) *If the lower right Dini derivative of f at 0 is finite, $D_+f(0) < \infty$, then f is differentiable a.e. on $\mathbb{R}^{\oplus}$ and there is a (right) derivative $f'(0)$ of f at 0 and*

$$D_+f(0) = f'(0) > 0. \tag{3.27}$$

Proof. Statement (i) follows from Lemma 3.15. The inequality $D_+f(0) < \infty$ implies, by Lemma 3.15, that f is Lipschitz. Consequently, by the Rademacher theorem, f is differentiable a.e. on $\mathbb{R}^{\oplus}$. It follows from (3.19) that

$$D^+f(0) := \limsup_{\substack{x \to 0 \\ x > 0}} \frac{f(x) - f(0)}{x - 0} \le D_+f(0).$$

The converse inequality $D^+f(0) \ge D_+f(0)$ is trivial. Hence $D_+f(0) = D^+f(0)$, that implies the existence of the (right) derivative $f'(0)$. If $f'(0) = 0$, then, using (3.19), we obtain $f(t) = f(0) = 0$ for all $t \in \mathbb{R}^{\oplus}$, contrary to the condition $f \in \mathcal{F}$. □

3.17 Remark. Theorem 3.8 and Proposition 3.16 are closely related to Theorem 7.11.2 in the book [6].

3.18 Definition. *Let S be a semigroup. A bilateral ideal A of S is prime if*

$$xy \in A \text{ implies } (x \in A) \text{ or } (y \in A)$$

for each $x, y \in S$.

3.19 Corollary. *The set $\mathcal{I} := \{f \in \mathcal{F} : D_+f(0) = \infty\}$ is a prime bilateral ideal of the semigroup $(\mathcal{F}, \circ)$.*

Proof. Let f and g be elements of $\mathcal{F}$. Statement (ii) of Proposition 3.16 implies that there is a constant $c > 0$ such that

$$(D_+f(0)) \wedge (D_+g(0)) > c.$$

Using (3.17) we can obtain the inequality

$$D_+(f \circ g)(0) \ge D_+f(0) \cdot D_+g(0).$$

Hence if $D_+f(0) = \infty$, or $D_+(g(0)) = \infty$ then $D_+(f \circ g)(0) = \infty$ i.e., $\mathcal{I}$ is a bilateral ideal of $(\mathcal{F}, \circ)$.

Furthermore, Statement (ii) of Proposition 3.16 implies also that f and g are differentiable at 0 if $f, g \notin \mathcal{I}$. In this case it follows from the Chain Rule that

$$D_+(f \circ g)(0) = f'(0) \cdot g'(0) < \infty.$$

Consequently $\mathcal{I}$ is prime. □

As we have seen from Example 2.2 that a function $f \in \mathcal{F}$ can be discontinuous everywhere. By Statement (i) of Proposition 3.16 a nonvoid set of discontinuity of $f \in \mathcal{F}$ must contain 0. Moreover it is well known, that the set of discontinuity of each real-valued function f is a F_σ-subset of the domain of definition of f.

3.20 Proposition. *The following two statements are equivalent for each nonvoid set $A \subseteq \mathbb{R}^{\oplus}$.*

(i) *There is $f \in \mathcal{F}$ such that A is the set of discontinuity of f.*

(ii) *A is of type F_σ and $0 \in A$.*

Proof. It is sufficient to prove (ii)$\Rightarrow$(i). Let A be a F_σ-subset of $\mathbb{R}^{\oplus}$ and let $0 \in A$. There is a function $f : (0, \infty) \to \mathbb{R}$ such that the F_σ-set $A \setminus \{0\}$ is the set of discontinuity of f. A simple construction of such function can be found in [http://en.wikipedia.org/wiki/Thomeae's_function]. Let $h : \mathbb{R} \to (1, 2)$ be a homeomorphism. The function

$$g(t) = \begin{cases} 0 & \text{if } t = 0 \\ h(f(t)) & \text{if } t > 0 \end{cases}$$

belongs to $\mathcal{F}$, see Example 2.2, and has A as the set of discontinuity. □

