TOEPLITZ OPERATORS WITH DISTRIBUTIONAL SYMBOLS ON BERGMAN SPACES

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ABSTRACT. We study the boundedness and compactness of Toeplitz operators T_a on Bergman spaces $A^p(\mathbb{D})$, $1 < p < \infty$. The novelty is that distributional symbols are allowed. It turns out that the belonging of the symbol to a weighted Sobolev space $W_\nu^{-m,\infty}(\mathbb{D})$ of negative order is sufficient for the boundedness of T_a. We show the natural relation of the hyperbolic geometry of the disc and the order of the distribution. A corresponding sufficient condition for the compactness is also derived.

1. INTRODUCTION.

The aim of this work is to find a general class of distributions a on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, such that a Toeplitz operator with symbol a becomes well defined and bounded on (reflexive) Bergman spaces. We provide a sufficient condition for boundedness, which involves a natural connection of the order or singularity of the distribution on one hand, and of the hyperbolic geometry of $\mathbb{D}$ on the other hand: roughly, if the symbol a is a kth order distributional derivative of a bounded function $b : \mathbb{D} \to \mathbb{C}$, then the function $b(z)\nu(z)^{-k} := b(z)(1 - |z|^2)^{-k}$ should be bounded (Theorem 3.1). More precisely, the sufficient condition is expressed in terms of a belonging to a weighted Sobolev space, see Definition 2.2. The corresponding sufficient condition for compactness is given in Theorem 4.2.

Consider the space $L^p := L^p(\mathbb{D})$, $1 \le p \le \infty$, defined using the normalized area measure dA on $\mathbb{D}$, and the Bergman space A^p, which is the closed subspace of L^p consisting of analytic functions. The Bergman projection P is the orthogonal projection of L^2 onto A^2, and it has the integral representation

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta).$$

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It is also known to be a bounded projection of L^p onto A^p, when $1 < p < \infty$. For an integrable function $a : \mathbb{D} \to \mathbb{C}$ and, say, bounded analytic functions f, the Toeplitz operator T_a with symbol a is defined by setting

$$T_a f = P(af).$$

Since P is bounded, it follows easily that T_a extends to a bounded operator $A^p \to A^p$ for $1 < p < \infty$, whenever a is a bounded measurable function. A more general sufficient condition for the boundedness of T_a was recently given in [9]. The condition is a rather weak requirement of the boundedness of certain "averages" of a over hyperbolic rectangles (see Theorem 2.3 of [9]).

As for distributional symbols, if a is a compactly supported distribution on $\mathbb{D}$, one can directly define the corresponding Toeplitz operator by

$$T_a f(z) = \langle f(\zeta)(1 - z\bar{\zeta})^{-2}, a\rangle_\zeta \tag{1.1}$$

where $\langle \cdot, \cdot \rangle_\zeta$ denotes the dual paring of the test function and distribution spaces and the test function is considered as a function of the variable ζ with z being a parameter. This definition is however not satisfactorily general, since L^1–functions which are compactly supported distributions, are also compactly supported as functions. Below we give a more general definition for distributional symbols, the supports of which do not have any artificial restrictions, and the order of which may be arbitrary, though finite.

Obviously our sufficient condition for boundedness seems quite different from the one in [9], although both conditions are related to the hyperbolic geometry. The exact relation of the both conditions remains an open problem; the same holds of course for the problem of how close any of the conditions is to being also necessary.

For an account of recent developments in the study of boundedness of Toeplitz operators, see [7]. Distribution symbols have been used in the reference [2], but for different purposes than in the present work. Other related works are [3], [5], [8] (the case $p = 1$), [11], [12], [14]; see also the references in [7] and [10].