4 Monotone functions transferring metrics to metrics

Write $\mathcal{F}_m$ and, respectively, $\mathcal{F}_{mc}$ for the set of all increasing functions $f \in \mathcal{F}$ and, respectively, for the set of all continuous functions $f \in \mathcal{F}_m$. It is clear that $(\mathcal{F}_m, \circ)$ and $(\mathcal{F}_{mc}, \circ)$ are subsemigroups of $(\mathcal{F}, \circ)$ and that a function f belongs to $\mathcal{F}_m$ if and only if f is monotone and $f \in \mathcal{F}$.

4.1 Theorem. *Let $f : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ be an increasing function such that $f(0) = 0$ and $f(t) > 0$ for all $t > 0$. Then $f \in \mathcal{F}_m$ if and only if the inequality*

$$f(x + y) \le f(x) + f(y) \tag{4.1}$$

holds for all $x, y \in \mathbb{R}^{\oplus}$.

Proof. In view of Lemma 2.17 it is sufficient to show that subadditivity (4.1) implies $f \in \mathcal{F}$.

Suppose f to be subadditive. Let $a, b, c \in \mathbb{R}^{\oplus}$ such that $a \le b + c$. Using the increase of f and (4.1) we obtain

$$f(a) \le f(b + c) \le f(b) + f(c).$$

Hence the inequality $2(a \vee b \vee c) \le a + b + c$ implies

$$2(f(a) \vee f(b) \vee f(c)) \le f(a) + f(b) + f(c).$$

Consequently, by Theorem 1.1, f belongs to $\mathcal{F}$. □

Another way to prove the subadditivity of an increasing function $f : \mathbb{R}^\oplus \to \mathbb{R}^\oplus$ with $f(0) = 0$ is to show that the function $t^{-1}f(t)$ is decreasing on $(0, \infty)$ [9, Section 3.2.3]. The last condition is true in the case of concave functions.

4.2 Corollary. *If a function $f : \mathbb{R}^\oplus \to \mathbb{R}^\oplus$ is increasing, concave and $f(0) = 0$ and $f(x) > 0$ for all $x > 0$, then $f \in \mathcal{F}_m$.*

In this case, f is continuous on $(0, \infty)$ because each bounded concave function is continuous on open intervals belonging to its domain of definition, see, for example [7, Chapter X, §5, Theorem 5].

It is interesting to note that the functions belonging to $\mathcal{F}_m$ can be also characterized as the "first moduli of continuity" of bounded real-valued functions.

Let us define the set $\mathcal{W}$ of functions $f : \mathbb{R}^\oplus \to \mathbb{R}^\oplus$ by the rule:

$$\begin{aligned}(f \in \mathcal{W}) \Leftrightarrow (&f \text{ is nonconstant and } \forall\, \varepsilon > 0\ \exists\, \delta > 0 \\ &\text{such that } |f(x) - f(y)| < \delta \text{ whenever } |x - y| < \varepsilon).\end{aligned} \tag{4.2}$$

It is clear that every uniformly continuous $f : \mathbb{R}^\oplus \to \mathbb{R}^\oplus$ belongs to $\mathcal{W}$. Write, for $f \in \mathcal{W}$

$$\omega(f, \varepsilon) := \sup_{\substack{|t-x|\le\varepsilon \\ t,x\in\mathbb{R}}} |f(t) - f(x)|. \tag{4.3}$$

Rule (4.2) simply means that $\omega(f, \varepsilon) < \infty$ for each $\varepsilon > 0$ if $f \in \mathcal{W}$. Lemmas 2.6 and 3.9 imply $\mathcal{F} \subseteq \mathcal{W}$. In the case where f is uniformly continuous on $\mathbb{R}$, the function $\omega(f, \cdot)$ is simply the first modulus of continuity of f. We shall apply this term to arbitrary functions $\omega(f, \cdot)$ of form (4.3).