Notation and terminology. We follow the definitions and terminology of [4] and [6] for general theory of distributions. As for Sobolev spaces, we refer to [1]. For operator theory and analytic function spaces, in particular Bergman spaces, see [13]. By C, C', C_1, c etc. (respectively, C_n etc.) we denote positive constants independent of functions, variables or indices occurring in the given calculations (respectively, depending only on n). These may vary from place to place, but not in the same group of inequalities. For the norm of an element f of a Banach function space X we use the notation $\|f; X\|$; for the operator norm of a bounded linear operator $T : X \to Y$ we write $\|T : X \to Y\|$. In the following we consider various function and distribution spaces, all of which are defined on $\mathbb{D}$, unless otherwise stated.

The standard space of distributions on the disc is denoted by $\mathcal{D}' = \mathcal{D}'(\mathbb{D})$. The order of a multi–index $\alpha \in \mathbb{N}^2$, where $\mathbb{N} := \{0, 1, 2, , \ldots\}$, is denoted by $|\alpha| := \alpha_1 + \alpha_2$. The notation $\alpha \geq \beta$ for the multi–indices α, β means that $\alpha_j \geq \beta_j$ for $j = 1, 2$. As for derivatives, the notation $D^\alpha f$ stands for

$$\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}} f,$$

if f is a function of $z = x + iy$, where x, $y \in \mathbb{R}$, and α is a multi–index. The same notation is used for both classical and distributional derivatives. We also write $D^\alpha_\zeta f$, if it is necessary to indicate the same differentation of a function f with respect to its variable ζ. For an analytic function f, we denote by $f^{(l)}$ the l:th derivative with respect to $z \in \mathbb{C}$, for all $l \in \mathbb{N}$.

2. Weighted Sobolev spaces.

We define the standard weight function $\nu : \mathbb{D} \to \mathbb{R}^+$ by

$$\nu(z) = 1 - |z|^2. \tag{2.1}$$

Given $m \in \mathbb{N}$, we denote by $W^{m,1}_\nu := W^{m,1}_\nu(\mathbb{D})$ the weighted Sobolev space consisting of measurable functions f on $\mathbb{D}$ such that the distributional derivatives satisfy

$$\|f; W^{m,1}_\nu\| := \sum_{|\alpha| \leq m} \int_{\mathbb{D}} |D^\alpha f(z)| \nu(z)^{|\alpha|} dA(z) < \infty. \tag{2.2}$$

The following fact is known, but we sketch the proof for the convenience of the reader.

Lemma 2.1. *The subspace $C^\infty_0 := C^\infty_0(\mathbb{D})$ of compactly supported infinitely smooth functions on the disc is dense in $W^{m,1}_\nu$.*

Proof. First we remark that if the support of a $g \in W^{m,1}_\nu(\mathbb{D})$ is contained in a compact disc $\Omega_r := \{|z| \leq r\}$ with $0 < r < 1$, then it can be approximated in $W^{m,1}_\nu(\mathbb{D})$ by an element of $C^\infty_0(\mathbb{D})$. This follows from Lemma 3.15 of [1] by choosing the set Ω' there to be a disc Ω_s with $s > r$: the convergence $\lim_{\varepsilon \to 0} J_\varepsilon * g = g$ in $W^{m,1}(\Omega')$ (in the notation of the citation) implies also the convergence in $W^{m,1}_\nu(\mathbb{D})$.

Consequently, it suffices to approximate an arbitrary $f \in W^{m,1}_\nu$ by a compactly supported element of $W^{m,1}_\nu$. To this end, notice that it is possible to define a sequence of radial cut-off functions $\chi_n \in C^\infty_0$, $n = 4, 5, 6, \ldots$, such that $\chi_n(z) = \chi_n(|z|)$ for $z \in \mathbb{D}$, $0 \leq \chi_n(r) \leq 1$ for all $0 \leq r < 1$, $\chi_n(r) = 1$ for $0 \leq r \leq 1 - 3/n$, $\chi_n(r) = 0$ for $1 - 1/n \leq r \leq 1$, and such that for all $k \in \mathbb{N}$

$$\Big| \frac{d^k \chi_n(r)}{dr^k} \Big| \leq C_k n^k \tag{2.3}$$

for all $0 < r < 1$: in fact one can define χ_n as the usual convolution

$$\chi_n(r) = \int_{-\infty}^{\infty} X_{[-1+2/n,1-2/n]}(\varrho) J_n(r-\varrho) d\varrho, \tag{2.4}$$

where $X_{[-1+2/n,1-2/n]}$ is the characteristic function of the interval $[-1 + 2/n, 1 - 2/n]$ and J_n is the standard mollifier

$$J_n(r) = \begin{cases} Cne^{-1/(1-(nr)^2)}\ , & \text{if } |r| \leq 1/n \\ 0\ , & \text{if } |r| > 1/n. \end{cases}$$

and $C > 0$ is a constant independent of n making the integral $\int_{-\infty}^{\infty} J_n\, dr$ equal to 1. The property (2.3) now follows by differentiating J_n under the integral sign in (2.4).

A good approximation of a given $f \in W_\nu^{m,p}$ is then $\chi_n f$ for a large enough n. To see this for example in the case $m = 1$ one denotes $D_n := \{z \mid |z| \geq 1 - 3/n\}$ and estimates one of the terms ($\alpha = (1, 0)$) in (2.2) as

$$\int_{\mathbb{D}} \Big|\frac{\partial}{\partial x}(f - \chi_n f)\Big| \nu dA \leq \int_{\mathbb{D}} \Big|\frac{\partial f}{\partial x}\Big| |1 - \chi_n| \nu dA + \int_{\mathbb{D}} |f| \Big|\frac{\partial \chi_n}{\partial x}\Big| \nu dA$$

$$\leq C \int_{D_n} \Big|\frac{\partial f}{\partial x}\Big| \nu dA + C \int_{D_n} |f| n(1 - (1 - 3/n)) dA$$

$$= C \int_{D_n} \Big|\frac{\partial f}{\partial x}\Big| \nu dA + C' \int_{D_n} |f| dA, \tag{2.5}$$

where we used the facts that $\partial\chi_n/\partial x$ vanishes ouside D_n and $|\partial\chi_n/\partial x| \leq Cn$ on D_n by (2.3). Now (2.5) approaches 0 as $n \to \infty$, since the area of D_n tends to 0 and the integrals $\int_{\mathbb{D}} |f|\, dA$ and $\int_{\mathbb{D}} |\partial f/\partial x|\, \nu dA$ are bounded (by $\|f; W_\nu^{m,1}\|$). The other terms in (2.2) have similar or easier estimates, which proves the lemma for $m = 1$.

The idea for the higher m is similar: the higher powers of ν in (2.2) cancel out the growth of higher derivatives in (2.3). □

As a consequence of this it is possible to describe the dual space.

Definition 2.2. Given $m \in \mathbb{N}$ we denote by $W_\nu^{-m,\infty} := W_\nu^{-m,\infty}(\mathbb{D})$ the (weighted Sobolev) space consisting of distributions a on $\mathbb{D}$ which can be written in the form

$$a = \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha b_\alpha, \tag{2.6}$$

where

$$b_\alpha \in L^\infty_{\nu^{-|\alpha|}} := L^\infty_{\nu^{-|\alpha|}}(\mathbb{D}) \ \text{ i.e.}$$

$$\|b_\alpha; L^\infty_{\nu^{-|\alpha|}}\| := \operatorname*{ess\,sup}_{\mathbb{D}} \nu(z)^{-|\alpha|}|b_\alpha(z)| < \infty. \tag{2.7}$$

Here every b_α is considered as a distribution like a locally integrable function, and the identity (2.6) contains distributional derivatives.