4.3 Theorem. *The following statements hold for every function $F : \mathbb{R}^\oplus \to \mathbb{R}^\oplus$.*

(i) *F belongs to $\mathcal{F}_m$ if and only if there is $g \in \mathcal{W}$ such that $F = \omega(g, \cdot)$.*

(ii) *F belongs to $\mathcal{F}_{mc}$ if and only if there is uniformly continuous, nonconstant $g : \mathbb{R}^\oplus \to \mathbb{R}^\oplus$ such that $F = \omega(g, \cdot)$.*

In view of Theorem 4.1, statement (ii) is, in fact, equivalent to the classical Lebesgue–Nikolsky characterization of the first moduli of continuity as increasing, subadditive, continuous functions f with $f(0) = 0$. Statement (i) is an extension of this result on moduli of continuity (4.3) which are, generally speaking, discontinuous. The proof of these statements can be obtained as a simple variation of the standard one, see [9, Section 3.2] or [6, Section 7.10].

4.4 Corollary. *Let $g : \mathbb{R}^\oplus \to \mathbb{R}^\oplus$ be an increasing, nonconstant function. Then the superposition $g \circ d$ is a metric for each metric d if and only if $g \circ \omega$ is the first modulus of continuity for each first modulus of continuity ω.*

Proof. By statement (i) of Theorem 4.3, the function g belongs to $\mathcal{F}_m$ if and only if g is the first modulus of continuity. Since $(\mathcal{F}_m, \circ)$ is a semigroup with the unit, the set of all nonzero, first moduli of continuity forms also a semigroup with the unit. Hence the superposition of each two first modulus of continuity is a such modulus and every nonconstant function $g : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ transferring the set of these moduli into itself belongs to $\mathcal{F}_m$. $\square$

The known properties of the usual first moduli of continuity imply the corresponding properties for $f \in \mathcal{F}_m$. For instance we have

4.5 Corollary. *A function $f \in \mathcal{W}$ belongs to $\mathcal{F}_m$ if and only if f is a fixed point of the mapping*

$$\mathcal{W} \ni f \longmapsto \omega(f, \cdot) \in \mathcal{W}.$$

The proof is simple and it is omitted here. A similar property for classical moduli of continuity was firstly noted by S. M. Nikolsky in [8].

Consider now some examples of functions belonging to $\mathcal{F}_m$.

4.6 Example. If $\alpha \in (0,1)$, then the function $s_\alpha(t) = t^\alpha,\ t \in \mathbb{R}^{\oplus}$, belongs to $\mathcal{F}_m$. If (X, d) is a metric space, then $(X, s_\alpha \circ d)$ is so called the *snowflaked version* of (X, d). Letting $\alpha \to 0$ we obtain

$$\lim_{\alpha \to 0} t^\alpha = f_1(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0, \end{cases}$$

see Example 2.1.

4.7 Example. The function

$$\mu(x) = \begin{cases} x & \text{if } x \in \mathbb{N} \\ [x] + 1 & \text{if } x \in \mathbb{R}^{\oplus} \setminus \mathbb{N}, \end{cases}$$

where $[x]$ is the integral part of x is increasing and subadditive. This function belongs to $\mathcal{F}_m$, see Remark 2.20. Let $\mathcal{G}_{1m}$ be a family of all functions $f \in \mathcal{F}_m$ such that $f(1) = 1$. Then μ is the upper envelope of the family $\mathcal{G}_{1,m}$, i.e.,

$$\mu(x) = \bigvee_{f \in \mathcal{G}_{1,m}} f(x) \tag{4.4}$$

for each $x \in \mathbb{R}^{\oplus}$. If $x \in \mathbb{R}^{\oplus} \setminus (0,1)$, then formula (4.4) follows from Theorem 2.16, because $\mathcal{G}_{1,m} \subseteq \mathcal{G}_1$. In the case $x \in (0,1)$ we have

$$f(x) \le f(1) = 1 = \mu(x)$$

for each $f \in \mathcal{G}_{1,m}$. Consequently (4.4) holds for all $x \in \mathbb{R}^{\oplus}$.