Given such an a, the representation (2.6) is not unique in general. Hence, we define the norm of a by

$$\|a\| := \|a; W^{-m,\infty}_\nu\| := \inf \max_{0\le|\alpha|\le m} \|b_\alpha; L^\infty_{\nu^{-|\alpha|}}\|, \tag{2.8}$$

where the infimum is taken over all representations (2.6).

Lemma 2.3. *The dual of $W^{m,1}_\nu$ is isometrically isomorphic to $W^{-m,\infty}_\nu$ with respect to the dual paring*

$$\langle f, a\rangle := \sum_{0\le|\alpha|\le m} \int_{\mathbb{D}} (D^\alpha f) b_\alpha dA, \tag{2.9}$$

where $f \in W^{m,1}_\nu$, $a \in W^{-m,\infty}_\nu$, and the functions b_α are as in (2.6).

This can be proven using Lemma 2.1 and following the arguments of [1], Sections 3.8–3.10: It is clear that a defines a bounded linear functional on $W^{m,1}_\nu$ via (2.9). By the argument of 3.8 in [1], the dual norm of a coincides with (2.8). On the other hand, 3.9–3.10 of the reference show that every element of the dual space comes from a distribution satisfying (2.7).

Remark 2.4. Let $a \in W^{-m,\infty}_\nu$ be given. Although the representation (2.6) is not unique, the value of the expression on the right hand side of (2.9) is. This is so, since for every $\varphi \in C^\infty_0$, the value of

$$\sum_{0\le|\alpha|\le m} \int_{\mathbb{D}} (D^\alpha \varphi) b_\alpha dA$$

coincides with $\langle \varphi, a\rangle$, by the standard definition of distributional derivative. The uniqueness of (2.9) now follows from Lemma 2.1.

3. Boundedness of Toeplitz operators with distributional symbols.

In the following, a is a distribution on $\mathbb{D}$ having a finite order. Assuming a belongs to a weighted Sobolev space described in the previous section, we give the definition of a Toeplitz operator with symbol a, and prove its boundedness on reflexive Bergman spaces. We remark that this result can be applied for example to all compactly supported distributions, but since such distributions actually determine compact operators, this case is considered only in the next section.

Theorem 3.1. *Assume that the distribution $a \in \mathcal{D}'$ belongs to $W_\nu^{-m,\infty}$ for some m. Then the Toeplitz operator T_a, defined by the formula*

$$T_a f(z) = \sum_{0 \le |\alpha| \le m} \int_{\mathbb{D}} \Big(D_\zeta^\alpha \frac{f(\zeta)}{(1-z\bar\zeta)^2} \Big) b_\alpha(\zeta) dA(\zeta) \quad , \ f \in A^p, \tag{3.1}$$

is well defined and bounded $A^p \to A^p$ for all $1 < p < \infty$. The resulting operator is independent of the choice of the representation (2.6). Moreover, there is a constant $C > 0$ such that

$$\| T_a : A^p \to A^p \| \le C \| a; W_\nu^{-m,\infty} \|. \tag{3.2}$$

Proof. As for the boundedness, let us fix a representation (2.6) such that

$$\| a; W_\nu^{-m,\infty} \| \ge \frac{1}{2} \max_{0 \le |\alpha| \le m} \| b_\alpha; L^\infty_{\nu^{-|\alpha|}} \|, \tag{3.3}$$