4.8 Example. Let $x \in [0,1]$ and expand x as

$$x = \sum_{n=1} \frac{a_{nx}}{3^n}, \quad a_{nx} \in \{0,1,2\}.$$

Denote by N_x the smallest n with $a_{nx} = 1$ if it exists and put $N_x = \infty$ if there is no such a_{nx}. The Cantor function $G : [0,1] \to \mathbb{R}^{\oplus}$ can be defined as

$$G(x) := \frac{1}{2^{N_x}} + \frac{1}{2} \sum_{n=1}^{N_x - 1} \frac{a_{nx}}{2^n}.$$

Define an extended Cantor's function $\hat{G} : \mathbb{R}^{\oplus} \to \mathbb{R}^{\oplus}$ as follows

$$\hat{G}(x) = \begin{cases} G(x) & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

The extended Cantor function is subadditive, i.e.,

$$\hat{G}(x+y) \leq \hat{G}(x) + \hat{G}(y) \tag{4.5}$$

for all $x, y \in \mathbb{R}^{\oplus}$. Moreover $\hat{G}(0) = 0$, $\hat{G}(x) > 0$ for all $x > 0$ and $\hat{G}$ is a continuous increasing function. Hence $\hat{G} \in \mathcal{F}_{mc}$. The proof of inequality (4.5) can be found in the book [9, Section 3.2.4] or in the paper [5].

4.9 Example. For every $k > 0, a > 0$ and all $x \in \mathbb{R}^{\oplus}$ write

$$F_{k,a}(x) = (kx) \wedge a, \tag{4.6}$$

and put

$$F_{k,\infty}(x) := kx, \tag{4.7}$$

i.e., $F_{k,\infty} = g_k$, see Example 2.1. Corollary 4.4 implies that each $F_{k,a}$ belongs to $\mathcal{F}_{m,c}$. A simple calculation shows that

$$F_{k,a} \circ F_{c,b} = F_{kc,(kb) \wedge a} \tag{4.8}$$

for all $k, c \in (0, \infty)$ and $a, b \in (0, \infty]$. Hence the set of all functions $F_{k,a}$ forms a subsemigroup of the semigroup $\mathcal{F}_{m,c}$. We denote this subsemigroup by $\mathcal{F}_i$. We shall find a purely algebraic characterization of $\mathcal{F}_i$ as a subsemigroup of $\mathcal{F}_{m,c}$ in Theorem 4.13. To this purpose we start with the description of the idempotents in $\mathcal{F}_{m,c}$.

Recall that an element e of a semigroup S is called an idempotent of S if $ee = e$. It is well known that, for every nonvoid set X, an element α of the symmetric semigroup $\mathcal{T}_X$ of all functions $f : X \to X$ is an idempotent if and only if

$$\alpha(t) = t$$

for each $t \in \alpha(X) = \{\alpha(x) : x \in X\}$. See, for example, [4, §1.1, Exercise 9].

4.10 Proposition. *An element e of the semigroup $\mathcal{F}_{m,c}$ is idempotent if and only if there is $a \in (0, \infty]$ such that*

$$e = F_{1,a}. \tag{4.9}$$

Proof. Using (4.8) we see that

$$F_{1,a} \circ F_{1,a} = F_{1\cdot 1, a \wedge a} = F_{1,a}$$

for each $a \in (0, \infty]$. Hence each $F_{1,a}$ is an idempotent of $\mathcal{F}_{m,c}$.