Since $2|1 - z\bar\zeta| \ge 1 - |\zeta|$, we can estimate for every $f \in A^p$

$$\begin{aligned}
&\sum_{0 \le |\alpha| \le m} \Big| D_\zeta^\alpha \frac{f(\zeta)}{(1-z\bar\zeta)^2} \Big| \, |b_\alpha(\zeta)| \\
&\le C \sum_{|\alpha| \le m} \sum_{\beta \le \alpha} \Big| (D^\beta f)(\zeta) D_\zeta^{\alpha-\beta} (1 - z\bar\zeta)^{-2} \Big| \, |b_\alpha(\zeta)| \\
&\le C_1 \sum_{|\alpha| \le m} \sum_{\beta \le \alpha} \Big| (D^\beta f)(\zeta) (1 - z\bar\zeta)^{-2-|\alpha|+|\beta|} \Big| \, |b_\alpha(\zeta)| \\
&\le C_2 \sum_{|\alpha| \le m} \sum_{\beta \le \alpha} \Big| (D^\beta f)(\zeta) (1 - z\bar\zeta)^{-2} \Big| (1 - |\zeta|^2)^{-|\alpha|+|\beta|} |b_\alpha(\zeta)| \\
&\le C_3 \| a; W_\nu^{-m,\infty} \| \sum_{|\alpha| \le m} \sum_{\beta \le \alpha} \Big| (D^\beta f)(\zeta) (1 - z\bar\zeta)^{-2} \Big| (1 - |\zeta|^2)^{|\beta|}
\end{aligned}$$

$$\le C_4 \| a; W_\nu^{-m,\infty} \| \sum_{j=0}^m \frac{|f^{(j)}(\zeta)|(1-|\zeta|^2)^j}{|1 - z\bar\zeta|^2}; \tag{3.4}$$

notice that $|D^\beta f| = |f^{(|\beta|)}|$ for all analytic functions. We now recall that, given $g \in A^p$, the functions $|g^{(l)}(z)|(1 - |z|^2)^l$ belong to L^p with norms bounded by $C_l \|g\|_p$ (see [13], Theorem 4.28). Moreover, the maximal Bergman projection is bounded on L^p, i.e., for some constant $C > 0$

$$\Big\| \int_{\mathbb{D}} \frac{|g(\zeta)|}{|1 - z\bar\zeta|^2} dA(\zeta) \Big\|_p \le C \|g\|_p$$

for all $g \in L^p$ (see [13, Corollary 3.13]). These facts together with the definition (3.1) and the estimate (3.4) prove that $Tf \in A^p$ with the norm estimate (3.2).

The uniqueness of the definition (3.1) is a direct consequence of Remark 2.4, as soon as we prove that for all $f \in A^p$, every fixed $z \in \mathbb{D}$, the function

$$F_z(\zeta) := \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} \tag{3.5}$$

of the variable ζ belongs to the Sobolev space $W_\nu^{m,1}$. But this follows from the above cited result that $|f^{(l)}(\zeta)|(1 - |\zeta|^2)^l \in L^p \subset L^1$ for all $l \in \mathbb{N}$: we also get $|(D_\zeta^\alpha F_z)(\zeta)|(1 - |\zeta|^2)^{|\alpha|} \in L^1$ for all α, $|\alpha| \leq m$, since the factor $(1 - z\bar{\zeta})^{-2}$ and all of its derivatives are bounded functions of ζ (for a fixed z). □

A nontrivial example is the symbol

$$a := b_{(0,0)} + D^{(1,0)} b_{(1,0)} = b_{(0,0)} + \frac{\partial}{\partial x} b_{(1,0)}, \tag{3.6}$$

where, for $z = x + iy = re^{i\theta}$,

$$b_{(0,0)}(z) = \begin{cases} 0\ , & \text{if } x \leq 0 \\ 2x\ , & \text{if } x > 0 \end{cases}$$

and

$$b_{(1,0)}(z) = \begin{cases} 0\ , & \text{if } x \leq 0 \\ 1 - r^2\ , & \text{if } x > 0. \end{cases}$$

Notice that $b_{(1,0)}$ can be written as the product $Y(x)(1 - r^2)$, where Y is the usual step function of one real variable. Denoting by $\delta_0(x)$ the Dirac measure of 0 with respect to the variable $x \in \mathbb{R}$, we get

$$\begin{aligned} a &= b_{(0,0)} + D^{(1,0)} b_{(1,0)} \\ &= b_{(0,0)} + \big(D^{(1,0)} Y(x)\big)(1 - r^2) + Y(x) D^{(1,0)}(1 - r^2) \\ &= \delta_0(x)(1 - r^2) \\ &= \delta_0(x)(1 - y^2), \end{aligned}$$

since $D^{(1,0)}(1 - r^2) = -2x$. The symbol a is thus a weighted Dirac measure of the line segment $\{z \in \mathbb{D} \mid \operatorname{Re} z = 0\}$. Clearly, $a \in W_\nu^{-1,\infty}$ and hence defines a bounded Toeplitz operator $A^p \to A^p$. Notice that the support of a is not compact in $\mathbb{D}$.

4. Compactness of Toeplitz operators with distributional symbols.

We start with a remark on distributional symbols with compact supports.

Proposition 4.1. *Any distribution $a \in \mathcal{D}'$ with compact support belongs to the Sobolev space $W_\nu^{-m,\infty}$. Via the formula* (3.1)*, it defines a Toeplitz operator which is a compact operator $A^p \to A^p$.*

Proof of Proposition 4.1. A well-known basic result in distribution theory (see e.g. [6], Theorem 6.26) states that any compactly supported distribution a has a finite order m and can be presented as

$$a = \sum_{|\alpha| \le m} D^\alpha b_\alpha,$$

where the functions b_α can be assumed continuous and supported in an arbitrary neighbourhood V of $\operatorname{supp} a$. In particular we may assume that the closure $\overline{V}$ is a compact subset of $\mathbb{D}$. Trivially, such functions b_α satisfy (2.7), hence, $a \in W_\nu^{-m,\infty}$. Also the compactness of the supports of b_α imply that the operator (3.1) is compact on A^p. □

Theorem 4.2. *Assume that the symbol $a \in \mathcal{D}'$ belongs to $W_\nu^{-m,\infty}$ for some m. The Toeplitz operator T_a,* (3.1)*, is compact, if a has a representation* (2.6) *such that the functions b_α satisfy*

$$\lim_{r \to 1} \operatorname*{ess\,sup}_{|z| \ge r} \nu(z)^{-|\alpha|} |b_\alpha(z)| = 0. \tag{4.1}$$

Proof. If the functions b_α, $0 \le |\alpha| \le m$, are as in (4.1) and $0 < r < 1$, we define for all α the compactly supported functions

$$b_{\alpha,r}(z) = \begin{cases} b_\alpha(z) \ , & \text{if } |z| \le r, \\ 0 \ , & \text{if } |z| > r. \end{cases}$$

We also define the compactly supported distribution

$$a_r = \sum_{0 \le |\alpha| \le m} (-1)^{|\alpha|} D^\alpha b_{\alpha,r},$$

where the derivatives are again distributional. The Toeplitz operator $T_{a_r} : A^p \to A^p$ is compact for every r, by Proposition 4.1. On the other hand, due to the definition (2.8), the property (4.1) and the norm estimate (3.2), the operator norm $\|T_a - T_{a_r} : A^p \to A^p\| = \|T_{a-a_r} : A^p \to A^p\|$ can be made arbitrarily small choosing r close enough to 1. Hence, T_a must be a compact operator. □

Returning to the example (3.6), let us redefine the functions $b_{(0,0)}$ and $b_{(1,0)}$ by

$$b_{(0,0)}(z) = \begin{cases} 0 \ , & \text{if } x \le 0 \\ 2cx(1-r^2)^{c-1} \ , & \text{if } x > 0 \end{cases}$$

and

$$b_{(1,0)}(z) = \begin{cases} 0 \ , & \text{if } x \le 0 \\ (1-r^2)^{c} \ , & \text{if } x > 0, \end{cases}$$

where $c > 1$ is arbitrary. Again, the symbol is a weighted Dirac measure of the line segment $\{z \in \mathbb{D} \mid \operatorname{Re} z = 0\}$ with noncompact support; moreover, in this case the resulting Toeplitz operator $A^p \to A^p$ is even compact.

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