Conversely, suppose that e is an idempotent of $\mathcal{F}_{m,c}$. Then each point of the set $e(\mathbb{R}^{\oplus})$ is a fixed point of the function e. Since e is continuous and $e(0) = 0$ and $e(x) > 0$ for $x > 0$, the set $e(\mathbb{R}^{\oplus})$ is a nondegenerate interval containing 0. If this interval is unbounded, then $e = F_{1,\infty}$. In the opposite case there is $a > 0$ such that

$$\text{either } e(\mathbb{R}^{\oplus}) = [0, a] \quad \text{or } e(\mathbb{R}^{\oplus}) = [0, a).$$

The second equality is impossible because $e([0, a)) = [0, a)$ and, by continuity, we have

$$e(a) = \lim_{\substack{x \to a \\ x < a}} e(x) = \lim_{x \to a} x = a.$$

The equalities $e(\mathbb{R}^{\oplus}) = [0, a]$, $e(a) = a$ and the increase of e imply the equality $e(x) = a$ for all $x \geq a$. Thus there is $a > 0$ such that (4.9) holds. □

4.11 Corollary. *Every two idempotents of $\mathcal{F}_{m,c}$ are commuting.*

Proof. By (4.8) we have

$$F_{1,a} \circ F_{1,b} = F_{1,a \wedge b} = F_{1,b} \circ F_{1,a}$$

for each $a, b \in (0, \infty]$. □

4.12 Definition. *Let S be a semigroup. An element $e \in S$ is regular if there is $x \in S$ such that*

$$axa = a.$$

The semigroup S is regular if each $a \in S$ is regular. The semigroup S is inverse if for every $a \in S$ there is a unique $b \in S$ such that

$$aba = a \quad \text{and} \quad bab = b. \tag{4.10}$$

The following theorem shows that $\mathcal{F}_i$, see Example 4.9, is the largest inverse subsemigroup of $\mathcal{F}_{mc}$.

4.13 Theorem. *The semigroup $\mathcal{F}_i$ is inverse and each inverse subsemigroup of $\mathcal{F}_{mc}$ lies in $\mathcal{F}_i$.*

The following characterization of inverse semigroup shall be used in the proof of Theorem 4.13.

4.14 Lemma. *A semigroup S is inverse if and only if S regular and every two idempotents of S are commuting.*

For the proof see, for example, [4, §1.9 Theorem 1.17].

Proof of Theorem 4.13. Lemma 4.14 and Corollary 4.11 imply that $\mathcal{F}_i$ is inverse if and only if each $F_{k,a}$ is regular. Using (4.8) we obtain

$$F_{k,a} \circ F_{c,b} \circ F_{k,a} = F_{kc,(kb)\wedge a} \circ F_{k,a} = F_{kck,(kca)\wedge((kb)\wedge a)}.$$

In particular,

$$F_{k,a} \circ F_{\frac{1}{k},\frac{a}{k}} \circ F_{k,a} = F_{k,a}$$

because, for $c = \frac{1}{k}$ and $b = \frac{a}{k}$, we have

$$kck = a \quad \text{and} \quad (kca) \wedge ((kb) \wedge a) = a \wedge (a \wedge a) = a.$$

Thus $\mathcal{F}_i$ is an inverse semigroup.

Conversely, suppose that S is an inverse subsemigroup of $\mathcal{F}_{m,c}$. If $e, l \in S$ and (4.10) holds, then

$$(e \circ l \circ e) \circ l = e \circ l \quad \text{and} \quad (l \circ e \circ l) \circ e = l \circ e.$$

Hence $e \circ l$ and $l \circ e$ are idempotents. Consequently, by Proposition 4.10, there are $a, b \in (0, \infty]$ such that

$$l(e(t)) = \begin{cases} t & \text{if } 0 \le t \le b \\ b & \text{if } t > b \end{cases} \quad \text{and} \quad e(l(t)) = \begin{cases} t & \text{if } 0 \le t \le a \\ a & \text{if } t > a. \end{cases} \tag{4.11}$$

Since

$$e \circ (l \circ e) = e \quad \text{and} \quad l \circ (e \circ l) = l,$$

we obtain from (4.11) that

$$e(t) = e(l(e(t))) = e(b),$$

for $t \ge b$, and, similarly, for $t \ge a$,

$$l(t) = l(e(l(t))) = l(a).$$

Since $D_+(e \circ l)(0) = 1$, Corollary 3.19 and formula (4.11) show that

$$D_+e(0) < \infty \quad \text{and} \quad D_+l(0) < \infty.$$

Proposition 3.16 gives that e and l are differentiable at 0 and

$$l'(0) > 0 \quad \text{and} \quad e'(0) > 0.$$

Write $k := l'(0)$. The Chain Rule and (4.11) imply $e'(0) = \frac{1}{k}$. By Lemma 3.15, the functions e and l are Lipshitz and

$$Lip(e) = \frac{1}{k} \quad \text{and} \quad Lip(l) = k. \tag{4.12}$$

Consequently

$$l(t) \leq F_{k,l(a)}(t) \quad \text{and} \quad e(t) \leq F_{\frac{1}{k},e(b)}(t) \tag{4.13}$$

for all $t \in \mathbb{R}^{\oplus}$. We claim that the strict inequalities are possible neither in the left part nor in the right part of (4.13). Indeed, suppose there is $t \in (0, a]$ such that

$$l(t) < F_{k,l(a)}(t).$$

Rewriting the last inequality as

$$|l(t) - l(0)| < k|t - 0|$$

and using (4.11)–(4.13) we obtain

$$t = e(l(t)) = e(l(t)) - e(l(0)) \leq \frac{1}{k}|l(t) - l(0)| < \frac{k}{k}|t - 0| = t.$$

This contradiction shows that the equality

$$l(t) = F_{k,l(a)}(t) \tag{4.14}$$

holds for each $t \in [0, a]$. Since l and $F_{k,l(a)}$ are constant functions on $[a, \infty)$, equality (4.14) holds also on $[a, \infty)$. Hence we have $l = F_{k,l(a)}$. The equality $e = F_{\frac{1}{k},e(b)}$ can be obtained similarly. □

The function μ, see Example 4.7, is a discontinuous idempotent of the semigroup $(\mathcal{F}_m, \circ)$. Hence $\mathcal{F}_i$ is not the largest inverse subsemigroup of $\mathcal{F}_m$. Nevertheless the following proposition holds.

4.15 Proposition. *The semigroup $\mathcal{F}_i$ is a maximal inverse subsemigroup of $(\mathcal{F}_m, \circ)$ in the sense that the inclusions $\mathcal{F}_i \subseteq \mathcal{F}' \subseteq \mathcal{F}_m$ imply the equality $\mathcal{F}_i = \mathcal{F}'$ for every inverse semigroup $\mathcal{F}'$.*

Proof. Suppose that $\mathcal{F}'$ is an inverse subsemigroup of $\mathcal{F}_m$ and $\mathcal{F}_i \subseteq \mathcal{F}'$ and that

$$\mathcal{F}' \setminus \mathcal{F}_i \neq \emptyset.$$

Let $f \in \mathcal{F}' \setminus \mathcal{F}_i$. Since $\mathcal{F}_i$ is the largest inverse semigroup in $\mathcal{F}_{ic}$, the function f is not continuous. Theorem 2.11 implies that there is $a > 0$ such that

$$f(x) \geq a$$

for all $x > 0$. A simple calculation shows that $f_a \circ f = f_a$ where

$$f_a(t) = \begin{cases} 0 & \text{if } t = 0 \\ a & \text{if } t > 0, \end{cases}$$

see Example 2.1. Moreover we have

$$F_{k,\infty} \circ f_a = f_{ka}$$

for each $k \in (0, \infty)$. Consequently the set $\mathcal{A}$ of all left zeros of $\mathcal{F}$ is a subset of $\mathcal{F}'$. It is clear that each left zero is an idempotent,

$$f_a \circ f_a = f_a,$$

and that

$$f_a \circ f_b = f_a \neq f_b = f_b \circ f_a$$

for distinct a and b. Thus these idempotents are not commutative. By Lemma 4.14 the semigroup $\mathcal{F}'$ cannot be inverse, contrary to the supposition.

□

4.16 Corollary. *The semigroup $\mathcal{F}_i$ does not have the largest inverse subsemigroup.*

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